Journal of the Korean Institute of Industrial Engineers
Vol. 33, No. 1, pp. 26-32, March 2007.

Waiting-time Dependent Backordering Rate Under Partial Backlogging and Finite Production Rate

Ki Seung Jung · Hark Hwang†

Department of Industrial Engineering, Korea Advanced Institute of Science and Technology

품절 발생시 대기시간에 따른 Backorder 전환 비율에 관한 연구

정기승 · 황 학

한국과학기술원 산업공학과

This study deals with waiting-time dependent backordering rate during stock-out period in the Economic Production Quantity (EPQ) model. Assuming that the backordering rate follows an exponentially decreasing function of the waiting time, the backorder rate is developed under First-Come-First-Served (FCFS) and Last-Come-First-Served (LCFS) policy. The mathematical models are developed based on differential equations. Through numerical examples, the validity of the developed models is illustrated.

Keywords: Inventory, Partial Backorder, EPQ, FCFS, LCFS

1. Introduction

An important characteristic of the demand generating process in the inventory system is what happens when a demand occurs and the system is out of stock. Basically, there are two possibilities, either all the demand occurring during the stockout period is backordered or lost. These are referred to as the backorder case and the lost sale case, respectively. Another possibility is the case of partial backordering where some demands occurring during the stockout period are backordered while others are lost.

There exist numerous research papers in the literature that deal with the unsatisfied demand under shortage (Hadley and Whitin, 1963). Elsayed and Teresi (1983) assumed complete backlogging of demand under stockout situation. The assumption made it possible to analyze inventory systems with relative ease. Unfortunately, in free markets many sources are usually available and it is not likely that customers will wait until the procurement arrives. In other words, the assumption of complete backlogging is valid only for monopolistic markets. Later, Wee (1993) assumed that unsatisfied demand under shortage is partially backordered with a constant rate. We find that his assumption still does not reflect the customers' behaviour under shortage although his model is more realistic than the model of complete backlogging. In the real world situation, the customers' behavior might depend on how long they have to wait until the next replenishment arrives, i.e. the willingness of a customer to wait in the queue of backorder might be inversely proportional to the length of the waiting time. In Abad (1996),

This work was supported by grant No. R01-2003- 000-10077-0 from the Korea Science and Engineering Foundation.
† Corresponding author : Hark Hwang, KAIST, 373-1 Kusong-dong Yuseong-gu, Daejeon 305-701, Korea, Tel：+82-42-869-3113, Fax：+82-42-869-3110, E-mail：harkhwang@kaist.ac.kr
Received March 2006; revision received July 2006; accepted August 2006.

Chang *et al*. (1999), and Ouyang (2005), the waiting time (or the length of the queue of backorder) was considered as the major factor influencing customer's decision. With infinite production rate and FCFS rule they developed an inventory model assuming that only a fraction of demand is backordered where the fraction is a decreasing function of the waiting time. But their models are still limited to the EOQ model. Recently, Abad (2003) extended his research to the case of finite production rate under FCFS policy. Though his approach to model the backorder rate under finite production rate is very unique and mathematically elegant, we find that it is inadequate to examine the backordering phenomenon under LCFS policy.

This paper proposes another approach to express the backlogging rate analytically under FCFS and LCFS rules in the traditional EPQ setting. We assume that customers are impatient and the lost sales do not affect the customer's future behavior. Hence, when a stockout situation occurs, only a fraction of demand occurring at a given time is backordered. The fraction, moreover, is assumed to be a decreasing function of the waiting time, which is defined as the length of time customers have to wait until their backorders are satisfied. This paper is organized as follow: In section 2 and 3, the models are developed under FCFS and LCFS policy, respectively. The results are verified through numerical examples in section 4 and conclusions appear in Section 5.

