Analysis of a Queueing Model with Time Phased Arrivals

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Abstract A single-server queueing model with infinite buffer and batch arrival of customers is considered. In contrast to the standard batch arrival when a whole batch arrives into the system at one epoch, we assume that the customers of an accepted batch arrive one-by one in exponentially distributed times. Service time is exponentially distributed. Flow of batches is the stationary Poisson arrival process. Batch size distribution is geometric. The number of batches, which can be admitted into the system simultaneously, is subject of control. Analysis of the joint distribution of the number batches and customers in the system and sojourn time distribution is implemented by means of the matrix technique and method of catastrophes. Effect of control on the main performance measures of the system is demonstrated numerically.

Key Words : Reactive Admission Control, Queue, Matrix Analytic Methods, Performance Modeling

1. Introduction

Queueing systems suit for description of a variety of real-life processes, in particular, description of operation of telecommunication networks and they have got a lot of attention in probabilistic literature. Important class of queueing systems assumes that customers arrive into the system in batches. It is usually assumed that, at a batch arrival epoch, all customers of this batch arrive into the system simultaneously.

However, the typical feature of many nowadays communication networks is that customers arrive in batches, but arrival of customers is not instantaneous but is distributed in time. The first customer of a batch arrives at the batch arrival epoch while the rest of customers arrive one-by-one in random intervals. The batch size is random and it may be not known a priori at the batch arrival epoch. Such a situation is typical, e.g., in modeling transmission of video and multimedia information. This situation is also discussed in [8] with respect to the modeling Scheme of Alternative Packet Overflow Routing in IP networks. In [8], performance measures of this scheme of routing in IP networks are evaluated by means of computer simulation. Analogous queueing model arises in modeling queries processing in data bases where, besides the CPU, some additional threads or connections should be provided to start the query processing.

In the paper ([3], [4] and [5]), the Markovian queueing model with a finite buffer that suits for performance evaluation of this routing scheme as well of other real life systems with time distributed arrival of customers in a batch is considered. To the best of our knowledge, such kind of queueing models was not

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considered and investigated in literature previously. In [5], the problem of the system throughput maximization subject to restriction of loss probability for customers from accepted batches is solved.

In the present paper, we extend analysis given in [5] to the case of the system with infinite buffer. In this case, customers from accepted batch never are lost. So, we may solve the problem of the system throughput maximization subject to restriction of average sojourn time for accepted batches.

The rest of the paper is organized as follows. In section 2, the model is described. Stability condition is derived and the steady state joint distribution of the number of batches and customers in the system is analyzed in section 3. Section 4 is devoted to consideration of the batch sojourn time distribution. Section 5 contains numerical illustrations and section 6 concludes the paper.

2. Mathematical Model

We consider a single server queueing system with an infinite buffer. Service time has exponential distribution with parameter. The customers arrive to the system in batches. Batches arrive into the system according to a stationary Poisson arrival process with intensity λ. Following [7], we assume that admission of batches (they are called *flows* in [7] and called threads, connections, windows, etc. in different real-life applications) is restricted by means of *tokens*. The total number of available tokens is assumed to be $K, K \geq 1$. Further we consider the number K as control parameter and can solve the corresponding optimization problem. If there is no token available at a batch arrival epoch the batch is rejected. It leaves the system forever. If the number of available tokens at the batch arrival epoch is positive this batch is admitted into the system and the number of available tokens decreases by one. We assume that one customer of a batch arrives at the batch arrival epoch and if it meets free server, it occupies the server and is processed. If the server is busy, the customer moves to a buffer and later it is picked up for the service according to the First Came First Served discipline. After admission of the batch, the next customer of this batch can arrive into the system in exponentially distributed with parameter γ time. The number of customers in the batch has geometrical distribution with parameter $\theta, 0 \leq \theta \leq 1$, *i. e.* probability that the batch consists of k customers is equal to $\theta^{k-1}(1-\theta), k \geq 1$. Mean size of the batch is equal to $(1-\theta)^{-1}$. If the exponentially distributed with parameter γ time since arrival of the previous customer of a batch expires and new customer does not arrive, it means that the arrival of the batch is fished. The token, which was obtained by this batch on arrival, is returned into the pool of available tokens. The customers of this batch, which stay in the system at the epoch of returning the token, should be completely processed by the system. When the last customer is served, sojourn time of the batch in the system is considered finished. It is intuitively clear that this mechanism of arrivals restriction by means of tokens is reasonable. At the expense of rejecting some batches, it allows to decrease sojourn time and jitter for admitted batches. It is important in modeling real-life systems because quality of transmission of accepted information units should satisfy imposed requirements of Quality of Service.

