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SUFFICIENT CONDITIONS FOR STARLIKENESS AND STRONGLY-STARLIKENESS

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Abstract. In this paper we derive certain sufficient conditions for starlikeness and strongly-starlikeness of analytic functions in U, by using the method of differential subordination.

1. Introduction

Let A_n denote the class of functions of f of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots , \quad z \in U,$$

which are analytic in the unit disc U.

Let $A = A_1$ and let $S^*(\beta) = \left\{ f \in A \middle| \operatorname{Re} \dfrac{zf'(z)}{f(z)} > \beta, 0 \le \beta < 1,\ z \in U \right\}$ be the class of starlike functions of order β in U.

For $\lambda \in (0, 1]$ let

$$\widetilde{S}^*(\lambda) = \left\{ f \in A \middle| \left| \arg \frac{zf'(z)}{f(z)} \right| < \lambda \frac{\pi}{2},\ z \in U \right\}$$

denote the class of strongly starlike functions.

We will use the following notions and lemmas to prove our results.

If f and g are analytic functions in U, then we say that f is subordinate to g, and write $f \prec g$ or $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function $f(z)$ in A is said to be in the class $S^*(C, D)$ if satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + CZ}{1 + DZ}$$

for some C and D $(-1 \le D < C \le 1)$.

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Lemma A ([1], [2], [3]). *Let q be univalent in $\overline{U}$ with $q(\xi) \neq 0$, $|\xi| = 1$, $q(0) = a$ and $p(z) = a + p_n z^n + \cdots$ be analytic in U, $p(z) \not\equiv a$, and $n \geq 1$.*

If $p(z) \not\prec q(z)$ then there exist points $z_0 \in U$, $\xi_o \in \partial U$ and there is $m \geq n$ such that:

$$\text{(i)} \quad p(z_0) = q(\xi_0)$$

$$\text{(ii)} \quad z_0 p'(z_0) = m\xi_0 q'(\xi_0).$$

The function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ is analytic and univalent in U for any $t \geq 0$ and if $L(z,t_1) \prec L(z,t_2)$ when $0 \leq t_1 \leq t_2$.

Lemma B ([5]). *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in (0,1]$ and $M > 0$ such that:*

(i) *$L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.*

(ii) *There is a function $p(z,t)$ analytic in U for any $t \geq 0$ measurable in $[0,\infty)$ for any $z \in U$ with $\operatorname{Re} p(z,t) > 0$ for $z \in U$, $t \geq 0$ such that*

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t) \quad \text{for } |z| < r$$

and for almost any $t \geq 0$.

The object of this paper is to derive some sufficient conditions for starlikeness and strongly-starlikeness of functions in A.

2. Main results

Theorem 2.1. *Let $\alpha > 0$ and let q be a convex function in U, with $q(0) = 1$, $\operatorname{Re} q(z) > \dfrac{\alpha - 1}{\alpha}$ and let*

$$h(z) = \alpha n z q'(z) + \alpha q^2(z) + (2 - 2\alpha)q(z), \tag{1}$$

where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ satisfies the condition

$$\alpha z p'(z) + \alpha p^2(z) + (2 - 2\alpha)p(z) \prec h(z), \tag{2}$$

where h is given by (1) then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Proof. Let

$$\begin{aligned} L(z,t) &= \alpha(n+t)zq'(z) + \alpha q^2(z) + (2 - 2\alpha)q(z) \\ &= \psi(q(z), (n+t)zq'(z)). \end{aligned} \tag{3}$$

If $t = 0$, we have $L(z,0) = \alpha n z q'(z) + \alpha q^2(z) + (2 - 2\alpha)q(z) = h(z)$. We will show that condition (2) implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

From (3) we deduce

$$\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} = (n+t)\left[1+\frac{zq''(z)}{q'(z)}\right] + 2q(z) + \frac{2-2\alpha}{\alpha}$$

and by using the convexity of $q(z)$ and $\operatorname{Re} q(z) > \dfrac{\alpha-1}{\alpha}$, we obtain

$$\operatorname{Re} \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} > 0.$$

Hence by Lemma B, we deduce that $L(z,t)$ is a subordination chain. In particular, the function $h(z) = L(z,0)$ is univalent and $h(z) \prec L(z,t)$, for $t \geq 0$. If we suppose that $p(z) \not\prec q(z)$ then, from Lemma A, there exist $z_0 \in U$, and $\xi_0 \in \partial U$ such that $p(z_0) = q(\xi_0)$ with $|\xi_0| = 1$, and $z_0 p'(z_0) = (n+t)\xi_0 q'(\xi_0)$ with $t \geq 0$. Hence

$$\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\xi_0), (n+t)\xi_0 q'(\xi_0)) = L(\xi_0, t) \quad t \geq 0.$$