2. Development of the Backordering Rate Under FCFS Service Policy

<Figure 1> shows the fluctuation of inventory level during the shortage period in a cycle of the EPQ model. Backlogged demand occurs at time 0 and builds up until T_1 where production begins. During $[T_1, T_2]$ the backlogged demands are satisfied by the

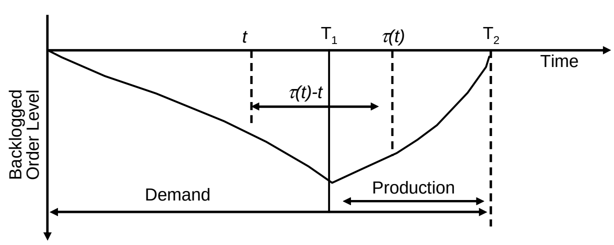

Figure 1. Backordered inventory level under the EPQ model with partial backlogging

items being produced. Suppose customer arrives at time t under FCFS policy. Then he has to wait until $\tau(t)$ where $\tau(t)$ is defined to be the time point where the backlogged demand of customer who arrives at time t is satisfied and his waiting time becomes $(\tau(t) - t)$. Let $B(t)$ be the fraction of the demand that arrives at time t and turns into backorder.

For the development of the model the notations are introduced as follow:

- $I(t)$ Inventory level at time t
- P production rate for an item
- D demand rate of an item ($P > D$)
- $\tau(t)$ the time point where the backlogged demand of customer who arrives at time t is satisfied
- $B(t)$ fraction of the demand which turns into backorder at time t
- $[0, T_1]$ the time interval during which backordered demand builds up (given)
- $[T_1, T_2]$ the time interval where backordered demand is satisfied

The following assumptions are adopted:

1. EPQ model with partial backordering is considered.
2. A single product is produced, backordered and consumed.
3. The demand is constant and deterministic.
4. The production rate is finite and larger than the demand rate.
5. Units from production are immediately available.
6. Unsatisfied demand is backordered at a rate of $B(t) = k_0\, e^{-k_1\{\tau(t) - t\}}$, $0 < k_0 \le 1,\ k_1 \ge 0$

2.1 Backordering Rate During $[0, T_1]$

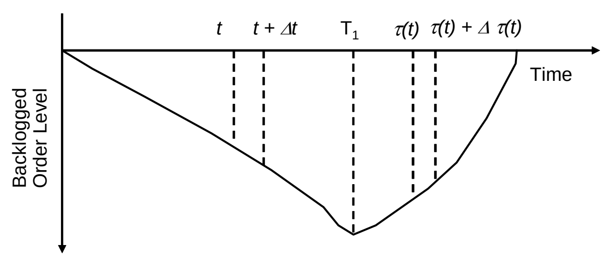

Figure 2. Inventory fluctuation during $[0, T_1]$

In <Figure 2> it can be argued that the backordering quantities occurred during $[t, t+\triangle t]$ are satisfied by the items produced during $[\tau(t), \tau(t)+\triangle(t)]$ i.e., (Demand rate D) × (Backordering rate $B(t)$) × $\triangle t$ = (Production rate P) × $\triangle\tau(t)$. Thus

$$D \times k_0\, e^{-k_1\{\tau(t)-t\}} \times \triangle t = P \times \triangle\tau(t) \quad (1)$$

$$\frac{\Delta\tau(t)}{\Delta t} = \frac{D}{P} \times k_0\, e^{-k_1\{\tau(t)-t\}} \quad (2)$$

If we assume the time duration $\triangle t$ and $\triangle\tau(t)$ are infinitesimally small, equation (2) can be rewritten as

$$\frac{d\tau(t)}{dt} = \frac{D}{P} \times k_0 e^{-k_1\{\tau(t)-t\}} \quad (3)$$

With $k_0 D/P = A$, equation (3) becomes

$$\frac{d\tau(t)}{dt} = Ae^{-k_1\{\tau(t)-t\}} = Ae^{-k_1\tau(t)}e^{k_1 t} \quad (4)$$