Note, that situation when the new batch is rejected while the system is empty of customers is theoretically possible. Probably, this is shortcoming of the considered scheme. But the mechanism of tokens creates better conditions (shorter delay and smaller jitter) for transmission of customers from the accepted batches. Quantitative analysis of advantages and shortcomings of this mechanism requires calculation of the main performance measures of the system under the fixed value K of tokens in the system. These measures can be calculated basing in the knowledge of stationary distribution of the random process describing dynamics of the system under study.

3. Joint Distribution of the Number of Batches and Customers in the System

Let the number K of tokens be fixed, i_t be the total number of customers in the system at epoch $t, t \ge 0, i_t \ge 0$, and k_t be the number of batches having token for admission to the system at epoch $t, t \ge 0, k_t = \overline{0,K}$. It is obvious that the two-dimensional process $\xi_t = \{i_t, k_t\}, t \ge 0$ is the irreducible regular continuous time Markov chain.

Introduce the following notation:

- $\gamma^{-1} = \gamma(1-\theta), \gamma^{+} = \gamma\theta$,
- $C_K = diag\{0,1,\cdots,K\}$ i.e., the diagonal matrix with the diagonal entries $\{0,1,\cdots,K\}$
- I is identity matrix, $\vec{e}$ is column vector consisting of 1's, $\vec{0}$ is row vector consisting of 0's. When dimension of the matrix or the vector is not clear from context, it is indicated by suffix, e.g., $\vec{e}_{K+1}$ denotes the unit column vector of dimension $K+1$

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \gamma^{-1} & -\gamma & 0 & \cdots & 0 & 0 \\ 0 & 2\gamma^{-1} & -2\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & K\gamma^{-1} & -K\gamma \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -\gamma & 0 & \cdots & 0 & 0 \\ \gamma^{-} & -2\gamma & \cdots & 0 & 0 \\ 0 & 2\gamma^{-} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (K-1)\gamma^{-} & -K\gamma \end{pmatrix}$$

$$E^{+} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \hat{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

- $\delta_{i,j}$ is Kronecker delta. It is equal to 1 if $i = j$ and equal to 0 otherwise, $\otimes$ is the symbol of Kronecker product of matrices, $\vec{b}^T$ means transposition of the vector $\vec{b}$.

Let Q be the generator of the Markov chain $\xi_t, t \ge 0$, with blocks $Q_{i,j}$ consisting of intensities $(Q_{i,j})_{k,k'}$ of transitions of the Markov chain $\xi_t, t \ge 0$, from the state (i,k) into the state $(j,k'), k,k' = \overline{0,K}$.

The block $Q_{i,j}, i,j \ge 0$ has dimension $(K+1)\times(K+1)$. The diagonal entries of the matrix $Q_{i,i}$ are negative and the modulus of the entry of $(Q_{i,i})_{k,k}$ defines the total intensity of leaving the state (i,k) of the Markov chain.

Lemma. The generator Q has the three-block-diagonal structure:

$$Q=\begin{pmatrix} Q_{0,0} & Q_0 & 0 & 0 & \cdots \\ Q_2 & Q_1 & Q_0 & 0 & \cdots \\ 0 & Q_2 & Q_1 & Q_0 & \cdots \\ 0 & 0 & Q_2 & Q_1 & \cdots \\ \vdots & \vdots & 0 & \vdots & \ddots \end{pmatrix}$$

Proof of the lemma consists of analysis of transitions of the Markov chain $\xi_t, t \geq 0$, during the infinitesimal interval of time and further combining intensities of corresponding transitions into the matrix blocks. Value γ^- is the intensity of tokens releasing due to the finish of a batch arrival, γ^+ is the intensity of new customers in a batch arrival.

It follows from Lemma that the Markov chain $\xi_t = \{i_t, k_t\}, t \geq 0$, belongs to the class of Quasi-Birth-and-Death processes. So, well-known technique by M. Neuts, see (Neuts 1981), can be applied to derive ergodicity condition for this Markov chain and to find its stationary distribution.

