Commun. Korean Math. Soc. **22** (2007), No. 1, pp. 65–74

NOTES ON CARLESON TYPE MEASURES ON BOUNDED SYMMETRIC DOMAIN

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ABSTRACT. Suppose that μ is a finite positive Borel measure on bounded symmetric domain $\Omega \subset \mathbb{C}^n$ and ν is the Euclidean volume measure such that $\nu(\Omega) = 1$. Suppose $1 < p < \infty$ and $r > 0$. In this paper, we will show that the norms $\sup\{\int_\Omega |k_z(w)|^2 d\mu(w) : z \in \Omega\}$, $\sup\{\int_\Omega |h(w)|^p d\mu(w)/\int_\Omega |h(w)|^p d\nu(w) : h \in L_a^p(\Omega, d\nu), h \neq 0\}$ and

$$\sup\{\frac{\mu(E(z,r))}{\nu(E(z,r))} : z \in \Omega\}$$

are all equivalent. We will also show that the inclusion mapping $i_p : L_a^p(\Omega, d\nu) \to L^p(\Omega, d\mu)$ is compact if and only if $\lim_{w\to\partial\Omega} \frac{\mu(E(w,r))}{\nu(E(w,r))} = 0$.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 = \langle z, z\rangle$. For $w \in \mathbb{C}^n, r > 0$, let $B(w,r) = \{z \in \mathbb{C}^n : |z - w| < r\}$.

If Ω is a domain in $\mathbb{C}^n$, we let $\mathrm{Aut}(\Omega)$ denote the group of biholomorphic mappings of Ω onto itself. The domain Ω is said to be homogeneous if $\mathrm{Aut}(\Omega)$ is transitive on Ω, i.e., for $z_1, z_2 \in \Omega$, there exists $\psi \in \mathrm{Aut}(\Omega)$ such that $\psi(z_1) = z_2$. A homogeneous domain Ω is symmetric if for any $z_0 \in \Omega$, there exists $\psi \in \mathrm{Aut}(\Omega)$ such that (a) $\psi(z_0) = z_0$ but $\psi(z) \neq z$ for all $z \neq z_0$, and (b) $\psi \circ \psi =$ identity on Ω. In this paper, Ω will be the bounded symmetric domain with its standard realization in $\mathbb{C}^n$ (See [6, 10, 12]).

Let ν be the usual Euclidean volume measure on $\mathbb{C}^n$ such that $\nu(\Omega) = 1$. We let $L^2(\Omega, d\nu)$ be the usual space of Lebesgue square-integrable complex valued functions on Ω. The Bergman space $L_a^2(\Omega, d\nu)$ is defined to be the subspace of $L^2(\Omega, d\nu)$ consisting of analytic functions.

Received July 11, 2006.
2000 *Mathematics Subject Classification.* 32H25, 32E25, 30C40.
Key words and phrases. Bergman space, Bergman projection.

Fix a point $z \in \Omega$. Since the functional e_z given by $e_z(f) = f(z), f \in L^2_a(\Omega, d\nu)$, is continuous, there exists a function $K(\cdot, z) \in L^2_a(\Omega, d\nu)$ such that

$$f(z) = \int_\Omega f(w)\overline{K(w,z)}d\nu(w)$$

by the Riesz representation theorem. $K(\cdot, z)$ is called the Bergman reproducing kernel in $L^2_a(\Omega, d\nu)$.

The functions $K(\cdot,\cdot)$ are well understood on bounded symmetric domains and have many useful properties. The function $K(\cdot,\cdot)$ is actually defined and continuous on $\Omega\times\overline{\Omega}$ (where $\overline{\Omega}$ is the closure of Ω in $\mathbb{C}^n$). For bounded symmetric domains Ω in the standard representation, with normalized volume measure, the kernel functions $K(\cdot,\cdot)$ have the special properties (See [1, 10]):

(1) $K(0,a) = 1 = K(a,0)$,
(2) $K(z,a) \neq 0, z \in \Omega, a \in \overline{\Omega}$,
(3) $\lim_{a\to\partial\Omega} K(a,a) = +\infty$,
(4) $K(z,a)^{-1}$ is a smooth function on $\mathbb{C}^n \times \mathbb{C}^n$.