By applying the variable separation method, the solution of the differential equation (4) is derived as follows;

$$e^{k_1\tau(t)}\, d\tau(t) = Ae^{k_1 t}\, dt \quad (5a)$$

$$\int e^{k_1\tau(t)} d\tau(t) = A\int e^{k_1 t} dt \quad (5b)$$

$$\frac{1}{k_1}e^{k_1\tau(t)} = \frac{A}{k_1}e^{k_1 t} + C_1 \quad (5c)$$

$$e^{k_1\tau(t)} = Ae^{k_1 t} + C \quad (5d)$$

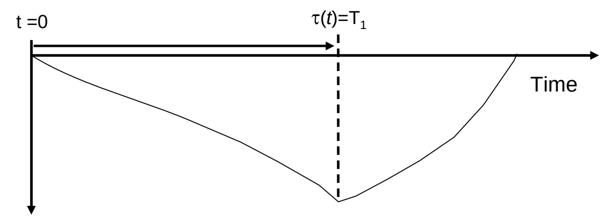

Figure 3. The boundary condition of differential equation, Eq.5

As shown in <Figure 3>, the demand of the customer who arrived at the beginning of the shortage will be satisfied at T_1, i.e. $\tau(t) = T_1$. With the boundary condition of $\tau(t) = T_1$ at $t = 0$, equation (5c) becomes

$$e^{k_1\tau(t)} = Ae^{k_1 t} + e^{k_1 T_1} - A \quad (6)$$

And $\tau(t)$ can be easily found from equation (6) and

$$\text{In}\ e^{k_1\tau(t)} = \text{In}(Ae^{k_1 t} + e^{k_1 T_1} - A) \quad (7a)$$

$$\tau(t) = \frac{1}{k_1}\text{In}\ \{A(e^{k_1 t} - 1) + e^{k_1 T_1}\} \quad (7b)$$

$$\tau(t) = \frac{1}{k_1}\text{In}\ \{\frac{k_0 D}{P}(e^{k_1 t} - 1) + e^{k_1 T_1}\} \quad (7c)$$

The backordering rate $B(t)$ in the interval $[0, T_1]$ can be easily expressed once $\tau(t)$ is known. The rate of change of the inventory level can be written as

$$\frac{dI(t)}{dt} = -DB(\tau(t)-t) = -Dk_0\, e^{-k_1(\tau(t)-t)} \quad (8a)$$

Thus the inventory level in the interval $[0, T_1]$ becomes

$$I(t) = -\int_0^t Dk_0\, e^{-k_1(\tau(t)-t)}\, dt,\ 0 \le t \le T_1 \quad (8b)$$

2.2 Backordering rate during $[T_1, T_2]$

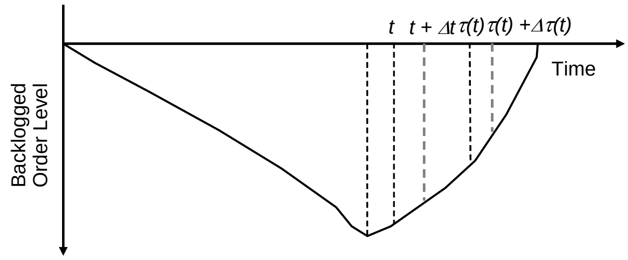

Figure 4. Inventory fluctuation during $[T_1, T_2]$

With the same argument as 2.1, the backorders occurred during $[t, t+\triangle t]$ are satisfied by the items produced during $[\tau(t), \tau(t)+\triangle\tau(t)]$ <Figure 4>. Thus equation (5d) is still applicable for $[T_1, T_2]$.