Theorem 1. Markov chain $\xi_t = \{i_t, k_t\}, t \geq 0$, is ergodic if and only if the following inequality is fulfilled:

$$\mu \geq \lambda(1 - yK) + \gamma^+ \sum_{k=1}^{K} k y_k \tag{1}$$

where

$$y_k = \sigma_k \left(\sum_{l=0}^{K} \sigma_l \right)^{-1}, \quad \sigma_k = \frac{(\lambda(\gamma^-)^{-1})^k}{k!}, \quad k = \overline{0, K} \tag{2}$$

Proof. It follows from (Neuts 1981) that ergodicity condition of Markov chain $\xi_t = \{i_t, k_t\}, t \geq 0$, is fulfillment of inequality

$$\vec{y} Q_2 \vec{e} \rangle \vec{y} Q_0 \vec{e} \tag{3}$$

where the row vector $\vec{y}$ is solution to the system of linear algebraic equations

$$\vec{y}(Q_0 + Q_1 + Q_2) = \vec{0}, \ \vec{y}\vec{e} = 1 \tag{4}$$

Explicit form of the matrix $(Q_0 + Q_1 + Q_2)$ is the following:

$$\begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \gamma^- & -(\gamma^- + \lambda) & \lambda & \cdots & 0 & 0 \\ 0 & 2\gamma^- & -(2\gamma^- + \lambda) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & K\gamma^- & -K\gamma^- \end{pmatrix}$$

It can be easy verified that this matrix coincides with generator of the process of number of customers in the classical Erlang's $M/M/K/0$ system with customers arrival rate λ and service rate γ^-. So, the entries $y_k, k = \overline{0, K}$ of the vector $\vec{y}$, which is solution of the system (4), are computed by formulae(2). Substituting the vector $\vec{y}$ defined by these components and explicit form of matrices Q_0, Q_2 into (3), we get inequality (1). Theorem is proven.

Inequality (1) has intuitively evident meaning: the Markov chain $\xi_t = \{i_t, k_t\}, t \geq 0$, is ergodic if and only if the service rate exceeds the total customer arrival rate. The first summand in the right hand side of (1) is the rate of first customers in batches arrival. The second one is the rate of customers from already accepted batches.

It should be noted that condition (1) can be used for rough estimation of the admissible number K of tokens in the system under the fixed values of the arrival and service processes parameters. More exact estimation, which takes into account sojourn time of accepted batches in the system and the batch loss probability, requires calculation of stationary distributions of the Markov chain $\xi_t = \{i_t, k_t\},\ t \geq 0$, and sojourn time of accepted batches in the system. In what follows we assume that condition (1) is fulfilled. Then the following limits (stationary probabilities) exist:

$$\pi(i,k) = \lim_{t\to\infty} P\{i_t = i, k_t = k\}, i \geq 0, k = \overline{0,N}$$

Let us combine these probabilities into the row vectors

$$\boldsymbol{\pi}_i = (\pi(i,0), \pi(i,1), \cdots, \pi(i,K)), i \geq 0$$

It is well known that the vector $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)$ is the unique solution to the following system of linear algebraic equations:

$$(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)Q = \mathbf{0},\quad (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)\mathbf{e} = 1$$

The following statement directly follows from the results by M. Neuts in (Neuts 1981).

Theorem 2. The stationary probability vectors $\boldsymbol{\pi}_i, i \geq 0$, are computed by

$$\boldsymbol{\pi}_i = \pi_0 R^i,\ i \geq 0$$

where the matrix R is the minimal non-negative solution to the system

$$R^2 Q_2 + RQ_1 + Q_0 = O$$

and the vector $\boldsymbol{\pi}_0$ is the unique solution to the system of linear algebraic equations

$$\boldsymbol{\pi}_0 (Q_{0,0} + RQ_2) = \mathbf{0},\quad \boldsymbol{\pi}_0 (I - R)^{-1}\mathbf{e} = 1$$

Having stationary probability vectors $\boldsymbol{\pi}_i,\ i = \overline{0,N}$, been computed one can calculate different performance measures of the system. Some of them are given in the following statements.

Corollary 1. Probability distribution of the number of customers in the system is computed by

$$\lim_{t\to\infty} P\{i_t = i\} = \boldsymbol{\pi}_i \mathbf{e}, i \geq 0$$

Average number L of customers in the system is computed by

$$L = \sum_{i=0}^{\infty} i\boldsymbol{\pi}_i \mathbf{e} = \boldsymbol{\pi}_0 R(I - R)^{-2}\mathbf{e}$$

Corollary 2. Probability distribution of the number of batches in the system is computed by

$$\lim_{t\to\infty} P\{k_t = k\} = \boldsymbol{\pi}_0 (I - R)^{-1}\mathbf{e}^{(k)}, k = \overline{0,K}$$

where the column vector $\mathbf{e}^{(k)}$ has all zero entries except the k^{th} one, which is equal to $1, k = \overline{0,K}$

Average number B of batches in the system is computed by

$$B = \sum_{k=1}^{K}\sum_{i=0}^{\infty} k\pi(i,k) = \boldsymbol{\pi}_0 \sum_{k=1}^{K} k(I - R)^{-1}\mathbf{e}^{(k)}$$

Remark 1. In contrast to the model with the finite buffer, see (Lee et al. 2007), where the arriving batch can be rejected not only due to the tokens absence but also due to the buffer overloading, distribution of the number of batches in the model under study does not depend on the number of customers in the system. So,

$$\lim_{t\to\infty} P\{k_t = k\} = \sigma_k (\sum_{l=0}^{K} \sigma_l)^{-1},\ k = \overline{0,K}$$

i.e., the marginal distribution of the number of batches in the system coincides with distribution of busy servers in Erlang loss model $M/M/K/0$. However, distribution $\pi(i,k), i \geq 0, k = \overline{0,K}$, does not have multiplicative form because the number of customers in the system depends on the number of batches currently presenting in the system.

