Commun. Korean Math. Soc. **22** (2007), No. 1, pp. 9–14

EXTENDED SPECIAL SETS IN IMPLICATIVE SEMIGROUPS

KEUMSEONG BANG AND KEUM SOOK SO*

ABSTRACT. In this paper, we introduce more extended special sets in implicative semigroups, and obtain some relations with ordered filters.

1. Introduction

Chan and Shum ([3]) introduced the notion of implicative semigroups and ordered filter on it. It is a generalization of implicative semilattice (see Nemitz ([8]) and Blyth ([2])) and has a deep relation with implication in mathematical logic and set difference (see Birkhoff ([1]) and Curry ([4])). The ordered filters play an important role in the study of implicative semilattice. Chan and Shum ([3]) obtained some elementary properties, and constructed quotient structure of implicative semigroup using the ordered filters. Jun et al. ([5]) discussed ordered filters of implicative filters. Jun ([6]) introduced a special set in an implicative semigroup, and obtained an equivalent condition of an ordered filter, and proved that an ordered filter can be represented by the union of such sets. Jun ([7]) introduced a generalized special set, and obtained more general results discussed in [6].

In this paper, we introduce more extended special sets in implicative semigroups, and obtain some relations with ordered filters.

2. Preliminaries

By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*, we mean a set S with a partial ordering "$\leq$" and a binary operation "$\cdot$" such that for all $x, y, z \in S$, we have

(*i*) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(*ii*) $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,
(*iii*) $x \cdot y \leq x$ and $x \cdot y \leq y$,

for all x, y, z in S.

Received November 29, 2006.
2000 *Mathematics Subject Classification.* 20M12, 06F05, 06A06, 06A12.
Key words and phrases. implicative semigroup, commutative, self distributive, ordered section, ordered filter, (extended) special set.
*This research was supported by Hallym University Research Fund, 2001(HRF-2001-26).

An n.p.o. semigroup $(S;\leq,\cdot)$ is said to be *implicative* if there exists an additional binary operation $*: S\times S\to S$ such that for any elements x,y,z of S,

$$z\leq x*y \text{ if and only if } z\cdot x\leq y \tag{iv}$$

The operation $*$ is said to be *implication.* From now on, an implicative n.p.o. semigroup is called simply an *implicative semigroup.* An implicative semigroup $(S;\leq,\cdot)$ is said to be *commutative* if $x\cdot y=y\cdot x$ for all $x,y\in S$, i.e., $(S;\cdot)$ is a commutative semigroup. In any implicative semigroup $(S;\leq,\cdot)$ we have $x*x=y*y$, $\forall x,y\in S$, and this element is the greatest element in S, and we denote it by 1.

Proposition 2.1. ([3]) *Let S be an implicative semigroup. Then the followings hold for any $x,y,z\in S$:*

(1) $x\leq 1,\ x*x=1,\ x=1*x,$
(2) *if* $x\leq y$, *then* $x*z\geq y*z$ *and* $z*x\leq z*y,$
(3) $x\leq y$ *if and only if* $x*y=1,$
(4) $x*(y*z)=(x\cdot y)*z,$

for all x,y,z in S.

Definition 2.2. ([3]) Let S be an implicative semigroup and let F be a non-empty set of S. Then F is called an *ordered filter* of S if

$(F1)$ $x\cdot y\in F$ for any $x,y\in F$,
$(F2)$ If $x\in F$ and $x\leq y$, then $y\in F$.

The following result is very important equivalent condition for the ordered filter in implicative semigroups.

Proposition 2.3. ([5]) *Let S be an implicative semigroup. Then a non-empty subset F of S is an ordered filter of S if and only if it satisfies the following conditions*:

$(F3)$ $1\in F,$
$(F4)$ $x*y\in F$ *and* $x\in F$ *imply* $y\in F.$

Note that if S is a commutative implicative semigroup, then

$$x*(y*z)=y*(x*z)$$

for any $x,y,z\in S$.

3. Ordered sections and special sets

In what follows let S and $\mathbf{N}$ denote an implicative semigroup and the set of all positive integers, respectively, unless otherwise specified.

