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C-INTEGRAL AND DENJOY-C INTEGRAL

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Abstract. In this paper, we define and study the C-integral of functions mapping an interval [a,b] into a Banach space X and discuss the relations among Henstock integral, C-integral and McShane integral. We also study the Denjoy extension of the C-integral.

1. Introduction

In 1986 A. M. Bruckner, R. J. Fleissner and J. Fordan [3] researched the following function

$$F(x) = \begin{cases} x \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.1}$$

It is a primitive for the Riemann improper integral and therefore for the Henstock integral (restricted Denjoy integral and Perron integral), but it is neither a Lebesgue primitive, neither a differentiable function, nor a sum of Lebesgue primitive and a differentiable function. It is natural to ask whether there is a minimal integral includes the Lebesgue integral and the derivative.

In 1996 B. Bongiorno provided a new solution to the problem of recovering a function from its derivative by integration by introducing a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function. B. Bongiorno and L. Di Piazza [1, 2, 12] discussed some properties of the C-integral of real-valued functions.

In this paper, we define and study the C-integral of functions mapping an interval [a,b] into a Banach space X, we also discuss the relations among Henstock integral, C-integral and McShane integral. Finally, we study the Denjoy extension of the C-integral.

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2. Definitions and basic properties

Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. A partition D is a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0. $\delta(\xi)$ is a positive function on I_0, i.e. $\delta(\xi) : I_0 \to \mathrm{R}^+$. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a partial partition of I_0 if $\bigcup_{i=1}^n I_i \subset I_0$,

(2) a partition of I_0 if $\bigcup_{i=1}^n I_i = I_0$,

(3) δ - fine McShane partition of I_0 if $I_i \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_0$ for all $i = 1, 2, \dots, n$.

(4) δ - fine C-partition of I_0 if it is a δ - fine McShane partition of I_0 and satisfying the condition

$$\sum_{i=1}^{n} \mathrm{dist}(\xi_i, I_i) < \frac{1}{\varepsilon},$$

here $\mathrm{dist}(\xi_i, I_i) = \inf\{|t_i - \xi_i| : t_i \in I_i\}$,

(5) δ - fine partition of I_0 if $\xi_i \in I_i \subset B(\xi_i, \delta(\xi_i))$ for all $i = 1, 2, \dots, n$.

Given a δ - fine C-partition (McShane partition) $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)|I_i|$$

for integral sums over D, whenever $f : I_0 \to X$.

Definition 2.1. A function $f : I_0 \to X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the *C-integral* of f on I_0, and we write $A = \int_{I_0} f$ or $A = (\mathrm{C}) \int_{I_0} f$.

The function f is C-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is C-integrable on I_0. We write $\int_E f = \int_{I_0} f\chi_E$.

We can easily get the following two theorems.

Theorem 2.2. *A function $f : I_0 \to X$ is C-integrable if and only if for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$\|S(f, D_1) - S(f, D_2)\| < \epsilon$$

for arbitrary δ - fine C-partition D_1 and D_2 of I_0.

Theorem 2.3. *Let $f : I_0 \to X$ and $g : I_0 \to X$.*

(1) *if f is C-integrable on I_0, then f is C-integrable on every subinterval of I_0,*

(2) *if f is C-integrable on each of the intervals I_1 and I_2, where I_i are non-overlapping and $I_1 \bigcup I_2 = I_0$, then f is C-integrable on I_0 and $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$,*

(3) *if* f *and* g *are C-integrable on* I_0 *and if* α *and* β *are real numbers, then* $\alpha f+\beta g$ *is C-integrable on* I_0 *and* $\int_{I_0}(\alpha f+\beta g)=\alpha\int_{I_0} f+\beta\int_{I_0} g$.

Lemma 2.4. (Saks-Henstock) *Let* $f: I_0\to X$ *be C-integrable on* I_0. *Then for* $\varepsilon>0$ *there is a positive function* $\delta(\xi): I_0\to R^+$ *such that*

$$\|S(f,D)-\int_{I_0} f\|<\epsilon$$

for each δ *- fine C-partition* $D=\{(I,\xi)\}$ *of* I_0. *Particularly, if* $D'=\{(I_i,\xi_i)\}_{i=1}^m$ *is an arbitrary* δ *- fine partial C-partition of* I_0, *we have*

$$\|S(f,D')-\sum_{i=1}^m\int_{I_i} f(\xi_i)\|\leq\epsilon.$$

Proof. Assume $D'=\{(I_i,\xi_i)\}_{i=1}^m$ is an arbitrary δ - fine partial C-partition of I_0, then the complement $I_0\backslash\bigcup_{i=1}^m I_i$ can be expressed as a fine collection of closed subintervals and we denote $I_0\backslash\bigcup_{i=1}^m I_i=\bigcup_{j=1}^k I_j'$.