Corollary 3. Mean number T of customers processed by the system at unit of time (throughput) is computed by

$$T = \mu(1 - \boldsymbol{\pi}_0 \mathbf{e})$$

The probability $P_b^{(loss)}$ of an arbitrary batch rejection is computed by

$$P_b^{(loss)} = \sum_{i=0}^{\infty} \pi(i,K) = \boldsymbol{\pi}_0 (I - R)^{-1} \mathbf{e}^{(K)} = y_K$$

where probability y_K is given by formula (2).

4. Distribution of the Sojourn Times

Let $V_b(x)$ be distribution function of an arbitrary batch sojourn time in the system under study and $v_b(s)$ be its Laplace-Stieltjes Transform (*LST*):

$$v_b(s) = \int_0^{\infty} e^{-sx} dV_b(x), \ \text{Re}\ s \geq 0$$

Recall that sojourn time of an arbitrary batch in the system starts since the epoch of the batch arrival into the system until the moment when all customers belonging to this batch leave the system. We will derive expression for the *LST* $v_b(s)$ by means of the method of collective marks (method of additional event, method of catastrophes). To this end, we assume that the variable s is real and interpret it as the intensity of some virtual stationary Poisson flow of catastrophes. So, $v_b(s)$ has meaning of probability that no one catastrophe arrives during the sojourn time of an arbitrary batch. The expression for complex s is easy obtained by means of analytic continuation.

We will tag an arbitrary batch and will keep track of its staying in the system. Let $v(s,i,l,k)$ be probability that catastrophe will not arrive during the rest of the tagged batch sojourn time in the system conditional that, at the given moment, the number of batches processed in the system is equal to $k, k = \overline{0,K}$, the number of customers is equal to $i, i \geq 0$, and the last (in the order of arrival) customer of the tagged batch has position number $l, l = \overline{0,i}$, in the system. Position number 0 means that currently there is no one customer of the tagged batch in the system.

Theorem 3. The *LST* $v_b(s)$ of an arbitrary batch sojourn time is computed by formula

$$v_b(s) = P_b^{(loss)} + \sum_{i=0}^{\infty}\sum_{k=0}^{K-1} \pi(i,k) v(s, i+1, l+1, k+1)$$

where the *LST*s $v(s,i,l,k),\ l = \overline{0,i}, i \geq 0, k = \overline{1,K}$, are defined as components of the vector $\mathbf{v}(s)$ having form

$$\mathbf{v}(s) = -\boldsymbol{\Omega}^{-1}(s)\boldsymbol{\beta}(s) \qquad (6)$$

where the vector $\boldsymbol{\beta}(s)$ is defined by

$$\boldsymbol{\beta}(s) = (\boldsymbol{\beta}_0(s), \boldsymbol{\beta}_1(s), \cdots, \boldsymbol{\beta}_N(s), \cdots)^T$$

$$\boldsymbol{\beta}_i(s) = \gamma^-(1, \frac{\mu}{\mu+s}, \cdots, (\frac{\mu}{\mu+s})^i)^T \otimes \mathbf{e}_K,\ i \geq 0$$

the matrix $\boldsymbol{\Omega}(s)$ is the blocking three-diagonal matrix with non-zero blocks

$$\Omega_{i,j}(s), j = \max\{0, i-1\}, i, i+1, i \geq 0$$

defined by

$$\Omega_{i,i}(s) = -I_{i+1} \otimes (sI - \hat{Q}_{i,i})$$

$$\Omega_{i,i+1}(s) = D_1^{(i)} \otimes \hat{Q}_{i,i+1} + D_2^{(i)} \otimes \gamma^+ I_K$$

$$\Omega_{i,i-1}(s) = D_3^{(i)} \otimes \hat{Q}_{i,i-1},\ i \geq 0$$

where the matrix $D_1^{(i)}$ of dimension $(i+1) \times (i+2)$ is obtained from identity matrix I_{i+1} by means of supplementing it from the right with a column 0_{i+1}^T.