Since $h(z_0) = L(z_0, 0)$, we deduce that $\psi_0 \notin h(U)$, which contradicts condition (2). Therefore we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

If we let $p(z) = \dfrac{zf'(z)}{f(z)}$, where $f \in A_n$, then Theorem 2.1 can be written in the following equivalent form:

Theorem 2.2. *Let $q(z)$ be convex function with $q(0) = 1$, $\operatorname{Re} q(z) > \dfrac{\alpha-1}{\alpha}$ and $\alpha > 0$. If $f \in A_n$, with $\dfrac{f(z)}{z} \neq 0$, $z \in U$, satisfies the condition:*

$$\alpha\frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U,$$

where h is given by (1) then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

3. Some applications

Theorem 3.1. *If $f(z) \in A$ satisfies $\dfrac{f(z)}{z} \neq 0$ in U and*

$$\alpha\frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z),$$

where

(4)
$$h(z) = \frac{(\alpha C^2 + CD(2-2\alpha))z^2 + (\alpha(C-D) + 2C + D(2-2\alpha))z + 2 - \alpha}{(1+Dz)^2},$$

(5)
$$-1 \le D < C \le 1, \quad \frac{1-C}{1-D} \ge \frac{\alpha - 1}{\alpha} \quad \text{and } \alpha > 0,$$

then $f(z) \in S^*(C, D)$.

Proof. Let us define the analytic function $p(z)$ in U by $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem 2.1. If $q(z) = \dfrac{1+Cz}{1+Dz}$ $(-1 \le D < C \le 1)$, then $q(z)$ is a convex function, and

$$\begin{aligned} h(z) &= \alpha z \left(\frac{1+Cz}{1+Dz}\right)' + \alpha \left(\frac{1+Cz}{1+Dz}\right)^2 + (2-2\alpha)\left(\frac{1+Cz}{1+Dz}\right) \\ &= \frac{(\alpha C^2 + CD(2-2\alpha))z^2 + (\alpha(C-D) + 2C + D(2-2\alpha))z + 2 - \alpha}{(1+Dz)^2}. \end{aligned}$$

By Theorem 2.2, we get $f(z) \in S^*(C, D)$. □

Corollary 3.2. *If* $f(z) \in A$ *satisfies* $\dfrac{f(z)}{z} \neq 0$ *in* U *and*

(6)
$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec \alpha C^2 z^2 + C(2+\alpha)z + 2 - \alpha,$$

where $0 < C \le 1$, $1 - C \ge \dfrac{\alpha-1}{\alpha}$ *and* $\alpha > 0$, *then*

(7)
$$\left|\frac{zf'(z)}{f(z)} - 1\right| < C \quad (z \in U)$$

and the bound C *in* (7) *is sharp.*

Proof. Letting $D = 0$ in Theorem 3.1 and using (6), we have the inequality (7). If we take $f(z) = ze^{Cz}$, then

$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} = \alpha C^2 z^2 + C(2+\alpha)z + 2 - \alpha$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = C|z| \;\to\; C$$

as $|z| \to 1$. □

Corollary 3.3. *If* $f(z) \in A$ *satisfies* $\dfrac{f(z)}{z} \neq 0$ *in* U *and*

(8)
$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z)$$

where

$$h(z) = \frac{(1-2\beta)(3\alpha-2\alpha\beta-2)z^2+(4\alpha-2\alpha\beta-4\beta)z+2-\alpha}{(1-z)^2}, \tag{9}$$

$0 \le \beta < 1$ *and* $0 < \alpha \le \frac{1}{1-\beta}$, *then* $f(z) \in S^*(\beta)$.

Proof. Setting $C = 1-2\beta$ $(0 \le \beta < 1)$ and $D = -1$ in Theorem 3.1, it follows from (8) and (9) that $f(z) \in S^*(\beta)$. □

Taking $\alpha = 1$ in Corollary 3.3, we have following corollary.

Corollary 3.4. *If* $f(z) \in A$ *satisfies* $\frac{f(z)}{z} \neq 0$ *in* U, $0 \le \beta < 1$ *and*

$$\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{(1-2\beta)^2 z^2 + (4-6\beta)z + 1}{(1-z)^2}, \tag{10}$$

then $f(z) \in S^*(\beta)$.

Remark 1. Setting $\beta = \frac{1}{2}$ in Corollary 3.4, we obtain a result due to K. S. Padmanabham [4]. That is, if $\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1+z}{(1-z)^2}$ in U then $S^*(\frac{1}{2})$.