The normalized (in $L^2_a(\Omega, d\nu)$) reproducing kernel is denoted by $k_z(\cdot) = K(z,z)^{-1/2}K(\cdot,z)$. For μ a finite positive Borel measure on Ω and g measurable, we write

$$\| g \|^2_\mu = \int_\Omega |g(z)|^2 d\mu(z).$$

We also define the Berezin transform of μ by

$$\tilde{\mu}(z) = \int_\Omega |k_z(w)|^2 d\mu(w)$$

and consider the usual supremum $\| \tilde{\mu} \|_\infty \equiv \sup_{z\in\Omega} |\tilde{\mu}(z)|$.

The Carleson norm of μ is defined by

$$\| \mu \|_s = \sup\{\| h \|^2_\mu \ / \ \| h \|^2 : h \in L^2_a(\Omega, d\nu), h \neq 0\}$$

where $\| h \|^2 = \int_\Omega |h(w)|^2 d\nu(w)$. μ is called a Carleson type measure on Ω if $\| \mu \|_s < +\infty$.

For any $\Omega \subset \mathbb{C}^n$, we define a Hermitian metric on Ω by

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z,z), z \in \Omega.$$

If $\gamma : [0,1] \to \Omega$ is a C^1-curve, the Bergman length of γ is defined by $|\gamma|_B = \int_0^1 (\sum_{i,j} g_{i,j}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)})^{1/2}dt$. For $z, w \in \Omega$, let $\beta(z,w) = \inf\{|\gamma|_B : \gamma(0) = z, \gamma(1) = w\}$ where the infimum is taken over all C^1-curves from z to w. β is called the Bergman metric on Ω.

The Bergman metric $\beta(\cdot,\cdot)$ gives the usual topology on Ω (See [9, p52]). Moreover, the closed metric balls

$$E(z,r) = \{w : \beta(z,w) \leq r\}$$

are compact (See [9, p56]). For any fixed $r > 0$, we define

$$\| \mu \|_r = \sup_{z\in\Omega} \frac{\mu(E(z,r))}{\nu(E(z,r))}.$$

Suppose $1 < p < +\infty$ and $r > 0$. Let $L^p_a(\Omega, d\nu)$ be the subspace of $L^p(\Omega, d\nu)$ consisting of analytic functions. Let

$$\| \mu \|_{s,p} = \sup\{\| h \|^p_\mu \,/\, \| h \|^p : h \in L^p_a(\Omega, d\nu), h \neq 0\},$$

where $\| h \|^p = \int_\Omega |h(w)|^p d\nu(w)$ and $\| h \|^p_\mu = \int_\Omega |h(w)|^p d\mu(w)$. In Section 2, we will show that the norms $\| \mu \|_{s,p}$, $\| \tilde{\mu} \|_\infty$ and $\| \mu \|_r$ are all equivalent.

By $z \to \partial\Omega$, we mean that the usual distance function

$$d(z, \partial\Omega) \equiv \inf\{|z - w| : w \in \partial\Omega\}$$

has the property that $d(z, \partial\Omega) \to 0$. f_n and f are in $L^p_a(\Omega, d\nu)$ for all $n \in N$ where $1 < p < +\infty$. In Section 3, we will prove that $f_n \to f$ weakly in $L^p(\Omega, d\nu)$ if and only if $\{\| f_n \|_p : n \in N\}$ is bounded and f_n converges to f uniformly on each compact set of Ω. Using the above result, we will also show that the inclusion mapping $i_p : L^p_a(\Omega, d\nu) \to L^p(\Omega, d\mu)$ is compact if and only if $\lim_{z\to\partial\Omega} \frac{\mu(E(z,r))}{\nu(E(z,r))} = 0$.

2. Carleson Type Measures

Proposition 1. *For a, b in Ω with $\beta(a, b) \leq R$ and $r, s > 0$, there are constants $M(R, r, s), m(R, r, s)$ so that*

$$0 < m(R, r, s) \leq \frac{\nu(E(a, r))}{\nu(E(b, s))} \leq M(R, r, s) < \infty.$$

Proof. See [1, Lemma 6]. □

Proposition 2. *For $r > 0$, there are constants $M(r), n(r)$ so that*

$$\infty > M(r) \geq |k_z(w)|^2 \nu(E(z, r)) \geq n(r) > 0$$

for all $z, w \in \Omega$ with $\beta(z, w) \leq r$.