The matrix $D_2^{(i)}$ of the same dimension has all entries equal to 0 except the entries which are located in the last column and are equal to 1. The matrix $D_3^{(i)}$ of dimension $(i+1)\times i$ is obtained from identity matrix I_i by means of supplementing it from above with a row $\{1,0,\cdots,0\}$, the matrices $\hat{Q}_{i,j}$ are defined by

Proof. Formula (5) for the *LST* $v_b(s)$ obviously follows from formula of total probability. So, to prove the theorem we have to derive equation (6). To this end, we derive the system of linear algebraic equations for the *LST*s $v(s,i,l,k)$, based on the formula of total probability:

$$v(s,i,l,k)=\frac{1}{s+\lambda(1-\delta_{k,K})(1-\delta_{i,N})+\mu+k\gamma}\times$$
$$\times[\lambda(1-\delta_{k,K})(1-\delta_{i,N})v(s,i+1,l,k+1)+$$
$$+\mu_i v(s,i-1,l-1,k)+\gamma^+ v(s,i+1,l+1,k)+$$
$$+\gamma^+(k-1)v(s,i+1,l,k)+\gamma^-(\frac{\mu}{\mu+s})^l+$$
$$+\gamma^-(k-1)v(s,i,l,k-1)]$$
$$l=\overline{0,i},\ i\geq 0,\ k=\overline{1,K} \qquad (7)$$

Brief explanation of formula (7) is the following. Denominator of the right side of (7) is equal to the total intensity of the events which can happen after the arbitrary time moment: catastrophe arrival, new batch arrival, service completion, and expiring the time till the moment of possible customer arrival from batches already admitted into the system. The first term in the square brackets in (7) corresponds to the case when the new batch arrives. The second one corresponds to the case when service completion takes place. The third term corresponds to the case when the new customer of the tagged batch arrives into the system. In this case, the position of the last customer of a tagged batch in the system is reinstalled from l to $i+1$. The fourth term corresponds to the case when the new customer from another batch, which was already admitted to the system, arrives. The fifth term corresponds to the case when the expected new customer of the tagged batch does not arrive into the system and arrival of customers of the tagged batch is stopped. This batch will not more counted as arriving into the system and the tagged customer finishes its sojourn time when the last customer, who is currently the lth in the system, will leave the system. The sixth term corresponds to the case when some other batch is stopped.

Let us introduce column vectors $\mathbf{v}(s,i,l)=(v(s,i,l,1),\cdots,(s,i,l,K))^T, l=\overline{0,i},\ i\geq 0$ of dimension K.

The system of linear algebraic equations (7) can be rewritten in the following matrix form:

$$-(sI-\hat{Q}_{i,i})\mathbf{v}(s,i,l)+\hat{Q}_{i,i+1}\mathbf{v}(s,i+1,l)+$$
$$+\hat{Q}_{i,i-1}\mathbf{v}(s,i-1,l-1)+\gamma^+\mathbf{v}(s,i+1,l)+$$
$$+\gamma^-(\frac{\mu}{\mu+s})^l\mathbf{e}_K=\mathbf{0}_K^T,\ l=\overline{0,i},\ i\geq 0 \qquad (8)$$

Now, let us introduce column vectors $\mathbf{v}(s,i)=(\mathbf{v}(s,i,0),\cdots,\mathbf{v}(s,N))^T$ of dimension $K(i+1), i\geq 0$, and column vector

$$\mathbf{v}(s)=(\mathbf{v}(s,0),\cdots,\mathbf{v}(s,N),\cdots)^T$$

Using this notation, we rewrite the system (8) to the form

$$\mathbf{\Omega}_{i,i}\mathbf{v}(s,i)+\mathbf{\Omega}_{i,i+1}\mathbf{v}(s,i+1)+\mathbf{\Omega}_{i,i-1}\mathbf{v}(s,i-1)+$$
$$+\boldsymbol{\beta}_i(s)=\mathbf{0}_{K(i+1)}^T,\ i\geq 0 \qquad (9)$$

and then into the form

$$\mathbf{\Omega}(s)\mathbf{v}(s)+\boldsymbol{\beta}(s)=\mathbf{0}^T \qquad (10)$$

Formula (6) evidently follows from (10). Inverse matrix in (6) exists for any $s, \mathrm{Re}\, s > 0$ and $s=0$ because the diagonal entries of the matrix $\mathbf{\Omega}(s)$ dominate in rows of this matrix. Theorem is proven.