Let $\eta>0$ be arbitrary. From Theorem 2.3 we know $\int_{I_j'} f$ exists, then there exists a positive function δ_j on I_j' such that if D_j is a δ_j - fine C-partition of I_j', then

$$\|S(f,D_j)-\int_{I_j'} f\|<\frac{\eta}{k}.$$

Assume that $\delta_j(\xi)\leq\delta(\xi)$ for all $\xi\in I_0$. Let $D_0=D'+D_1+D_2+\cdots+D_k$, obviously, D_0 is δ - fine C-partition of I_0, We have

$$\|S(f,D_0)-\int_{I_0} f\|=\|S(f,D')+\sum_{j=1}^k S(f,D_j)-\int_{I_0} f\|<\epsilon.$$

Consequently, we obtain

$$\begin{aligned}\|S(f,D')-\sum_{i=1}^m\int_{I_i} f(\xi_i)\| &= \|S(f,D_0)-\sum_{j=1}^k S(f,D_j)-(\int_{I_0} f-\sum_{j=1}^k\int_{I_j'} f)\|\\ &\leq \|S(f,D_0)-\int_{I_0} f\|+\sum_{j=1}^k\|S(f,D_j)-\int_{I_j'} f\|\\ &< \epsilon+\frac{k\eta}{k}=\epsilon+\eta.\end{aligned}$$

Since $\eta>0$ is arbitrary, we have

$$\|S(f,D')-\sum_{i=1}^m\int_{I_i} f(\xi_i)\|\leq\epsilon.$$

□

Theorem 2.5. *Let $f : I_0 \to X$. If $f = \theta$ almost everywhere on I_0, then f is C-integrable on I_0 and $\int_{I_0} f = \theta$.*

Proof. Assume $E = \{\xi \in I_0 : f(\xi) \neq \theta\}$ and $E = \bigcup E_n \subset I_0$ where $E_n = \{\xi \in I_0 : n - 1 \leq \|f(\xi)\| < n\}$. Obviously, $\mu(E) = 0$ and therefore $\mu(E_n) = 0$, then there are open sets $G_n \subset I_0$ such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n}$. We define a positive function $\delta(\xi) : I_0 \to R^+$ in such a way that $\delta(\xi) = 1$ for $\xi \in I_0 \backslash E$ and $B(\xi, \delta(\xi)) \subset G_n$ if $\xi \in E_n$.

Suppose that $D = \{(I, \xi)\}$ is a δ - fine C-partition of I_0, then

$$\|\sum f(\xi)|I|\| \leq \sum n \cdot \frac{\epsilon}{n \cdot 2^n} \leq \epsilon.$$

Hence f is C-integrable on I_0 and $\int_{I_0} f = \theta$. □

Corollary 2.6. *Let $f : I_0 \to X$ be C-integrable on I_0. If $f = g$ almost everywhere on I_0, then g is C-integrable on I_0 and $\int_{I_0} f = \int_{I_0} g$ almost everywhere on I_0.*

Theorem 2.7. *Let $f : I_0 \to X$ be C-integrable on I_0.*

(1) *for each $x^* \in X^*$, the function $x^* f$ is C-integrable on I_0 and $\int_{I_0} x^* f = x^*(\int_{I_0} f)$.*

(2) *If $T : X \to Y$ is a continuous linear operator, then Tf is C-integrable on I_0 and $\int_{I_0} Tf = T(\int_{I_0} f)$.*

Proof. (1) Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to \mathrm{R}^+$ such that

$$\|S(f, D) - \int_{I_0} f\| < \frac{\epsilon}{\|x^*\|}$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence for each $x^* \in X^*$, we have

$$|S(x^* f, D) - x^*(\int_{I_0} f)| \leq \|x^*\| \cdot \|S(f, D) - \int_{I_0} f\| < \epsilon.$$

(2) $T : X \to Y$ is a continuous linear operator, then there exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for each $x \in X$. Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f, D) - \int_{I_0} f\| < \frac{\epsilon}{M}$$

for each δ - fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence we have

$$\|S(Tf, D) - T(\int_{I_0} f)\| \leq M \cdot \|S(f, D) - \int_{I_0} f\| < \epsilon.$$

□

Definition 2.8. Let $\{f_k\}$ be a sequence of integrable function defined on I_0 and X valued. The sequence $\{f_k\}$ is said C-equi-integrable on I_0 if $\{f_k\}$ is C-integrable and for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f_k, D) - \int_{I_0} f_k\| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0.