Theorem 3.5. *If* $f(z) \in A$ *satisfies* $\frac{f(z)}{z} \neq 0$ *in* U *and*

$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z),$$

where

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\lambda-1} \left[\frac{(2\alpha-2)z^2 + 2\alpha\lambda z + 2 - 2\alpha}{(1-z)^2} + \alpha\left(\frac{1+z}{1-z}\right)^{\lambda+1}\right] \tag{11}$$

$0 < \lambda \le 1$ *and* $0 < \alpha \le 1$, *then* $f(z) \in \widetilde{S}^*(\lambda)$.

Proof. If $q(z) = \left(\frac{1+z}{1-z}\right)^{\lambda}$ in Theorem 2.1, then

$$\begin{aligned} h(z) &= \alpha z \left(\left(\frac{1+z}{1-z}\right)^{\lambda}\right)' + \alpha\left(\left(\frac{1+z}{1-z}\right)^{\lambda}\right)^2 + (2-2\alpha)\left(\frac{1+z}{1-z}\right)^{\lambda} \\ &= \left(\frac{1+z}{1-z}\right)^{\lambda-1} \left[\frac{(2\alpha-2)z^2 + 2\alpha\lambda z + 2 - 2\alpha}{(1-z)^2} + \alpha\left(\frac{1+z}{1-z}\right)^{\lambda+1}\right]. \end{aligned}$$

By Theorem 2.2, we get $f(z) \in \widetilde{S}^*(\lambda)$. □

Remark 2. For $\alpha = 1$ and $\lambda = \frac{1}{2}$, Theorem 3.5 refines a result by Ramesha et al. [6]. That is, if $\operatorname{Re}\left\{\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0$ then $\widetilde{S}^*(\frac{1}{2})$.

By choosing certain subdomains of $h(U)$, we can deduce the following particular criteria for strongly-starlikeness.

Corollary 3.6. *Let* $0 < \lambda \leq 1$ *and* $0 < \alpha \leq 1$. *If* $f \in A$ *with* $\frac{f(z)}{z} \neq 0$, *satisfies the condition*

$$\left|\arg\left\{\alpha\frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)}\right\}\right| < \phi_0(\alpha,\lambda), \tag{12}$$

where

$$\phi_0(\alpha,\lambda) = \frac{\lambda\pi}{2} + \tan^{-1}\frac{\frac{\alpha\lambda}{2}(1+t_0^2) + \alpha\sin\frac{\lambda\pi}{2}t_0^{\lambda+1}}{\alpha\cos\frac{\lambda\pi}{2}\cdot t_0^{\lambda+1} + (2-2\alpha)t_0}, \tag{13}$$

and t_0 *is the root of the equation:*

$$\begin{aligned}&\left(\alpha^2\lambda\cdot\cos\frac{\lambda\pi}{2} - \frac{\alpha^2\lambda}{2}(\lambda+1)\cos\frac{\lambda\pi}{2}\right)t^{\lambda+2} + \alpha\lambda(2-2\alpha)t^2\\ &+\alpha\lambda(2-2\alpha)\sin\frac{\lambda\pi}{2}t^{\lambda+1} - \frac{\alpha^2\lambda}{2}(\lambda+1)\cos\frac{\lambda\pi}{2}t^{\lambda} - \alpha(1-\alpha)\lambda = 0\end{aligned} \tag{14}$$

then $f(z) \in \widetilde{S}^*(\lambda)$.

Proof. The domain $h(U)$, where h is given by (11), is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leq \theta \leq \pi$.

Letting $\cot\frac{\theta}{2} = t$ and $h(e^{i\theta}) = u + iv$, we find:

$$\begin{cases} u(t) = t^{\lambda}\left[-\frac{\alpha\lambda a}{2t}(1+t^2) + (2-2\alpha)b + \alpha(b^2-a^2)t^{\lambda}\right] \\ v(t) = t^{\lambda}\left[\frac{\alpha\lambda b}{2t}(1+t^2) + (2-2\alpha)a + 2\alpha abt^{\lambda}\right], \end{cases} \tag{15}$$

where $a = \sin\frac{\lambda\pi}{2}$ and $b = \cos\frac{\lambda\pi}{2}$.

We also have

$$\begin{aligned}\phi = \phi(t) &= \arg h(e^{i\theta})\\ &= \frac{\lambda\pi}{2} + \tan^{-1}\frac{\frac{\alpha\lambda}{2}(1+t^2) + \alpha\cdot\sin\frac{\lambda\pi}{2}t^{\lambda+1}}{\alpha\cdot\cos\frac{\lambda\pi}{2}\cdot t^{\lambda+1} + (2-2\alpha)t}.\end{aligned} \tag{16}$$

From (15) it is easy to show that the equation $\phi'(t) = 0$, has the root t_0, which is the root of the equation (14). Hence

$$\min_{t\geq 0}\phi(t) = \phi(t_0) = \phi_0(\alpha,\lambda),$$

where $\phi_0(\alpha,\lambda)$ is given by (13).