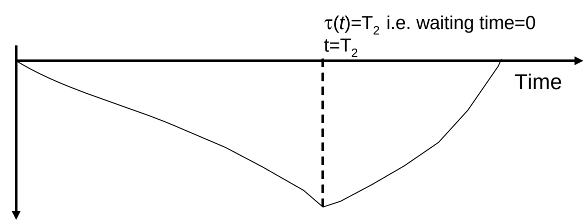

Figure 5. The boundary condition of the differential equation

The demand of the customer who arrives at the end of the shortage period will be immediately satisfied. This observation gives the boundary condition of the differential equation, i.e., $\tau(t) = T_2$ at $t = T_2$, and $C = e^{k_1 T_2}(1-A)$. Thus equation (5d) is rewritten as

$$e^{k_1\tau(t)} = Ae^{k_1 t} + e^{k_1 T_2}(1-A) \tag{9}$$

And $\tau(t)$ is derived from equation (9) as follows.

$$\mathrm{In}\ e^{k_1\tau(t)} = \mathrm{In}\{Ae^{k_1 t} + e^{k_1 T_2}(1-A)\} \tag{10a}$$

$$\tau(t) = \frac{1}{k_1}\mathrm{In}\{Ae^{k_1 t} + e^{k_1 T_2}(1-A)\} \tag{10b}$$

$$\tau(t) = \frac{1}{k_1}\mathrm{In}\{\frac{k_0 D}{P}e^{k_1 t} + e^{k_1 T_2}\frac{P-k_0 D}{P}\} \tag{10c}$$

With equations in (10) the backordering rate in the interval [T_1, T_2] under FCFS policy can be determined. Note that T_2 is the moment at which all the backlogged orders are satisfied. We utilize the result on T_2 of Abad(2003) [6] and

$$T_2 = \frac{1}{k_1}\mathrm{In}\left[\frac{P\,e^{k_1 T_1} - Dk_0}{P-Dk_0}\right] \tag{11}$$

The rate of change of the inventory level can be expressed as

$$\frac{dI(t)}{dt} = P - DB(\tau(t)-t) = P - Dk_0\,e^{k_1(\tau(t)-t)} \tag{12a}$$

Thus we can find the inventory level in the interval [T_1, T_2] and

$$I(t) = \int_{T_1}^{t}[P - Dk_0\,e^{k_1(\tau(t)-t)}]\,dt,\ \ T_1 \le t \le T_2 \tag{12b}$$

3. Development of Backordering Rate Under LCFS Service Policy

3.1 Backordering Rate During [0, T_1]

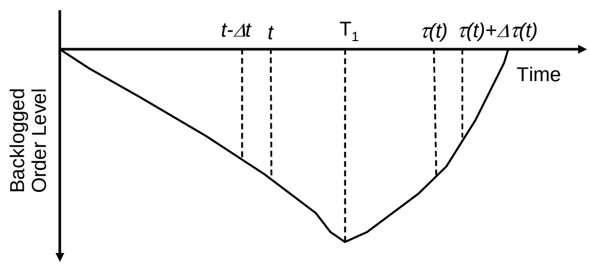

Figure 6. Inventory fluctuation during [0, T_1]

Under LCFS, the backorders occurred in the interval [t-$\triangle t$, t] are satisfied by the items produced during [$\tau(t)$, $\tau(t)+\triangle\tau(t)$]. The interval [t-$\triangle t$, t] implies that the backordered demand is satisfied inversely to the order of arrival. Hence, the mathematical relationship can be expressed as (Demand rate D) × (Backordering rate $B(t)$) × (-$\triangle t$) = (Production rate P - Demand rate D) × $\triangle\tau(t)$, i.e.,

$$D\times k_0^{-k_1\{\tau(t)-t\}} \times(-\triangle t) = (P-D)\times\triangle\tau(t) \tag{13}$$

$$\frac{d\tau(t)}{dt} = \frac{D}{D-P}\times k_0\,e^{-k_1\{\tau(t)-t\}} \tag{14}$$

The solution of the differential equation (14) is derived as the same manner as 2.1

$$e^{k_1\tau(t)} = Ae^{k_1 t} + C \ \ \text{where}\ \ A = \frac{k_0 D}{D-P} \tag{15}$$

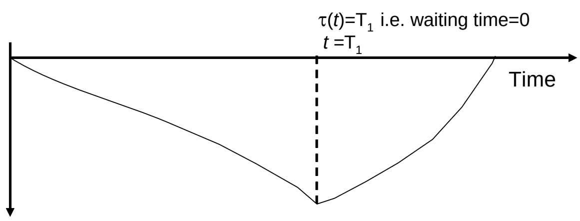

Figure 7. The boundary condition of differential equation, Eq.12

As shown in <Figure 7>, the demand of the customer who arrives at the beginning of the production period will be satisfied immediately under LCFS queue discipline, i.e. $\tau(t) = T_1$.