Proof. See [1, Lemma 8]. □

Proposition 3. *For fixed $r > 0$, there is a sequence $\{w_j\}$ in Ω such that*

(1) $\cup_{j=1}^\infty E(w_j, r) = \Omega$,

(2) *there is a positive integer N such that, for any z in Ω, z is contained in at most N number of the sets $E(w_k, 2r)$.*

For the above sequence $\{w_j\}$ and any positive Borel measure m, we have

$$\sum_{k=1}^{\infty} m(E(w_k, 2r)) \leq N m(\Omega).$$

Proof. See [2, Lemma 5, 6]. □

Theorem 4. *For $r > 0$ and $p > 1$, there is a constant $C_r > 0$ so that for all $f \in L^p_a(\Omega, d\nu)$ and $z \in \Omega$*

$$|f(z)|^p \leq \frac{C_r}{\nu(E(z,r))} \int_{E(z,r)} |f(w)|^p d\nu(w).$$

Proof. Since the Bergman metric induces the usual Euclidean topology on Ω, $E(0,r)$ contains Euclidean ball $B(0,t) = \{z \in C^n : |z| < t\}$ for some t. If f is holomorphic in $\Omega \in C^n$, then $|f|^p$ is subharmonic for all $p > 0$ (See [11, Corollary 2.1.15]). For holomorphic function f on Ω,

$$|f(0)|^p \leq \frac{1}{t^{2n}} \int_{B(0,t)} |f(w)|^p d\nu(w) \leq \frac{1}{t^{2n}} \int_{E(0,r)} |f(w)|^p d\nu(w).$$

Replacing f by $f \circ \varphi_z$, we have

$$\begin{aligned} |f(z)|^p &\leq \frac{1}{t^{2n}} \int_{E(0,r)} |f \circ \varphi_z(w)|^p d\nu(w) \\ &\leq \frac{1}{t^{2n}} \int_{E(z,r)} |f(w)|^p |k_z(w)|^2 d\nu(w). \end{aligned}$$

By Proposition 2,

$$|f(z)|^p \leq \frac{M(r)}{t^{2n}\nu(E(z,r))} \int_{E(z,r)} |f(w)|^p d\nu(w).$$

□

Proposition 5. *The net $\{k_\alpha\}$ converges to 0 weakly in $L^2_a(\Omega, d\nu)$ as $\alpha \to \partial\Omega$.*

Proof. See [1, p928]. □

It was proved in [2, Theorem 8] that the norms $\| \mu \|_r$, $\| \tilde{\mu} \|_\infty$ and $\| \mu \|_{s,2}$ $(=\| \mu \|_s)$ are all equivalent for finite positive Borel measure μ. In the following Theorem 6, we will extend the above result to the case of $p > 1$.

Theorem 6. *Suppose μ is a finite positive Borel measure on Ω, $p > 1$ and $r > 0$. Then the norms $\| \mu \|_r$, $\| \tilde{\mu} \|_\infty$ and $\| \mu \|_{s,p}$ are all equivalent.*

Proof. To show that $\| \mu \|_r$ is dominated by $\| \tilde{\mu} \|_\infty$, note that we can find $n(r)$ by Proposition 2 such that

$$\begin{aligned} \frac{n(r)\mu(E(w,r))}{\nu(E(w,r))} &= \frac{n(r)}{\nu(E(w,r))} \int_{E(w,r)} d\mu(z) \leq \int_{E(w,r)} |k_w(z)|^2 d\mu(z) \\ &\leq \int_\Omega |k_w(z)|^2 d\mu(z) = \tilde{\mu}(w) \leq \| \tilde{\mu} \|_\infty . \end{aligned}$$

Since $\frac{n(r)\mu(E(w,r))}{\nu(E(w,r))} \leq \| \tilde{\mu} \|_\infty$ for all $w \in \Omega$, $n(r) \| \mu \|_r \leq \| \tilde{\mu} \|_\infty$.