Definition 3.1. For any $x,y\in S$ and $n\in\mathbf{N}$ we define

$$S(x):=\{z\in S\,|\,x*z=1\},$$
$$S(x,y):=\{z\in S\,|\,x*(y*z)=1\}.$$

The set $S(x)$ is called an *ordered section* of $x \in S$, and $S(x,y)$ is called a *special set* of S. Obviously, $1, y \in S(x,y)$ for any $x, y \in S$. Note that if S is commutative, then $x \in S(x,y)$ for all $x, y \in S$.

Proposition 3.2. *If S is commutative, then $S(x) \subseteq S(x,y)$ for any $y \in S$.*

Proof. If $z \in S(x)$ then $x * z = 1$ and hence $x * (y * z) = y * (x * z) = y * 1 = 1$, since S is commutative, which means that $z \in S(x,y)$. □

By Proposition 3.2 we obtain $S(x) \subseteq \cap_{y \in S} S(x,y)$. We prove the reverse inequality also holds:

Proposition 3.3. *If S is commutative and $x \in S$, then $S(x) = \cap_{y \in S} S(x,y)$.*

Proof. If $z \in \cap_{y \in S} S(x,y)$, then $z \in S(x,y)$ for any $y \in S$. Hence $x * (y * z) = 1$ for any $y \in S$. Taking $y := 1$, we obtain $1 = x * (1 * z) = x * z$, which proves $z \in S(x)$. □

Using Proposition 2.1 and Proposition 3.3 we obtain the following:

Corollary 3.4. *If S is commutative, then for any $x \in S$, we have $S(x) = S(x,1) = \cap_{y \in S} S(x,y)$.*

An implicative semigroup S is said to be *self distributive* ([6]) if $x * (y * z) = (x * y) * (x * z)$ for any x, y, z in S. The following example shows that S is self distributive.

Example 3.5. ([6]) Let $S := \{1, a, b, c, d\}$ be an implicative semigroup with the following tables:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then it is a self distributive implicative semigroup.

Proposition 3.6. *If S is self distributive implicative semigroup and $x \in S$, then $S(x)$ is an ordered filter of S.*

Proof. Since $x * 1 = 1$, $1 \in S(x)$. If $\alpha * \beta, \alpha \in S(x)$, then $x * (\alpha * \beta) = 1$ and $x * \alpha = 1$. Hence, $1 = x * (\alpha * \beta) = (x * \alpha) * (x * \beta) = 1 * (x * \beta) = (x * \beta)$ and hence $\beta \in S(x)$. □

Note that if $x * y = 1$, then $S(y) \subseteq S(x)$.

Theorem 3.7. *If S is a commutative self distributive semigroup, then $S(x) = \cup_{\substack{y \in S \\ x \leq y}} S(x,y)$.*

Proof. Let $y \in S$ with $x \leq y$. For any $\beta \in S(x,y)$, $1 = x * (y * \beta) = (x * y) * (x * \beta) = 1 * (x * \beta) = x * \beta$. Hence, $\beta \in S(x)$. This means that $\cup_{\substack{y\in S\\ x\leq y}} S(x,y) \subseteq S(x)$. Since S is commutative, by applying Proposition 3.2, we obtain $S(x) \subseteq \cup_{\substack{y\in S\\ x\leq y}} S(x,y)$, proving the theorem. □

Example 3.5 is a commutative self distributive semigroup. It is easy to check that $S(c) = \{1, a, c\} = S(c,c) \cup S(c,a) \cup S(c,1)$.

Proposition 3.8. *If S is commutative, then $S(x) \cup S(y) \subseteq S(x,y) \subseteq S(x \cdot y)$ for any $x, y \in S$.*

Proof. Since S is commutative, it is easy to show that $S(x) \cup S(y) \subseteq S(x,y)$. For any $\beta \in S(x,y)$, we have $1 = x * (y * \beta) = (x \cdot y) * \beta$ by Proposition 2.1-(4), proving that $S(x,y) \subseteq S(x \cdot y)$. □

4. The extended special set $S(x_1, x_2, \ldots, x_n)$

For any elements $x, y, x_1, \ldots, x_n \in S$ and $n \in \mathbf{N}$, we use the notation $\prod_{i=1}^{n} x_i * z := x_1 * (x_2 * (\cdots * (x_n * z)) \cdots)$. Denote by $S(x_1, \ldots, x_n) := \{z \in S \mid \prod_{i=1}^{n} x_i * z = 1\}$, and we call it a *extended special set.* Obviously, $1, x_n \in S(x_1, \ldots, x_n)$. If S is commutative, then $x_i \in S(x_1, \ldots, x_n)$ for any $i = 1, \ldots, n$.