Thus with the boundary condition of $\tau(t) = T_1$ at $t = T_1$, equation (15) becomes

$$e^{k_1\tau(t)} = Ae^{k_1 t_1} + e^{k_1 T_1}(1-A) \tag{16}$$

And $\tau(t)$ is derived from equation (16) as follows.

$$\tau(t) = \frac{1}{k_1}\mathrm{In}\{\frac{k_0 D}{D-P}\,e^{k_1 t} + e^{k_1 T_1}\frac{D(1-k_0)-P}{D-P}\} \tag{17}$$

The inventory level under LCFS policy during the interval [0, T_1] can be obtained by utilizing equation (8b) developed in 2.1.

3.2 Backordering Rate During [T_1, T_2]

Note that under LCFS policy additional backorders do not occur in the interval [T_1, T_2] since the demand occuring in [T_1, T_2] is satisfied prior to the backorders in the waiting queue. And so the level of backorder decreases linearly with the decreasing rate of ($D-P$). In this case, T_2 is easily determined, once T_1 is known.

The inventory level in the interval [T_1, T_2] can be easily found from $dI(t)/dt = P - D$ and

$$I(t) = \int_{T_1}^{t} (P - D)\, dt, \quad T_1 \le t \le T_2 \tag{18}$$

4. Numerical Examples

We considered an EPQ model with the following parameter values: demand rate $D = 20$units/day, production rate $P = 30$units /day, $k_0 = 1.0$, $k_1 = 0.5$ and $T_1 = 8^{th}$ day. <Figure 8 ~ Figure 10> show the computational results on the waiting time, backordering rate, and backorder quantities, respectively. They were obtained by numerical analysis.

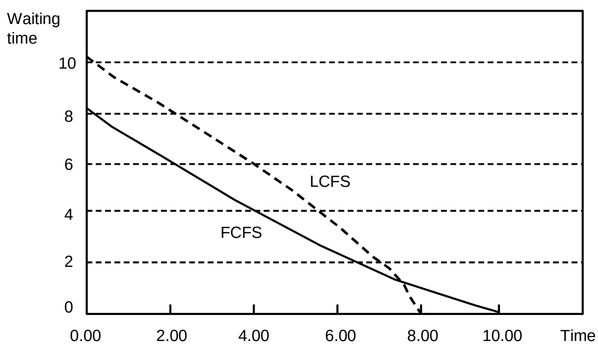

Figure 8. Waiting time under $D = 20$units/day, $P = 30$units /day, $k_0 = 1.0$, $k_1 = 0.5$

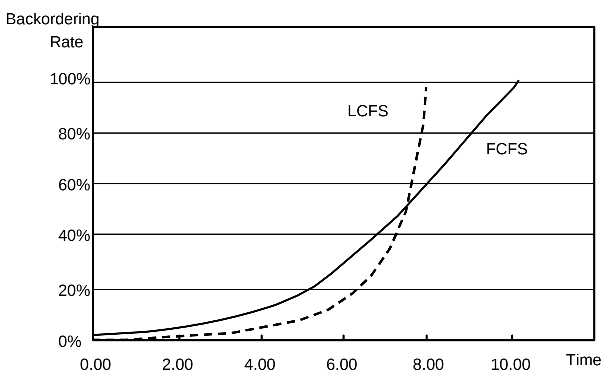

Figure 9. Backordering rate under $D = 20$units/day, $P = 30$units /day, $k_0 = 1.0$, $k_1 = 0.5$