To show that $\| \tilde{\mu} \|_\infty$ is dominated by $\| \mu \|_{s,p}$, note that $k_w(z)^{2/p}$ is in $L^p_a(\Omega, d\nu)$ since $\{k_\alpha\}$ is in $L^2_a(\Omega, d\nu)$. For $f(z) = k_w(z)^{2/p}$,

$$\tilde{\mu}(z) = \int_\Omega |k_z(w)|^2 d\mu(w) = \frac{\int_\Omega |k_z(w)|^2 d\mu(w)}{\int_\Omega |k_z(w)|^2 d\nu(w)}$$
$$\leq \sup\{\frac{\| h \|^p_\mu}{\| h \|^p} : h \in L^p_a(\Omega, d\nu), h \neq 0\}.$$

This shows that $\| \tilde{\mu} \|_\infty \leq \| \mu \|_{s,p}$.

To show that $\| \mu \|_{s,p}$ is dominated by $\| \mu \|_r$, choose sequence $\{\lambda_n\}$ as in Proposition 3.

$$\int_\Omega |f(z)|^p d\mu(z) \leq \sum_{n=1}^\infty \int_{E(\lambda_n, r)} |f(z)|^p d\mu(z)$$
$$\leq \sum_{n=1}^\infty \mu(E(\lambda_n, r)) \sup\{|f(z)|^p : z \in E(\lambda_n, r)\}.$$

By Theorem 4,

$$|f(z)|^p \leq \frac{C_r}{\nu(E(z,r))} \int_{E(z,r)} |f(w)|^p d\nu(w)$$
$$\leq \frac{C_r}{\nu(E(z,r))} \int_{E(\lambda_n, 2r)} |f(w)|^p d\nu(w)$$

for $z \in E(\lambda_n, r)$. By Proposition 1,

$$\frac{1}{\nu(E(z,r))} \leq \frac{C_1}{\nu(E(\lambda_n, r))}$$

for some constant C_1. This implies that

$$\sup\{|f(z)|^p : z \in E(\lambda_n, r)\} \leq \frac{C_r C_1}{\nu(E(\lambda_n, r))} \int_{E(\lambda_n, 2r)} |f(w)|^p d\nu(w).$$

Since $\| \mu \|_r = \sup\{\frac{\mu(E(w,r))}{\nu(E(w,r))} : w \in \Omega\}$,

$$\int_\Omega |f(z)|^p d\mu(z) \leq C_r C_1 \sum_{n=1}^\infty \frac{\mu(E(\lambda_n, r))}{\nu(E(\lambda_n, r))} \int_{E(\lambda_n, 2r)} |f(z)|^p d\nu(z)$$
$$\leq C_r C_1 \| \mu \|_r \sum_{n=1}^\infty \int_{E(\lambda_n, 2r)} |f(z)|^p d\nu(z)$$
$$\leq C_r C_1 \| \mu \|_r N \int_\Omega |f(z)|^p d\nu(z)$$

where the last inequality follows from Proposition 3. Since

$$\frac{\int_\Omega |f(z)|^p d\mu(z)}{\int_\Omega |f(z)|^p d\nu(z)} \leq C_r C_1 N \| \mu \|_r$$

for all $f \in L^p_a(\Omega, d\nu)$, $\| \mu \|_{s,p} \leq C_r C_1 N \| \mu \|_r$. □

3. Vanishing Carleson type measures

Proposition 7 (Principle of uniform boundedness). *Let X be a Banach space. Let Γ be a family of bounded linear transformations from X to some normed linear space Y. Suppose that for each $x \in X$, $\{\| Tx \|_Y : T \in \Gamma\}$ is bounded. Then $\{\| T \| : T \in \Gamma\}$ is bounded.*

Proof. See [5, Theorem 14.1]. □

Theorem 8. *If $f_n \to f$ in $L^p(\Omega, d\nu)$, then f_n converges to f weakly.*

Proof. Let q be the integer such that $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q(\Omega, d\nu)$. Then

$$|\int_\Omega (f_n - f)(w)g(w)d\nu(w)| \leq \| f_n - f \|_p \| g \|_q \to 0$$

as $n \to \infty$. □

Theorem 9. *If $f_n \to f$ weakly in $L^p(\Omega, d\nu)$, then $\{\| f_n \|_p : n \in N\}$ is bounded.*