Proposition 4.1. *If S is a commutative implicative semigroup and $x_1, \ldots, x_n \in S$, then $S(x_1) \subseteq S(x_1, x_2) \subseteq \cdots \subseteq S(x_1, \ldots, x_n)$.*

Proof. If $z \in S(x_1, \ldots, x_i)$, then $x_1 * (x_2 * (\cdots (x_i * z)) \cdots) = 1$. Since S is commutative, we have $x_1 * (x_2 * (\cdots (x_{i+1} * z)) \cdots) = x_{i+1} * (x_1 * (x_2 * (\cdots (x_i * z)) \cdots) = x_{i+1} * 1 = 1$, proving that $z \in S(x_1, \ldots, x_{i+1})$. □

Since S is commutative, it is easy to show that $S(x_1, x_2) = S(x_1, x_2, 1) \subseteq S(x_1, x_2, x_3)$. More generally, we can see that $S(x_1, \ldots, x_i) = S(x_1, \ldots, x_i, 1) = \cap_{x\in S} S(x_1, \ldots, x_i, x) \subseteq S(x_1, \ldots, x_{i+1})$.

Proposition 4.2. *If S is a commutative implicative semigroup and $x_1, \ldots, x_n \in S$, then $S(x_1, \ldots, x_n) = S(x_{\pi(1)}, \ldots, x_{\pi(n)})$, where π is a permutation on $\{1, \ldots, n\}$.*

Proof. Straightforward. □

It is easy to show the following:

Proposition 4.3. *Let S be a commutative implicative semigroup and let $x_1, \ldots, x_n \in S$. If there is an $\alpha \in S$ such that $\alpha * x = 1$ for any $x \in S$, then $S = S(\alpha, x_1, \ldots, x_n) = S(x_1, \alpha, x_2, \ldots, x_n) = \cdots = S(x_1, \ldots, x_n, \alpha)$.*

Proposition 4.4. *Let S be a self distributive implicative semigroup and let $x_1, \ldots, x_n \in S$. Then $S(x_1, \ldots, x_n)$ is an ordered filter of S.*

Proof. It is easy to show that $1 \in S(x_1, \ldots, x_n)$. For any $\alpha, \beta \in S$ with $\alpha * \beta, \alpha \in S(x_1, \ldots, x_n)$ we have $1 = \prod_{i=1}^{n} x_i * \alpha$ and $1 = \prod_{i=1}^{n} x_i * (\alpha * \beta) = (\prod_{i=1}^{n} x_i * \alpha) * (\prod_{i=1}^{n} x_i * \beta) = 1 * (\prod_{i=1}^{n} x_i * \beta) = \prod_{i=1}^{n} x_i * \beta$. This means that $\beta \in S(x_1, \ldots, x_n)$, proving that $S(x_1, \ldots, x_n)$ is an ordered filter of S. □

Using the set $S(x_1, \ldots, x_n)$ we obtain an equivalent condition of an ordered filter of S.

Theorem 4.5. *Let F be a non-empty set of a commutative implicative semigroup S. Then F is an ordered filter of S if and only if $S(x_1, \ldots, x_n) \subseteq F$ for any $x_1, \ldots, x_n \in F$ where $n \in \mathbf{N}$.*

Proof. Let $x_1, \ldots, x_n \in F$. If $\beta \in S(x_1, \ldots, x_n)$, then $\prod_{i=1}^{n} x_i * \beta = 1$. Since $x_i \in F$, by $(F4)$, $\beta \in F$.