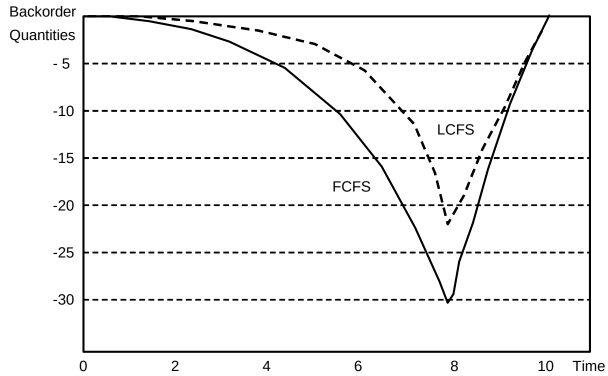

Figure 10. Backorder quantities under $D = 20$units/ day, $P = 30$units /day, $k_0 = 1.0$, $k_1 = 0.5$

Under FCFS queue discipline, the waiting time of the customer who arrives at the beginning of the shortage has to be T_1 where the production restarts. In <Figure 8> it can be confirmed that the maximum waiting time is 8 days while the waiting time decreases until it reaches zero at $T_2 = 10.17$.

The change of the backordering rate is shown in <Figure 9>. Under FCFS policy the backordering rate is only 1.83% at the beginning of stock out period due to the long waiting time of 8 days. After 4 days of stock out, the backordering rate increases rapidly. On the 5th day, it becomes 60.44% with the waiting time of 1.0067 days. And then finally it reaches 100% at $T_2 = 10.17$ days. Under LCFS policy, the backordering rate is only 0.6181% with the waiting time of 10.1727 days at the beginning of stockout period, which is quite smaller than those under FCFS discipline. The backordering rate increases rapidly after the 6th day due to the shorter waiting time under LCFS policy which reaches100% at $T_2 = 10.17$days.

The amount of backorder inventory was obtained through a numerical method with a time interval of 0.25 days and the results are shown in <Figure 10>. The backorder level under FCFS policy builds up quite rapidly and reaches the maximum level of 30.23 units at $T_1 = 8$ days. Theoretically the backorder amount at T_2 is zero while the numerically driven value is 0.04 units. The difference is due to the discrete analysis we adopted. Under the LCFS queue discipline the demand of customers who arrives after T_1 is satisfied immediately, which is illustrated by the straight dotted line during the production period.

Changing the value of P and k_1, another example problem was solved with the following parameter values: demand rate $D = 20$units/day, production rate $P = 24$units / day, $k_0 = 1.0$, $k_1 = 0.8$ and $T_1 = 8^{th}$ day. Note

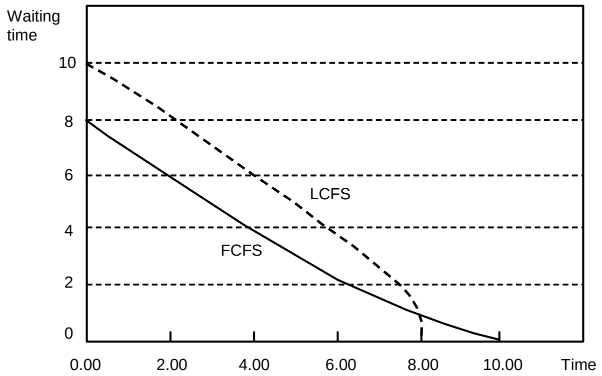

Figure 11. Waiting time under $D = 20$units/day, $P = 24$units /day, $k_0 = 1.0$, $k_1 = 0.8$

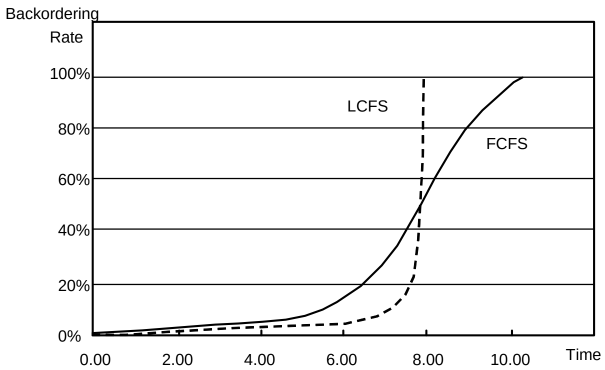

Figure 12. Backordering rate under $D = 20$units/day, $P = 24$units /day, $k_0 = 1.0$, $k_1 = 0.8$