Theorem 2.9. *Assume* $\{f_k\}$, $f_k : I_0 \to X$ *is C-equi-integrable on* I_0 *such that*

$$\lim_{k\to\infty} f_k(\xi) = f(\xi).$$

Then the function $f : I_0 \to X$ *is C-integrable on* I_0 *and we have*

$$\lim_{k\to\infty} \int_{I_0} f_k = \int_{I_0} f.$$

Proof. From the definition of C-equi-integrability of $\{f_k\}$, for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f_k, D) - \int_{I_0} f_k\| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. Assume D is fixed.

Since $\lim_{k\to\infty} f_k(\xi) = f(\xi)$, then there is a $N \in \mathbb{N}$ such that

$$\|S(f_k, D) - S(f, D)\| < \epsilon$$

for all $k > N$. Then we have

$$\begin{aligned} & \|\int_{I_0} f_k - \int_{I_0} f_l\| \\ \le\ & \|S(f, D) - \int_{I_0} f_k\| + \|S(f, D) - \int_{I_0} f_l\| \\ \le\ & \|S(f_k, D) - S(f, D)\| + \|S(f_k, D) - \int_{I_0} f_k\| \\ & + \|S(f_l, D) - S(f, D)\| + \|S(f_l, D) - \int_{I_0} f_l\| \\ <\ & 4\epsilon \end{aligned}$$

for all $k, l > N$. Hence the sequence $\int_{I_0} f_k$ of elements of X is Cauchy and therefore

$$\lim_{k\to\infty} \int_{I_0} f_k = A \in X \quad \text{exists.}$$

In other words, there is a $M \in \mathbb{N}$ such that$\| \int_{I_0} f_k - A\| < \epsilon$ for all $k > M$.

Take any δ - fine *C-partition* $D = \{(I, \xi)\}$ of I_0. Since $\lim_{k\to\infty} f_k(\xi) = f(\xi)$, then there is a $n > M$ such that

$$\|S(f_n, D) - S(f, D)\| < \epsilon.$$

Then we have

$$\begin{aligned} & \|S(f,D)-A\| \\ \leq\ & \|S(f,D)-S(f_n,D)\| + \|S(f_n,D)-\int_{I_0} f_n\| + \|\int_{I_0} f_n - A\| \\ <\ & 3\epsilon. \end{aligned}$$

Hence f is C-integrable on I_0 and $\lim_{k\to\infty}\int_{I_0} f_k = \int_{I_0} f$. □

Theorem 2.10. *A function $f: I_0 \to X$ is C-integrable on I_0 if and only if $\{x^*f : x^* \in B(X^*)\}$ is C-equi-integrable on I_0.*

Proof. (Necessity) If f is C-integrable on I_0, then for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|S(f,D) - \int_{I_0} f\| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}$ of I_0. Therefore

$$\begin{aligned} & |S(x^*f,D) - x^*(\int_{I_0} f)| \\ \leq\ & \|x^*\| \cdot \|S(f,D) - \int_{I_0} f\| \leq \epsilon \end{aligned}$$

and it follows that $\{x^*f : x^* \in B(X^*)\}$ is C-equi-integrable on I_0.

(Sufficiency) If $\{x^*f : x^* \in B(X^*)\}$ is C-equi-integrable on I_0, then for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$|S(x^*f,D) - \int_{I_0} x^*f| < \epsilon$$

for each δ - fine *C-partition* $D = \{(I_i, \xi_i)\}$ of I_0 and $x^* \in B(X^*)$.