Conversely, since $\prod_{i=1}^{n} x_i * 1 = 1$, $1 \in S(x_1, \ldots, x_n) \subseteq F$. For any $\alpha, \beta \in S$ with $\alpha*\beta, \alpha \in F$, we have $1 = (\alpha*\beta)*(\alpha*\beta) = \alpha*((\alpha*\beta)*\beta) = \prod_{i=1}^{n-2} 1*(\alpha*((\alpha*\beta)*\beta))$, since S is commutative. This proves that $\beta \in S(1, \ldots, 1, \alpha, \alpha*\beta) \subseteq F$. Hence F is an ordered filter of S. □

Theorem 4.6. *Let F be a non-empty set of a commutative implicative semigroup S. Then $F = \cup_{x_1,\ldots,x_n \in F} S(x_1, \ldots, x_n)$.*

Proof. If $\beta \in F$, then $\beta \in S(\beta, 1)$. Since S is commutative, by Proposition 4.1, we have $\beta \in S(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in F$. Hence $\beta \in \cup_{x_1,\ldots,x_n \in F} S(x_1, \ldots, x_n)$.

Conversely, if $\beta \in \cup_{x_1,\ldots,x_n \in F} S(x_1, \ldots, x_n)$, then $1 = x_1 * (x_2 * (\cdots * (x_n * \beta)) \cdots)$ for some $x_1, \ldots, x_n \in F$. Since $x_i \in F$, by $(F4)$, we obtain $\beta \in F$. □

Proposition 4.7. *Let F be a non-empty subset of a self distributive implicative semigroup S. If $x_1 \leq x_2 \leq \cdots \leq x_n$ be an n-element chain, then $S(x_1) = S(x_1, x_2) = \cdots = S(x_1, \ldots, x_n)$.*

Proof. If $\alpha \in S(x_1, \ldots, x_n)$, then $1 = \prod_{i=1}^{n} x_i*\alpha = \prod_{i=1}^{n-2} x_i*(x_{n-1}*(x_n*\alpha)) = \prod_{i=1}^{n-2} x_i*((x_{n-1}*x_n)*(x_{n-1}*\alpha)) = \prod_{i=1}^{n-2} x_i*(1*(x_{n-1}*\alpha)) = \prod_{i=1}^{n-1} x_i*\alpha = \cdots = \prod_{i=1}^{3} x_i*\alpha = x_1*(x_2*(x_3*\alpha)) = x_1*((x_2*x_3)*(x_2*\alpha)) = x_1*(x_x*\alpha) = x_1 * \alpha$. This means that $\alpha \in S(x_1)$. By applying Proposition 4.1, we have $S(x_1) = S(x_1, x_2) = \cdots = S(x_1, \ldots, x_n)$. □

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. **25**, Providence (1967).

[2] T. S. Blyth, *Pseudo-residuals in semigroups*, J. London Math. Soc. **40** (1965), 441–454.

[3] M. W. Chan and K. P. Shum, *Homomorphisms of implicative semigroups*, Semigroup Forum **46** (1993), 7–15.

[4] H. B. Curry, *Foundations of Mathematical Logics*, McGrow-Hill, New york, 1963.

[5] Y. B. Jun, J. Meng and X. L. Xin, *On ordered filters of implicative semigroup*, Semigroup Forum **54** (1997), 75–82.

[6] Y. B. Jun, *A note on ordered filters of implicative semigroups*, Bull. Korean Math. Soc. **34** (1997), no. 2, 185–191.

[7] ______, *Some results on ordered filters of implicative semigroup*, Int. J. Math. & Math. Sci. **26** (2001), 731–735.
[8] W. C. Nemitz, *Implicative semilattices*, Trans. Amer. Math. Soc. **117** (1965), 128–142.

KEUMSEONG BANG
DEPARTMENT OF MATHEMATICS
THE CATHOLIC UNIVERSITY OF KOREA
PUCHEON 420-743, KOREA
E-mail address: bang@catholic.ac.kr

KEUM SOOK SO
DEPARTMENT OF MATHEMATICS
HALLYM UNIVERSITY
CHUNCHEON 200-702, KOREA
E-mail address: ksso@hallym.ac.kr