Proof. Let q be the integer such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us define $T_{f_n} : L^q(\Omega, d\nu) \to \mathbb{C}$ such that $T_{f_n}(g) = \int_\Omega f_n(w)g(w)d\nu(w)$. Since $f_n \to f$ weakly in $L^p(\Omega, d\nu)$, $\int_\Omega f_n(w)g(w)d\nu(w) \to \int_\Omega f(w)g(w)d\nu(w)$. This implies that there exists n_0 such that if $n > n_0$,

$$|\int_\Omega f_n(w)g(w) - \int_\Omega f(w)g(w)d\nu(w)| < 1.$$

If $n > n_0$,

$$|\int_\Omega f_n(w)g(w)d\nu(w)| \leq 1 + \| f \|_p \| g \|_q .$$

Let M be

$$M = \max\{|\int_\Omega f_1(w)g(w)d\nu(w)|, \ldots, |\int_\Omega f_{n_0}(w)g(w)d\nu(w)|\}.$$

Then

$$|\int_\Omega f_n(w)g(w)d\nu(w)| \leq \max\{M, 1 + \| f \|_p \| g \|_q\}.$$

Since $\{\| T_{f_n} g \| : n \in N\}$ is bounded, $\{\| T_{f_n} \| = \| f_n \|_p : n \in N\}$ is bounded by Proposition 7. □

Proposition 10. *Let $K \subset \Omega \subset \mathbb{C}^n$ be compact. There is a constant $C_K > 0$, depending on K and on n, so that*

$$\sup_{z \in K} |f(z)| \leq C_K \| f \|_{L^p_a(\Omega, d\nu)}$$

for all $f \in L^p_a(\Omega, d\nu)$ where $p > 0$.

Proof. See [11, Lemma 1.4.1]. □

If V denotes the usual Euclidean volume measure in $\mathbb{C}^n$ and $\kappa = V(\Omega)$, then $\kappa d\nu = dV$.

Theorem 11. *Suppose* $1 \le p < +\infty$. f_n *and* f *are in* $L^p_a(\Omega, d\nu)$ *for all* $n \in N$. *If* f_n *converges to* f *weakly in* $L^p(\Omega, d\nu)$, *then* $f_n(z) \to f(z)$ *uniformly on each compact subset of* Ω.

Proof. Let K be a compact subset of Ω. Choose $r > 0$ such that $\overline{B(a,r)} \subset \Omega$ for all $a \in K$. Since holomorphic functions are harmonic,

$$|f_n(z) - f(z)| = \frac{1}{r^{2n}K}\left|\int_{B(z,r)} (f_n(t) - f(t))d\nu(t)\right|$$

(See [11, p.38]). For set $A = \overline{\cup_{z\in K} B(z,r)}$,

$$\begin{aligned}|f_n(z) - f(z)| &\le \frac{1}{r^{2n}K}\int_{B(z,r)} |f_n(t) - f(t)|d\nu(t)\\ &\le \frac{1}{r^{2n}K}\int_{\overline{\cup_{w\in K} B(w,r)}} |f_n(t) - f(t)|d\nu(t)\\ &= \frac{1}{r^{2n}K}\int_{\Omega} |f_n(t) - f(t)|\chi_A(t)d\nu(t).\end{aligned}$$

Since f_n converges to f weakly in $L^p(\Omega, d\nu)$,

$$\frac{1}{r^{2n}K}\int_{\Omega} |f_n(t) - f(t)|\chi_A(t)d\nu(t) \to 0$$

as $n \to \infty$. This shows that f_n converges uniformly to f on each compact subset of Ω. □

Theorem 12. $1 \le p < \infty$. *If* $\{\| f_n \|_p : n \in N\}$ *is bounded and* f_n *converges to* f *uniformly on each compact set* Ω, *then* f_n *converges to* f *weakly in* $L^p(\Omega, d\nu)$.