By Hahn-Banach Theorem we have

$$\|S(f,D) - \int_{I_0} f\| < \epsilon.$$

Therefore f is C-integrable on I_0. □

Definition 2.11. A function $F : I_0 \to X$ is differentiable at $\xi \in I_0$ if there is a $f(\xi) \in X$ such that

$$\lim_{\delta\to 0} \|\frac{F(\xi+\delta) - F(\xi)}{\delta} - f(\xi)\| = 0.$$

We denote $f(\xi) = F'(\xi)$ the derivative of F at ξ.

Theorem 2.12. *If the function $F : I_0 \to X$ is differentiable on I_0 with $f(\xi) = F'(\xi)$ for each $\xi \in I_0$, then $f : I_0 \to X$ is C-integrable.*

Proof. By the definition of derivative, for each $\xi \in I_0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|\frac{F(\zeta) - F(\xi)}{\zeta - \xi} - f(\xi)\| < \frac{\epsilon^2}{2(1 + \epsilon|I_0|)}$$

for all $\zeta \in I_0$ with $|\zeta - \xi| < \delta(\xi)$. Assume $D = \{(I_i, \xi_i)\}_{i=1}^n$ is an arbitrary δ - fine C-partition of I_0, we have

$$\begin{aligned}\|\sum_{i=1}^n f(\xi_i)|I_i| - F(I_i)\| &\leq \sum_{i=1}^n \|f(\xi_i)|I_i| - F(I_i)\| \\ &< \frac{\epsilon^2}{1 + \epsilon|I_0|} \sum_{i=1}^n (\mathrm{dist}(\xi_i, I_i) + |I_i|) \\ &< \frac{\epsilon^2}{1 + \epsilon|I_0|}(\frac{1}{\epsilon} + |I_0|) < \epsilon.\end{aligned}$$

Hence $f : I_0 \to X$ is C-integrable on I_0. □

Definition 2.13. Let $F : I_0 \to X$ and let E be a subset of I_0.

(a) F is said to be $^*AC_c$ on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\| \sum_i F(I_i)\| < \epsilon$ for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying the endpoints of I_i belonging to E and $\sum_i |I_i| < \eta$.

(b) F is said to be AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : I_0 \to R^+$ such that $\sum_i \|F(I_i)\| < \epsilon$ for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying the endpoints of I_i belonging to E and $\sum_i |I_i| < \eta$.

(c) F is said to be $^*ACG_c$ if F is continuous on E and E can be expressed as a countable union of sets on each of which F is $^*AC_c$.

(d) F is said to be ACG_c on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_c.

Theorem 2.14. *If a function $f : I_0 \to X$ is C-integrable on I_0 with the primitive $F : I_0 \to X$, then F is $^*ACG_c$ on I_0.*

Proof. By the definition of C-integral and the Saks-Henstock Lemma, for each $\varepsilon > 0$, there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$\|\sum_{i=1}^n f(\xi_i)|I_i| - F(I_i)\| \leq \epsilon$$

for each δ - fine partial C-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0.

Assume $E_n = \{\xi \in I_0 : \|f(\xi)\| \leq n\}$ for $n = 1, 2, \ldots$. So we have $I_0 = \bigcup E_n$. Fix a δ - fine partial C-partition $D_0 = \{(I_i, \xi_i)\}$. Assume that the endpoints of

I_i belonging to E_n and satisfying $\sum_i |I_i| < \frac{\epsilon}{n}$. We have

$$\begin{aligned} \|\sum_i F(I_i)\| &\leq \|\sum_i F(I_i) - f(\xi_i)|I_i|\| + \|\sum f(\xi_i) \cdot |I_i|\| \\ &\leq \|\sum_i F(I_i) - f(\xi_i)|I_i|\| + \|f(\xi_i)\| \cdot \sum |I_i| \\ &\leq \epsilon + n \cdot \sum |I_i| < 2\epsilon. \end{aligned}$$

Therefore F is $^*AC_c$ on E_n and consequently F is $^*ACG_c$ on I_0.

□

3. The Relations among Henstock, McShane and C-integrals

Definition 3.1. A function $f : I_0 \to X$ is McShane integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to \mathrm{R}^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine McShane partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the McShane integral of f on I_0, and we write $A = \int_{I_0} f$.

Definition 3.2. A function $f : I_0 \to X$ is Henstock integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to \mathrm{R}^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the Henstock integral of f on I_0, and we write $A = \int_{I_0} f$.

