ANALYTIC PROPERTIES OF THE $q$-VOLKENBORN INTEGRAL ON THE RING OF $p$-ADIC INTEGERS

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Abstract. In this paper, we consider the $q$-Volkenborn integral of uniformly differentiable functions on the $p$-adic integer ring. By using this integral, we obtain the generating functions of twisted $q$-generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

1. Introduction

Let $p$ be an odd prime. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will always denote, respectively, the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ the field of rational numbers) will denote the $p$-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $| \cdot |_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. We let $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid 1/x \in \mathbb{Z}_p\}$. A $p$-adic integer in $\mathbb{Z}_p^\times$ is sometimes called a $p$-adic unit. Let $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For each integer $N \geq 0$, $\mathbb{C}_p^N$ will denote the multiplicative group of the primitive $p^N$-th roots of unity in $\mathbb{C}_p^\times = \mathbb{C}_p \setminus \{0\}$. Set $\mathbf{T}_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^N} = 1 \text{ for some } N \geq 0\} = \bigcup_{N \geq 0} \mathbb{C}_p^N$.

The dual of $\mathbb{Z}_p$, in the sense of $p$-adic Pontryagin duality, is $\mathbf{T}_p = \mathbb{C}_p^\times_\mathbf{w}$, the direct limit (under inclusion) of cyclic groups $\mathbb{C}_p^N$ of order $p^N (N \geq 0)$, with the discrete topology. $\mathbf{T}_p$ admits a natural $\mathbb{Z}_p$-module structure which we shall write exponentially, viz $\omega^x$ for $\omega \in \mathbf{T}_p$ and $x \in \mathbb{Z}_p$. $\mathbf{T}_p$ can be embedded discretely in $\mathbb{C}_p$ as the multiplicative $p$-torsion subgroup and we now choose, for once and all, one such embedding. If $\omega \in \mathbf{T}_p$, then we denote by

\begin{equation}
\phi_\omega : (\mathbb{Z}_p, +) \longrightarrow (\mathbb{C}_p^\times, \cdot)
\end{equation}

for the locally constant character $x \mapsto \omega^x$, which is locally analytic character if $\omega \in \{\omega \in \mathbb{C}_p \mid v_p(\omega - 1) > 0\}$. Then $\phi_\omega$ has continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p^\times, \cdot)$ (see [24]).

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The indefinite sum operator $S$ is defined by $Sf(0) = 0$ and $Sf(n) = \sum_{i=0}^{n-1} f(i)$ for $n \geq 1$. It is well known that for $f \in U(D(Z_p, C_p)$, its Volkenborn integral is defined to be the limit of average

$$\frac{1}{p^N} \sum_{x=0}^{p^{N-1}} f(x) = \frac{Sf(p^N) - Sf(0)}{p^N}$$

as $N \to \infty$ (see [2], [19], [22]). We see that the uniform differentiability guarantees the existence of limits. Write down this integral as

$$(1.2) \quad I_0(f) = \int_{Z_p} f(x) \, dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^{N-1}} f(x) = (Sf)'(0) \in C_p$$

(see [7] and [19] for more details). For $f \in U(D(Z_p, C_p)$, the $I_0$-Fourier transform of $f$ is the function $\hat{f} : T_p \to C_p$ defined by

$$\hat{f}_\omega = I_0(f\delta_\omega) \quad \text{for all } \omega \in T_p.$$  

This can be found in [24, Definition 3.1]. In fact we can extend $\hat{f}$ to $\{\alpha \in C_p \mid v_p(\alpha - 1) > 0\}$ by some formula, where it turns out to be an analytic function whose Taylor expansion has logarithmic growth (see [24, Proposition 5.3]). The analogue with the classical (complex) theory is substantially complicated by the absence of a $p$-adic valued Haar measure on $Z_p$. So, C. F. Woodcock had to do various attempts to construct a satisfactory analogue of integral analysis for the spaces of functions on $Z_p$. One, in [23] and [24], can find the fully detailed study of the $I_0$-Fourier transform on the space of all uniformly differentiable functions $f : Z_p \to C_p$.

The $q$-extension of the $I_0$-Fourier transform has been constructed by T. Kim [11], called the $I_q$-Fourier transform. The $I_q$-Fourier transform has the form

$$(1.4) \quad (\hat{f}_\omega)_q = I_q(f\delta_\omega) \quad \text{for all } \omega \in T_p,$$

see Section 2, Eq. (2.1).