Remark 2. The *LST* $v_b(s)$ is the *LST* of an arbitrary batch sojourn time, including the batches which are rejected and do not enter the system. The *LST* $v_b^{(accept)}$ of an arbitrary accepted batch sojourn time is calculated by

$$v_b^{(accept)}(s) = \frac{\sum_{i=0}^{\infty}\sum_{k=0}^{K-1} \pi(i,k) v(s,i+1,l+1,k+1)}{1-P_b^{(loss)}}$$

Corollary 4. The mean sojourn time V_b of an arbitrary batch is computed by

$$V_b = -\sum_{i=0}^{\infty}\sum_{k=0}^{K-1} \pi(i,k) \left. \frac{\partial v(s,i+1,i+1,k+1)}{\partial s} \right|_{s=0} \qquad (11)$$

where the values $\left. \frac{\partial v(s,i+1,i+1,k+1)}{\partial s} \right|_{s=0}$ are computed as the corresponding entries of the vector $\left. \frac{d\mathbf{v}(\mathbf{s})}{ds} \right|_{s=0}$, which is calculated by

$$\left. \frac{d\mathbf{v}(\mathbf{s})}{ds} \right|_{s=0} = \Omega^{-1}(0) \left(\left. \frac{d\beta(s)}{ds} \right|_{s=0} + \mathbf{e} \right) \qquad (12)$$

where

$$\left. \frac{d\beta(s)}{ds} \right|_{s=0} =$$

$$= -\gamma^{-} \left(0,0,\frac{1}{\mu},0,\frac{1}{\mu},\frac{2}{\mu},\cdots,0,\frac{1}{\mu},\cdots,\frac{N}{\mu},\cdots \right) \otimes \mathbf{e}_K$$

Corollary 5. The mean sojourn time $V_b^{(accept)}$ of an arbitrary accepted batch is computed by

$$V_b^{(accept)} = \frac{V_b}{1-P_b^{(loss)}}$$

Remark 3. Formulae (6), (12) require inversion of infinite size matrices. Thus, to compute the required vectors, one should cut the corresponding matrices in a proper way. This does not lead to essential error in calculation of the *LST* $v_b(s)$ by formula (5) and the mean sojourn time V_b by formula (11) because $v(s,i,l,k) \to 0$ when $i \to \infty$ and probabilities $\pi(i,k)$ in right hand side of (5) and (11) become negligible for large i under assumption that the Markov chain $\xi_t = \{i_t, k_t\}, t \geq 0$ under study is ergodic.

5. Numerical Examples

The goals of this section are to demonstrate feasibility of the elaborated formulae for calculating the main performance measures of the system under study and to show effect of the number K of tokens in the system and effect of mean batch size. In the first numerical example, we fix the following parameters of the system: $\lambda = 1$, $\mu = 2.5$, $\gamma = 2$, $\theta = 0.5$.

Figures 1-4 show dependence of the system throughput T, average number L of customers in the system, batch loss probability $P_b^{(loss)}$ and average sojourn time $V_b^{(accept)}$ of accepted batches on the number K of tokens in the system.

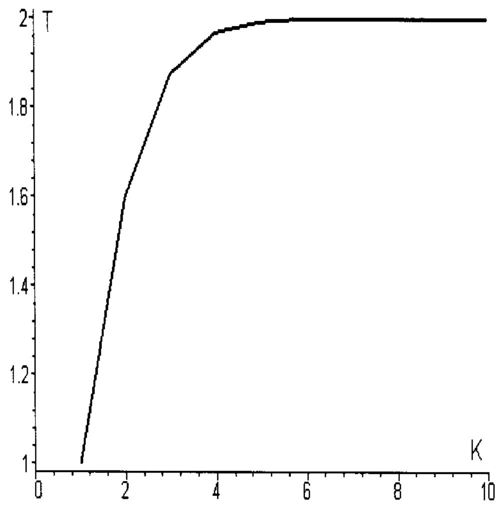

Fig. 1. Dependence of the throughput T on the maximal admissible number K of batches

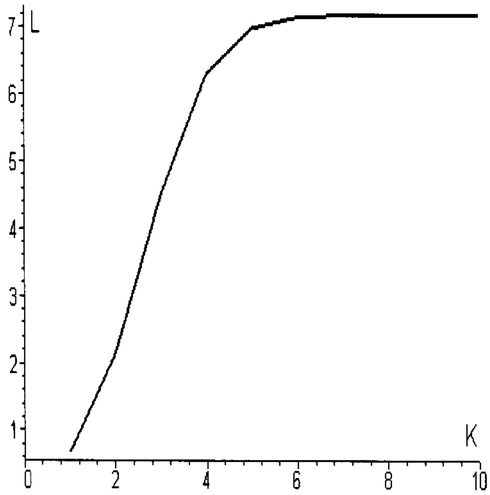

Fig. 2. Dependence of the average number L of customers in the system on the maximal admissible number K of batches

Looking at these figures, one can solve different optimization problems easy. For instance, if one would like to have the batch loss probability $P_b^{(loss)}$ less than 0.02, he should take the value K greater or equal to 4.