By the definitions of Henstock, C-integral and McShane integral and the fact that each δ - fine partition is also δ - fine C-partition and therefore is also δ - fine McShane partition, we get immediately the following theorem.

Theorem 3.3. *Let a function $f : I_0 \to X$.*

(a) *if f is McShane integrable on I_0, then f is C-integrable on I_0,*

(b) *if f is C-integrable on I_0, then f is Henstock integrable on I_0.*

Remark. The following two examples (when $X = R$) show that the converse of Theorem 3.3 is not true. In other words, there exists a function which is C-integrable but is not McShane integrable, there is a function which is Henstock integrable but is not C-integrable.

Example. (a) Assume a real-valued function f by

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.1}$$

It is easy to know the primitive of f is

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.2}$$

We have that $F(x)$ is differentiable and $F^{'}(x) = f(x)$ everywhere on $[0,1]$, then $f(x)$ is C-integrable from Theorem 2.12. But $F(x)$ is not absolutely continuous on $[0,1]$ and therefore $f(x)$ is not Lebesgue integrable (McShane integrable) on $[0,1]$.

(b) The function (1.1) is a primitive for the Riemann improper integral and therefore for the Henstock integral. From [12] we have $F(x)$ is not $^*ACG_c$ on I_0, hence $F(x)$ is not a C-primitive.

Theorem 3.4. *The function $f : I_0 \to X$ McShane integrable if and only if f is C-integrable and Pettis integrable.*

Proof. From [5] we have the function $f : I_0 \to X$ is McShane integrable if and only if f is Henstock integrable and Pettis integrable, then we get Theorem 3.4 by Theorem 3.3. □

From Theorem 3.4 and [7, Proposition 2B] we obtain the following theorem.

Theorem 3.5. *Let a function $f : I_0 \to X$. If f and $\|f\|$ are C-integrable on I_0 then f is McShane integrable on I_0.*

Proof. If $\|f\|$ is C-integrable on I_0 then $\|f\|$ is McShane integrable on I_0, the proof is similar to that of a non-negative real-valued Henstock integrable function is McShane integrable and we omit it. So we have

$$\int_E |x^* f| \leq \|x^*\| \cdot \int_E \|f\| < \infty$$

for each measurable set $E \subset I_0$ and for each $x^* \in X^*$. So f is Dunford integrable on E with its integral $(D)\int_E f$. $\|f\|$ is McShane integrable on I_0, then for each $\varepsilon > 0$ there exists a content $\eta > 0$, if $|E| < \eta$, we have

$$\|(D)\int_E f\| = \sup_{x^* \in B(X^*)} |\int_E x^* f| \leq \sup_{x^* \in B(X^*)} \int_E |x^* f| \leq \int_E \|f\| < \epsilon.$$

We also have $(D)\int_{I_i} f \in X$ for arbitrary subinterval $I_i \subset I_0$, then f is Pettis integrable on I_0. From Theorem 3.4 we have that f is McShane integrable on I_0. □

4. Denjoy-C integral

Definition 4.1. Let a function $f : I_0 \to X$ and $t \in I_0$. A vector $z \in X$ is the approximate derivative of F at t if there exists a measurable $E \subset I_0$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z.$$

we write $F'_{ap}(t) = z$ and say that F is approximate differentiable at the point t.

Definition 4.2. The function $f : I_0 \to R$ is $Denjoy_c$ integrable on I_0 if there exists an ACG_c function $F : I_0 \to R$ such that $F'_{ap} = f$ almost everywhere on I_0. The function f is $Denjoy_c$ integrable on the set $E \subset I_0$ if $f\chi_E$ is $Denjoy_c$ integrable on I_0.

Definition 4.3. Let the function $f : I_0 \to X$.

(a) f is Denjoy-Dunford integrable on I_0 if for each $x^* \in X^*$ the function x^*f is Denjoy integrable on I_0 and if for every interval $I \subset I_0$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = \int_I x^*f$ for all $x^* \in X^*$.

(b) f is Denjoy-Pettis integrable on I_0 if f is Denjoy-Dunford integrable on I_0 and if $x_I^{**} \in X$ for every interval $I \subset I_0$.

(c) f is Denjoy-Bochner integrable on I_0 if there exists an ACG function $F : I_0 \to X$ such that F is approximate differentiable almost everywhere on I_0 and $F'_{ap} = f$ almost everywhere on I_0.