We deduce that the sector $\{w : |\arg w| < \phi_0(\alpha, \lambda)\}$ is the largest sector which lies in $h(U)$. Hence (12) implies

$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z)$$

where h is given by (11). By Theorem 2.2, we get $f(z) \in \widetilde{S}^*(\lambda)$. □

Corollary 3.7. *Let* $0 < \lambda \leq 1$, $0 < \alpha \leq 1$. *If* $f \in A$ *with* $\frac{f(z)}{z} \neq 0$, *satisfies the condition*

$$\left|\operatorname{Im}\left(\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)}\right)\right| < V(\alpha, \lambda), \tag{17}$$

where $V(\alpha, \lambda) = v(t_0)$, *with* v *given by* (15) *and* t_0 *is the root of the equation:*

$$4\alpha \sin \lambda\pi t^{\lambda+1} + \alpha(\lambda+1)\cos\frac{\lambda\pi}{2}t^2 + 2(2-2\alpha)\sin\frac{\lambda\pi}{2}t + \alpha(\lambda-1)\cos\frac{\lambda\pi}{2} = 0 \tag{18}$$

then $f \in \widetilde{S}^*(\lambda)$.

Proof. From (15) we deduce:

$$v' = \lambda t^{\lambda-2}\left[\frac{\alpha(\lambda-1)b}{2} + (2-2\alpha)at + \frac{\alpha(\lambda+1)b}{2}t^2 + 4\alpha ab t^{\lambda+1}\right]$$

and the equation $v'(t) = 0$ become (18).

Hence

$$V(\alpha, \lambda) = v(t_0) = \min_{t \geq 0} v(t)$$

and we deduce that the strip $|v| < V(\alpha, \lambda)$ lies in $h(U)$. Therefore (17) implies

$$\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z)$$

which h is given by (11). By Theorem 2.2, we get $f \in \widetilde{S}^*(\lambda)$. □

Corollary 3.8. *Let* $0 < \lambda \leq 1$, $0 < \alpha \leq 1$. *If* $f \in A$ *with* $\frac{f(z)}{z} \neq 0$, *satisfies the condition*

$$\operatorname{Re}\left[\alpha \frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)}\right] > U(\alpha, \lambda), \tag{19}$$

where $U(\alpha, \lambda) = u(t_0)$, *with* u *given by* (15) *and* t_0 *is the root of the equation:*

$$4\alpha \cos \lambda\pi t^{\lambda+1} - \alpha(\lambda+1)\sin\frac{\lambda\pi}{2}t^2 + 2(2-2\alpha)\cos\frac{\lambda\pi}{2}t - \alpha(\lambda-1)\sin\frac{\lambda\pi}{2} = 0 \tag{20}$$

then $f \in \widetilde{S}^*(\lambda)$.

Proof. From (15) we deduce:

$$u' = \lambda t^{\lambda-2}\left[-\frac{\alpha a(\lambda-1)}{2} + (2-2\alpha)bt - \frac{\alpha a(\lambda+1)}{2}t^2 + 2\alpha(b^2-a^2)t^{\lambda+1}\right]$$

and the equation $u'(t) = 0$ become (18).

Hence

$$U(\alpha,\lambda) = u(t_0) = \max_{t\geq 0} u(t)$$

and we deduce that the half-plane $\{w : \text{Re } w > U(\alpha,\lambda)\}$ lies in $h(U)$. Therefore (19) implies

$$\alpha\frac{z^2 f''(z)}{f(z)} + (2-\alpha)\frac{zf'(z)}{f(z)} \prec h(z),$$

where $h(z)$ is given by (11). Hence we get $f(z) \in \widetilde{S}^*(\lambda)$ from Theorem 2.2. □

References

[1] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289–305.

[2] ———, *Differential subordinations and univalent functions*, Michigan Math. I. **28** (1981), 157–171.

[3] ———, *The theory and applications of second order differential subordinations*, Studia Univ. Babes-Bolyai, Math. **34** (1989), no. 4, 3–33.

[4] K. S. Padmanabhan, *On sufficient conditions for starlikeness*, Indian J. Pure Appl. Math. **32** (2001), no. 4, 543–550.

[5] Ch. Pommerenke, *Univalent Function*, Vandenhoeck Ruprecht in Göttngen, 1975.

[6] C. Ramesha, S. Kumar, and K. S. Padmanabhan, *A sufficient condition for starlikeness*, Chinese J. Math. **23** (1995), 167–171.

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