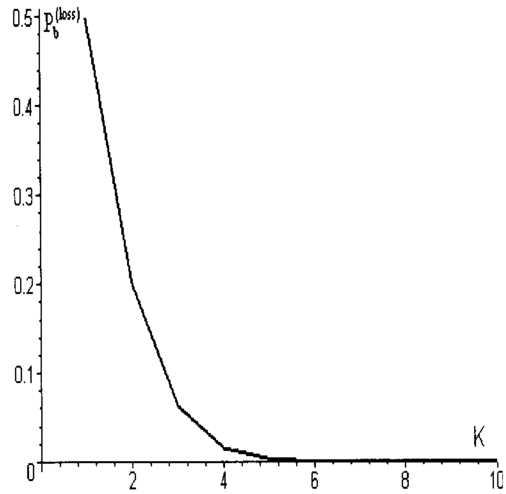

Fig. 3. Dependence of the batch loss probability $P_b^{(loss)}$ on the maximal admissible number K of batches in the system

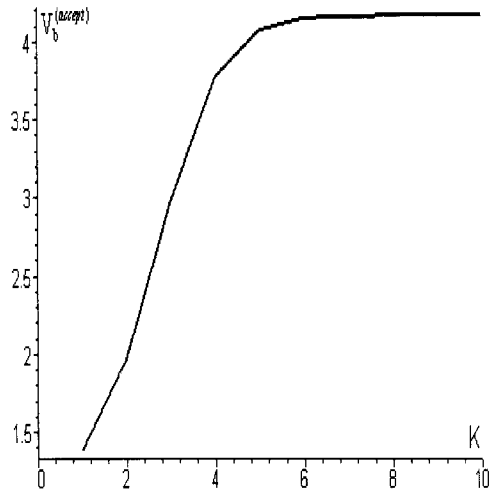

Fig. 4. Dependence of the average sojourn time $V_b^{(accept)}$ of accepted batches on the maximal admissible number K of batches

However, if one would like to have the average sojourn time $V_b^{(accept)}$ of accepted batches not exceeding 3.5, he should take the value K less or equal to 3.

In the previous example we fixed value $\theta = 0.5$ what implies that mean batch size is equal to 2. The second example has aim to demonstrate influence of parameter θ on the main performance measures of the system. We fix the following parameters of the system: $\lambda=2$, $\mu=4$, $\gamma=2$, $K=2$. Figures 5-8 show dependence of the system throughput T, average number L of customers in the system, batch loss probability $P_b^{(loss)}$ and average sojourn time $V_b^{(accept)}$ of accepted batches on the parameter θ.

Figures 5-8 evidently show great impact of the batch size and confirm usefulness of the results presented in this paper for accurate calculating the system performance measures, choosing suitable threshold K for batches accepting depending on the parameters of the system.

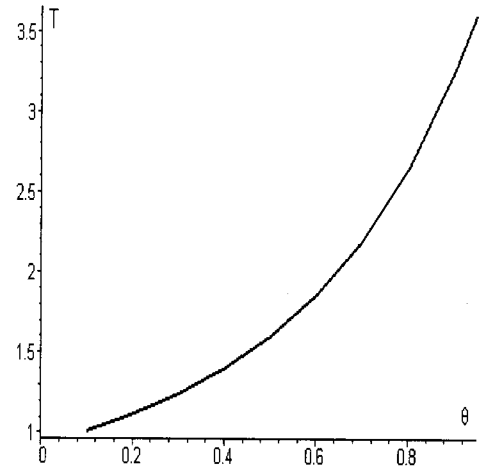

Fig. 5. Dependence of the throughput T on the parameter θ

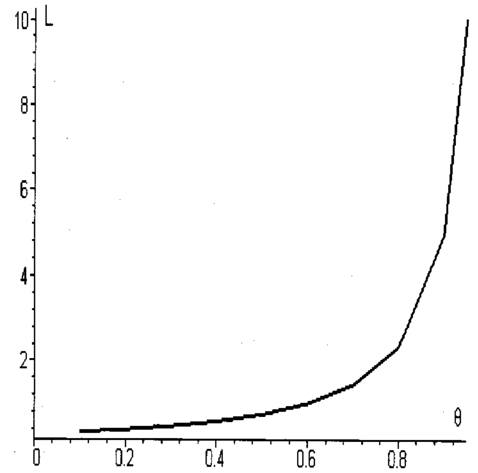

Fig. 6. Dependence of the average number L of customers in the system on the parameter θ