Definition 4.4. Let the function $f : I_0 \to X$.

(a) f is $Denjoy_c-Dunford$ integrable on I_0 if for each $x^* \in X^*$ the function x^*f is $Denjoy_c$ integrable on I_0 and if for every interval $I \subset I_0$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = \int_I x^*f$ for all $x^* \in X^*$.

(b) f is $Denjoy_c - Pettis$ integrable on I_0 if f is $Denjoy_c - Dunford$ integrable on I_0 and if $x_I^{**} \in X$ for every interval $I \subset I_0$.

(c) f is $Denjoy_c-Bochner$ integrable on I_0 if there exists an ACG_c function $F : I_0 \to X$ such that F is approximate differentiable almost everywhere on I_0 and $F'_{ap} = f$ almost everywhere on I_0.

We can easily get the following theorem.

Theorem 4.5. *Let the function $f : I_0 \to X$.*

(a) *if f is $Denjoy_c - Dunford$ integrable on I_0, then f is Denjoy-Dunford integrable on I_0,*

(b) *if f is $Denjoy_c - Pettis$ integrable on I_0, then f is Denjoy-Pettis integrable on I_0,*

(c) *if f is $Denjoy_c - Bochner$ integrable on I_0, then f is Denjoy-Bochner integrable on I_0.*

Definition 4.6. The function $f : I_0 \to X$ is Denjoy-McShane integrable on I_0 if there exists a continuous function $F : I_0 \to X$ such that

(a) x^*F is ACG for each $x^* \in X^*$ and

(b) x^*F is approximate differentiable almost everywhere on I_0 and $(x^*F)'_{ap} = x^*f$ almost everywhere on I_0.

Definition 4.7. The function $f : I_0 \to X$ is $Denjoy - C$ integrable on I_0 if there exists a continuous function $F : I_0 \to X$ such that

(a) x^*F is ACG_c for each $x^* \in X^*$ and

(b) x^*F is approximate differentiable almost everywhere on I_0 and $(x^*F)'_{ap} = x^*f$ almost everywhere on I_0.

Theorem 4.8. *Let $f : I_0 \to X$ be C-integrable on I_0. Then f is $Denjoy - C$ integrable on I_0.*

Proof. f is C-integrable on I_0, by Theorem 2.7 we have the function x^*f is C-integrable on I_0 for each $x^* \in X^*$ and $\int_{I_0} x^*f = x^*(\int_{I_0} f)$. Let $F(t)$ be the indefinite integral of f, then we have x^*F is ACG_c and $(x^*F)' = x^*f$. Hence f is $Denjoy - C$ integrable on I_0. □

We can easily get the following corollary from Theorem 3.3, Definition 4.6 and 4.7.

Corollary 4.9. *Let a function $f : I_0 \to X$.*
(a) if f is McShane integrable, then f is $Denjoy - C$ integrable on I_0,
(b) if f is $Denjoy - C$ integrable on I_0, then f is Denjoy-McShane integrable on I_0.

Theorem 4.10. *If a function $f : I_0 \to X$ is $Denjoy - C$ integrable on I_0, then f is $Denjoy_c - Pettis$ integrable on I_0.*

Proof. Assume f is $Denjoy - C$ integrable with the indefinite integral $F(t)$ on I_0, then x^*F is ACG_c for each $x^* \in X^*$, x^*F is approximate differentiable almost everywhere on I_0 and $(x^*F)'_{ap} = x^*f$ almost everywhere on I_0. Hence x^*f is $Denjoy_c$ integrable on I_0 from Definition 4.2. We have $x_I^{**} \in X$ for every interval $I \subset I_0$, then f is $Denjoy_c - Pettis$ integrable on I_0. □

Theorem 4.11. *If a function $f : I_0 \to X$ is $Denjoy_c - Bochner$ integrable on I_0, then f is $Denjoy - C$ integrable on I_0.*

Proof. By the definition of $Denjoy_c - Bochner$ integrable, there exists an ACG_c function $F : I_0 \to X$ such that F is approximate differentiable almost everywhere on I_0 and $F'_{ap} = f$ almost everywhere on I_0. So we have x^*F is ACG_c and $(x^*F)'_{ap} = x^*f$ almost everywhere on I_0. Hence f is $Denjoy - C$ integrable on I_0. □

Remark. The following example [8, Example 42] shows that the converse of Theorem 4.11 is not true. In other words, there is a function which is $Denjoy - C$ integrable but is not $Denjoy_c - Bochner$ integrable.