Proof. Put $\Omega = \cup_{n=1}^{+\infty} B(\lambda_n, r)$ and $A_n = \cup_{k=1}^{n} B(\lambda_n, r)$. For $g \in L^q(\Omega, d\nu)$,

$$\begin{aligned}\int_{\Omega} |g(w)|^q d\nu(w) &= \int_{\cup_{n=1}^{+\infty} B(\lambda_n,r)} |g(w)|^q d\nu(w)\\ &= \lim_{n\to\infty}\int |g(w)|^q \chi_{A_n}(w)d\nu(w)\\ &< +\infty.\end{aligned}$$

For arbitrary $\epsilon > 0$, choose n_0 such that $\int_{\Omega - \cup_{k=1}^{n_0} B(\lambda_n,r)} |g(w)|^q d\nu(w) < \epsilon$. Put $|g(w)|\chi_{\Omega - \cup_{k=1}^{n_0} B(\lambda_n,r)}(w) = g_{n_0}(w)$. Then,

$$\begin{aligned}&\int_{\Omega} |g(w)(f(w) - f_n(w))|d\nu(w)\\ &\le \int_{\Omega} |g(w) - g_{n_0}(w)||f(w) - f_n(w)|d\nu(w) + \int_{\Omega} |g_{n_0}(w)||f(w) - f_n(w)|d\nu(w)\end{aligned}$$

$$\leq \int_\Omega |g(w) - g_{n_0}(w)||f(w)|d\nu(w) + \int_\Omega |g_{n_0}(w)||f(w) - f_n(w)|d\nu(w)$$
$$+ \int_\Omega |g_{n_0}(w) - g(w)||f_n(w)|d\nu(w)$$
$$\leq \| g - g_{n_0} \|_q \| f \|_p + \int_\Omega |g_{n_0}(w)||f(w) - f_n(w)|d\nu(w) + \| g_{n_0} - g \|_q \| f_n \|_p .$$

This shows that f_n converges to f weakly in $L^p(\Omega, d\nu)$. □

Proposition 13. *A linear operator T on Hilbert space H is compact if and only if $\| Tx_n \| \to 0$ whenever $x_n \to 0$ weakly in H.*

Proof. See [13, Theorem 1.3.4]. □

In the proof of the following Theorem 14, we will follow the argument given in [13, ch6] which was for the case of unit disk D in the complex plane $\mathbb{C}$.

Theorem 14. *Suppose μ is a finite positive Borel measure on Ω, $p > 1$ and $r > 0$. The inclusion mapping $i_p : L^p_a(\Omega, d\nu) \to L^p(\Omega, d\mu)$ is compact if and only if $\lim_{w\to\partial\Omega} \frac{\mu(E(w,r))}{\nu(E(w,r))} = 0$.*

Proof. Suppose i_p is compact. Since k_w converges to 0 weakly in $L^2_a(\Omega, d\nu)$ as $w \to \partial\Omega$ (See Proposition 5), ${k_w}^{2/p}$ converges to 0 weakly in $L^p_a(\Omega, d\nu)$ as $w \to \partial\Omega$ for $p > 1$. Since i_p is compact, for $f(z) = k_w(z)^{2/p}$,

$$\int_{E(w,r)} |k_w(z)|^2 d\mu(z) \leq \int_\Omega |k_w(z)|^2 d\mu(z) = \int_\Omega |f(z)|^p d\mu(z)$$
$$\leq C \int_\Omega |f(z)|^p d\nu(z) = C \int_\Omega |k_w(z)|^2 d\nu(z)$$

for some constant C. By Proposition 2, we can find $n(r)$ such that

$$\frac{n(r)\mu(E(w,r))}{\nu(E(w,r))} = \frac{n(r)}{\nu(E(w,r))} \int_{E(w,r)} d\mu(z) \leq \int_{E(w,r)} |k_w(z)|^2 d\mu(z).$$

This implies that

$$\frac{n(r)\mu(E(w,r))}{\nu(E(w,r))} \leq C \int_\Omega |k_w(z)|^2 d\nu(z).$$

Since the net $\{k_\alpha\}$ converges to 0 weakly in $L^2_a(\Omega, d\nu)$ as $\alpha \to \partial\Omega$ by Proposition 5, $\int_\Omega |k_w(z)|^2 d\nu(z) \to 0$ as $w \to \partial\Omega$. This shows that $\lim_{w\to\partial\Omega} \frac{\mu(E(w,r))}{\nu(E(w,r))} \to 0$.