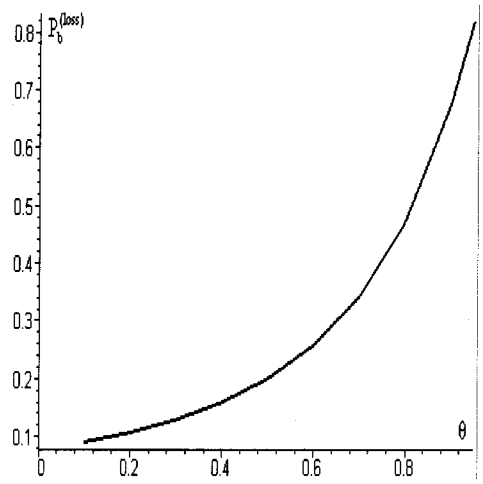

Fig. 7. Dependence of the batch loss probability $P_b^{(loss)}$ on the parameter

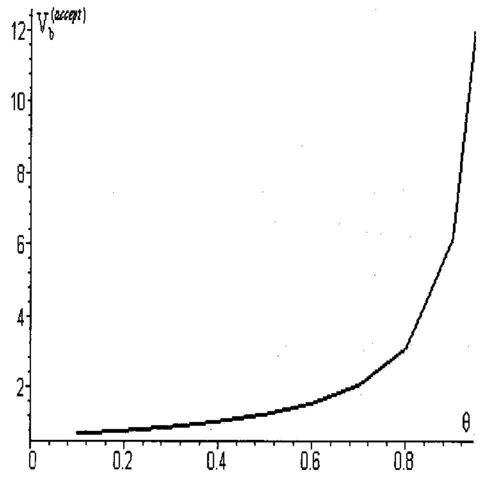

Fig. 8. Dependence of the average sojourn time $V_b^{(accept)}$ of accepted batches on the parameter θ

6. Conclusion

In this paper, novel infinite buffer queueing model with batch arrivals distributed in time is analyzed. Ergodicity condition is derived. Joint distribution of the number of customers in the system and number of currently admitted batches is computed. Sojourn time distribution of an arbitrary batch is given in terms of the Laplace-Stieltjes Transform. Usefulness of the presented results is illustrated numerically. Results are planned to be extended to the systems with more general batch arrival process (e.g., to *MAP* - Markovian Arrival Process), inter-arrival times of customers of a batch and service time distributions (e.g., to *PH* - Phase type distribution), possibility of standard batch customer arrival within an admitted batch, arbitrary distribution of the number of customers in a batch, service intensity depending on the number of customers in the system, etc.

References

[1] C. Blondia(1989), The *N/G/1* finite capacity queue, Communications in Statistics Stochastic Models 5, pp. 273-274.

[2] L. Breuer, V. Klimenok, A. Birukov, A. Dudin, U. Krieger(2005), Mobile networks modeling the access to a wireless network at hot spots, European Transactions on Telecommunications 16, pp.309-316.

[3] A.N. Dudin, S. Nishimura(2000), Optimal hysteretic control for a *BMAP/SM/1/N* queue with two operation modes, Mathematical Problems in Engineering 5, 397-420.

[4] A.N. Dudin, A.A. Shaban, V.I. Klimenok (2005), Analysis of a *BMAP/G/1/N* queue, International Journal of Simulation: Systems, Science and Technology 6, pp.13-23.

[5] Moon Ho Lee, Dudin S., Klimenok V.(2007), Queueing Model with Time-Phased Batch Arrivals". Managing Traffic Performance in Converged Networks The Interplay of Convergent and Divergent Forces. Proceedings of the 20th International Teletraffic Congress, 17 - 21 June 2007, Ottawa.

[6] W. Fisher, K.S. Meier-Hellstern(1993), The Markov-modulated Poisson process (MMPP) cookbook, Performance Evaluation 18, pp.149-171.

[7] D.P. Heyman, D. Lucantoni(2003), Modelling multiple IP traffic streams with rate limits, IEEE/ACM Transactions on Networking 11, pp. 948-958.26

[8] A.A. Kist, B. Lloyd-Smith, R.J. Harris(2005), A simple IP flow blocking model.Performance Challenges for Efficient Next Generation Networks. Proceedings of 19-th International Teletraffic Congress, 29 August 2 September 2005, Beijing, pp.355-364

[9] A. Klemm, C. Lindermann, M. Lohmann (2003), Modelling IP traffic using the batch Markovian arrival process, Performance Evaluation 54, pp.149-173.

[10] V. Klimenok, C.S. Kim, D. Orlovsky, A. Dudin(2005), Lack of invariant property of Erlang loss model in case of the MAP input, Queueing Systems 49, pp.187 213.

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