Example. Let $\{r_k\}$ be a listing of the rational numbers in [0,1] and for each pair of positive integers n and k let

$$I_n^k = (r_k + \frac{1}{n+1}, r_k + \frac{1}{n}).$$

For each k we define a function $f_k : [0,1] \to l_2$ by

$$f_k(t) = \{(n+1)\chi_{I_n^k}(t)\}.$$

For each positive integer j let $A_j = \bigcup\{t \in [0,1] : |t - r_k| < 2^{-(j+k)}\}$ and let $A = \bigcap_j A_j$. It is easy to know that $\mu(A) = 0$ and $\{r_k\} \subset A$. If $t \notin A$, then $t \notin A_{j_0}$ for some j_0 and $|t - r_k| > 2^{-(j_0+k)}$ for all k. So we have $\|f_k(t)\| \leq 2^{j_0+k}$

and $\sum_k \|4^{-k} f_k(t)\| \leq 2^{j_0}$. Hence the series $\sum_k 4^{-k} f_k$ is converges in l_2, in other words, the series $\sum_k 4^{-k} f_k$ is l_2- valued almost everywhere on $[0, 1]$. Define a function $g : [0, 1] \to l_2$ by

$$g(t) = \begin{cases} \sum_k 4^{-k} f_k(t) & \text{if } t \in [0,1]\backslash A, \\ \theta & \text{if } t \in A. \end{cases} \tag{4.1}$$

From [8] we know that g is McShane integrable but not Denjoy-Bochner integrable. So we have g is C-integrable on $[0, 1]$ and therefore $Denjoy - C$ integrable on $[0, 1]$. Since every $Denjoy_c - Bochner$ integrable function is Denjoy-Bochner integrable, we get g is not $Denjoy_c - Bochner$ integrable on $[0, 1]$.

From [8] we also have the following corollary.

Corollary 4.12. *Let a function $f : I_0 \to X$.*

(a) If f is $Denjoy_c - Bochner$ integrable on I_0, then each perfect set in $[0, 1]$ contains a portion on which f is Bochner integrable.

(b) If f is $Denjoy_c - Pettis$ integrable on I_0, then each perfect set in $[0, 1]$ contains a portion on which f is Pettis integrable.

References

[1] B. Bongiorno, *Un nvovo interale il problema dell primitive, Le Matematiche,* **51** (1996), No. 2, 299–313.

[2] B. Bongiorno, L. Di Piazza, and D. Preiss, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc. **(2) 62** (2000), no. 1, 117–126.

[3] A. M. Bruckner, R. J. Fleissner, and J. Fordan, *The minimal integral which includeds Lebesgue integrable functions and derivatives*, Collq. Mat. **50** (1986), 289–293.

[4] S. J. Chao, B. S. Lee, G. M. Lee, and D. S. Kim, *Denjoy-type integrals of Banach-valued functions,* Comm. Korean. Math. Soc. **13** (1998), No. 2, 307–316.

[5] D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. Math. **38** (1994), 471–479.

[6] ———, *The McShane, PU and Henstock integrals of Banach valued functions,* Cze. J. Math. **52** (127), (2002), 609–633.

[7] D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), 127–147.

[8] R. A. Gordon, *The Denjoy extension of the Bochner, Pettis and Dunford integrals,* Studia Math. **92** (1989), 73–91.

[9] R. Henstock, *The General Theory of Integration,* Oxford University Press, Oxford, 1991.

[10] C. K. Park, *On Denjoy-McShane-Stieltjes integral,* Commun. Korean. Math. Soc. **18** (2003), *no.* 4, 643–652.

[11] J. M. Park and D. H. Lee, *The Denjoy extension of the Riemann and McShane integrals,* Cze. J. Math. **50** (2000), no. 125, 615–625.

[12] L. Di Piazza, *A Riemann-type minimal integral for the classical problem of primitives,* Rend. Istit. Mat. Univ. Trieste Vol. XXXIV, (2002), 143–153.

[13] S. Schwabik and Guoju Ye, *Topics in Banach space integration*, World Scientific, 2005.

[14] L. P. Yee, *Lanzhou Lectures on Henstock Integration,* World Scientific, Singapore, 1989.

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