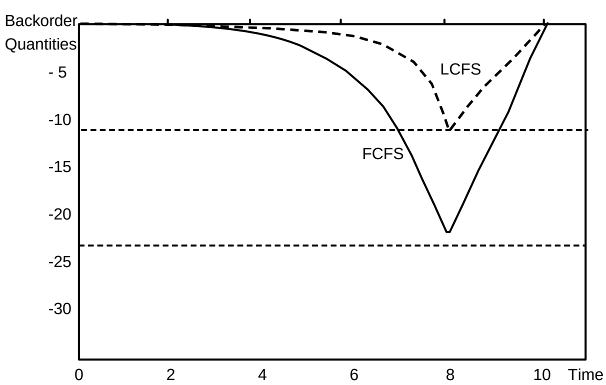

Figure 13. Backorder quantities under $D = 20$units/day, $P = 24$units /day, $k_0 = 1.0$, $k_1 = 0.8$

that compared with the previous example the pro duction rate becomes slightly larger than the demand rate by only 4 units per day, which implies that the insufficient capacity of manufacturer will lengthen the waiting time of the customers in the backorder queue. Also, the parameter k_1 is set to 0.8 which is much larger than that in the previous one. A larger value of k_1 implies that the customers are much more impatient and so they tend to look for other manufacturers who can satisfy their demand immediately. Consequently, we expect that the overall level of backordered inventory will be much smaller than those in the first example.

The waiting time, backordering rate and backorder amount for the example are shown in <Figure 11 ~ Figure 13>, respectively, under each of two queue disciplines. As expected, the overall inventory level turns out to be much smaller compared with the previous example regardless of queue discipline. It implies that the manufacturer tends to lose more customers. The backordering rate increases rapidly near the restart of production, which reflects the impatience of the customers. The results of the two example problems are consistent with our expectation and show the validity of the proposed models.

5. Conclusions

In this paper, we studied the partial backordering rate under stock out conditions with the assumption that the rate of the backordering follows an exponentially decreasing function of the waiting time of the customer. A major distinction of this study from the literature is that the partial backordering rate is quantified in mathematical form based on differential equations. Through solving examples problems we illustrated the effects of two issuing policies on the waiting time, backordering rate, and backorder quantities. As further studies, the current model could be extended by treating T_1 as decision variable with the objective of maximizing the average profit. Also, the case of stochastic demand could be suggested to make the model more realistic.

References

Abad, P. L. (1996), Optimal pricing and lot sizing under conditions of perishability and partial backordering, *Management Science*, **42**(8), 1093-1104.

Abad, P. L. (2003), Optimal pricing and lot-sizing under conditions of perishability, finite production and partial backordering and lost sale, *European Journal of Operational Research*, **144**(3), 677-685.

Elsayed, E. A. and Teresi, C. (1983), Analysis of inventory system with deteriorating items, *International Journal of Production*

Research, **21**(4), 449–460.

Hadley, G. and Whitin, T. (1963), *Analysis of Inventory Systems*, Prentice-Hall International Inc., Englewood Cliffs, NJ.

Chang, H. J. and Dye C. Y. (1999), An EOQ model for deteriorating items with time varying demand and partial backlogging, *Journal of the Operational Research Society*, **50**(11), 1176-1182.

Ouyang, L. Y. (2005), An EOQ model for perishable item under time-dependent partial backordering, *European Journal of Operational Research*, **163**(3), 776-783.

Wee, H. M. (1993), Economic production lot size model for deteriorating items with partial back-ordering, *Computers and Industrial Engineering*, **24**(3), 449–458.