For any integer $n \geq 1$, we shall use the following standard notation

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0.$$  

As $q \to 1, [n]_q \to n$, and this is the hallmark of a $q$-extension: the limit as $q \to 1$ recovers the classical object. If $q \in C$, one normally assume that $|q| < 1$. If $q \in C_p$, then we assume that $|q - 1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $x \in Z_p$. For $N \geq 1$, the $q$-extension $\mu_q$ (originally introduced by T. Kim [6]) of the $p$-adic Haar distribution $\mu_{\text{Haar}}$

$$(1.5) \quad \mu_q(a + p^N Z_p) = \frac{q^a}{[p^N]_q}$$

is known as a distribution on $Z_p$, where $\mu_{\text{Haar}}(a + p^N Z_p) = \frac{1}{p^N}$ and $a + p^N Z_p = \{x \in Q_p \mid |x-a|_p \leq p^{-N}\}$. Note that $\lim_{q \to 1} \mu_q = \mu_{\text{Haar}}$. We shall write $d\mu_q(x)$
to remind ourselves that $x$ is the variable of integration. This distribution $\mu_q$ yields an $I_q$-integral for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.6}$$

The $I_q$-integral for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ was defined by T. Kim ([6], [7], [8], [10], [11]) and basic properties were studied by many authors. Also, by (1.2) and (1.6), it is well-known that the numbers $B_n$, $\beta_n(q)$ and $B_n(q)$ are connected with $I_0$-integrals and $I_q$-integrals as follows.

- For any $n \geq 0$, $I_0(x^n) = \int_{\mathbb{Z}_p} x^n dx = B_n$, where $B_n$ is the ordinary Bernoulli numbers (see [19], [22]);
- For any $n \geq 0$, $I_q([x]_q^n) = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \beta_n(q)$, where $\beta_n(q)$ is the Carlitz’s $q$-Bernoulli numbers (see [6], [7], [8], [9], [11], [12]);
- For any $n \geq 0$, $I_q(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_n(q)$, where $B_n(q)$ is the modified Bernoulli numbers (see [3], [4]).

In this paper, we consider the $q$-Volkenborn integral of uniformly differentiable functions on $\mathbb{Z}_p$. By using this integral, we obtain the generating functions of twisted $q$-generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

2. The $q$-extension of the $I_0$-integral transform and related numbers and polynomials

Given $\omega \in \mathbb{T}_p$, we will denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p, x \mapsto \omega^x$, the locally constant extension of the power function from $\mathbb{Z}$ to $\mathbb{Z}_p$. In [11], the $I_q$-Fourier transform for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ is the function $I_q(f) : \mathbb{T}_p \to \mathbb{C}_p$ defined by

$$I_q(f(\phi_\omega)) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \phi_\omega(x) f(x) q^x. \tag{2.1}$$

This is an $q$-extension of the $I_0$-Fourier transform (1.3), that is, the $I_0$-Fourier transform of $f$ is the case $q \to 1$ of $I_q(f(\phi_\omega))$ in (2.1). In terms of integration, one can have the integral form

$$I_q(f(\phi_\omega)) = \int_{\mathbb{Z}_p} \phi_\omega(x) f(x) d\mu_q(x), \quad \omega \in \mathbb{T}_p. \tag{2.2}$$

Also its inverse $I_q$-Fourier transform seem to be equivalent to the limit

$$f(x) q^x = \frac{\log q}{q-1} \lim_{N \to \infty} \sum_{\omega \in \mathbb{C}_{pN}} \phi_{\omega^{-1}}(x) I_q(f(\phi_\omega)) \tag{2.3}$$

for all $x \in \mathbb{Z}_p$ (see [11]).

Note that the distribution $\mu_q$ on $\mathbb{Z}_p$ has the property

$$\mu_q(\alpha p + p^{N+1} \mathbb{Z}_p) = [p]_q^{-1} \mu_q(a + p^N \mathbb{Z}_p)$$
followed trivially from the definition (1.5). Since \( \mathbb{Z}_p^x = \mathbb{Z}_p \setminus p\mathbb{Z}_p \), this implies that for any \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \) on \( \mathbb{Z}_p^x \)

\[
(2.4) \quad \int_{\mathbb{Z}_p^x} \phi_\omega(x)f(x)\mu_q(x) = I_q(f \phi_\omega) - [p]_q^{-1}I_q(f(px)\phi_\omega).
\]

In the following proposition we obtain the shift versus integration of the \( I_q \)-Fourier transform for uniformly differentiable functions on \( \mathbb{Z}_p \).