Suppose that $\lim_{w\to\partial\Omega} \frac{\mu(E(w,r))}{\nu(E(w,r))} = 0$. Let $\{f_k\}$ be a sequence in $L^p_a(\Omega, d\nu)$ which goes to 0 weakly. By Theorem 9, there is a constant $M > 0$ such that $\| f_k \|_p \leq M$ for all $k \geq 1$, and $f_k(z) \to 0$ uniformly on compact sets of Ω. Fix any $r > 0$ and choose a sequence $\{\lambda_n\}$ as in Proposition 3 such that $\lambda_n \to \partial\Omega$

as $n \to \infty$. Since f_k is a function in $L^p_a(\Omega, d\nu)$ for each $k \in N$,

$$\begin{aligned}|f_k(z)|^p &\leq \frac{C_r}{\nu(E(z,r))}\int_{E(z,r)}|f_k(w)|^p d\nu(w)\\ &\leq \frac{C_r}{\nu(E(z,r))}\int_{E(\lambda_n,2r)}|f_k(w)|^p d\nu(w)\end{aligned}$$

for $z \in E(\lambda_n, r)$ by Theorem 4. There exists a constant C_1 such that

$$|f_k(z)|^p \leq \frac{C_r C_1}{\nu(E(\lambda_n,r))}\int_{E(\lambda_n,2r)}|f_k(w)|^p d\nu(w)$$

for all $z \in E(\lambda_n, r)$ by Proposition 1. This shows that

$$\begin{aligned}&\int_{E(\lambda_n,r)}|f_k(z)|^p d\mu(z)\\ &\leq \int_{E(\lambda_n,r)}\frac{C_r C_1}{\nu(E(\lambda_n,r))}\int_{E(\lambda_n,2r)}|f_k(w)|^p d\nu(w)d\mu(z)\\ &= C_r C_1 \frac{\mu(E(\lambda_n,r))}{\nu(E(\lambda_n,r))}\int_{E(\lambda_n,2r)}|f_k(w)|^p d\nu(w).\end{aligned}$$

Since $\lambda_n \to \partial\Omega$ as $n \to \infty$, $\lim_{n\to\infty}\frac{\mu(E(\lambda_n,r))}{\nu(E(\lambda_n,r))} \to 0$. Given ϵ, there exists positive integers n_0 such that $\frac{\mu(E(\lambda_n,r))}{\nu(E(\lambda_n,r))} < \epsilon$ if $n > n_0$.

$$\begin{aligned}&\sum_{n=n_0}^{\infty}\int_{E(\lambda_n,r)}|f_k(z)|^p d\mu(z)\\ &\leq C_r C_1 \sum_{n=n_0}^{\infty}\frac{\mu(E(\lambda_n,r))}{\nu(E(\lambda_n,r))}\int_{E(\lambda_n,2r)}|f_k(z)|^p d\nu(z)\\ &\leq C_r C_1 \epsilon \sum_{n=1}^{\infty}\int_{E(\lambda_n,2r)}|f_k(z)|^p d\nu(z)\\ &\leq \epsilon C_r C_1 N \int_{\Omega}|f_k(z)|^p d\nu(z)\\ &\leq \epsilon C_r C_1 N M^p\end{aligned}$$

where the third inequality follows from Proposition 3 and the last inequality follows from Theorem 9. Since $\cup_{n=1}^{\infty}E(\lambda_n, r) = \Omega$,

$$\begin{aligned}&\limsup_{k\to\infty}\int_{\Omega}|f_k(z)|^p d\mu(z)\\ &\leq \limsup_{k\to\infty}\sum_{n=1}^{n_0-1}\int_{E(\lambda_n,r)}|f_k(z)|^p d\nu(z) + \limsup_{k\to\infty}\sum_{n=n_0}^{\infty}\int_{E(\lambda_n,r)}|f_k(z)|^p d\nu(z).\end{aligned}$$

Since $\{f_k\}$ converges to 0 weakly, f_k converges to 0 uniformly on each compact sets of Ω by Theorem 11. This implies that

$$\lim_{k\to\infty}\sum_{n=1}^{n_0-1}\int_{E(\lambda_n,r)}|f_k(z)|^p d\nu(z)=0.$$

We have $\lim_{k\to\infty}\int_\Omega |f_k(z)|^p d\mu(z)=0$. By Proposition 13, i_p is compact.

□

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