**Proposition 2.1.** Let \( f \) be a uniformly differentiable function on \( \mathbb{Z}_p \). For some fixed \( s \in \mathbb{Z}_p \) and \( \omega \in T_p \),

\[
I_q(f(x + s + 1)\phi_\omega) - \frac{1}{\omega q}I_q(f(x + s)\phi_\omega) = \frac{q - 1}{\omega q \log q} (f'(s) + f(s) \log q).
\]

**Proof.** From the definition (2.1), it is easy to see that

\[
\frac{\omega q \log q}{q - 1} \int_{\mathbb{Z}_p} \phi_\omega(x)f(x + s + 1)\mu_q(x)
\]

\[
= \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_\omega(x)f(x + s)q^x + \lim_{N \to \infty} \frac{\phi_\omega(p^N)f(p^N + s)q^{p^N} - f(s)}{p^N}.
\]

It is easy to check that a uniformly differentiable function \( \phi_\omega(x)f(x + s)q^x \) on \( x \in \mathbb{Z}_p \) can be differentiated in the usual way:

\[
\lim_{p^N \to 0} \frac{\phi_\omega(p^N)f(p^N + s)q^{p^N} - f(s)}{p^N} = f'(s) + f(s) \log q \quad \text{for} \quad s \in \mathbb{Z}_p.
\]

The result now follows easily. \( \square \)

**Corollary 2.2.** 1. Suppose that \( \omega \in T_p \). Let \( I_q(e^{tx}\phi_\omega) \) be a power series about the origin as follows

\[
I_q(e^{tx}\phi_\omega) = \sum_{k=0}^{\infty} \frac{B_k(q, \omega)}{k!} t^k \in \mathbb{C}_p[[t]],
\]

where \( |t|_p < p^{-1/(p-1)} \) and \( t \neq 0 \in \mathbb{C}_p \). Then the coefficients of expansion \( \{B_k(q, \omega)\} \) can be written by

\[
B_k(q, \omega) = I_q(x^k \phi_\omega) = \left\{
\begin{array}{ll}
\frac{q-1}{\omega q-1} \left( H_k(\frac{1}{\omega q}) + \frac{k}{\log q} H_{k-1}(\frac{1}{\omega q}) \right), & \text{if } k \geq 1 \\
\frac{q-1}{\omega q-1}, & \text{if } k = 0.
\end{array}
\right.
\]

Here the generalized \( k \)-th Euler number \( H_k(u) \) attached to an algebraic \( u \neq 1 \) has been defined by Frobenius (1910); \( (1 - u)/(e^t - u) = \sum_{k=0}^{\infty} H_k(u)t^k/k! \).

2. For \( \omega \in T_p \) and \( k \geq 0 \),

\[
I_q(q^{kx}\phi_\omega) = \sum_{i=0}^{\infty} (q^k - 1)^i I_q(\left(\begin{array}{c} x \\ i \end{array}\right) \phi_\omega) = \frac{k + 1}{[k + 1]_q \omega},
\]
where \([k]_{q,\omega} = (\omega q^k - 1)/(q - 1)\) and \(\binom{x}{i}_q = \frac{x(x-1)\cdots(x-i+1)}{i!}.\) Moreover, the sequence \(\{q^k\}\) can be extended to a locally analytic function \(q^x\) for \(x \in \mathbb{Z}_p.\)

We define now the generating function of a new Bernoulli numbers by \(I_q\)-Fourier transform. The twisted \(q\)-extension of Bernoulli numbers is defined by

\[
I_q(e^{tx} \phi_\omega) = \frac{q - 1}{\log q} \frac{t + \log q}{qte^t - 1} = \sum_{k=0}^{\infty} B_k(q, \omega) \frac{t^k}{k!}, \quad \omega \in \mathbb{T}_p
\]

for \(|t|_p < q^{-1/(p-1)}\). If \(q \to 1\), then from application of L'Hospital's rule the expression (2.5) is reduced to

\[
\lim_{q \to 1} I_q(e^{tx} \phi_\omega) = \frac{t}{\omega e^t - 1}, \quad \omega \in \mathbb{T}_p.
\]

The exciting properties of this formula were shown by T. Kim (see [5]) and C. F. Woodcock ([23, Proposition 7.1 (i)] and [24]). If \(\omega = 1\), then

\[
I_q(e^{tx}) = \frac{q - 1}{\log q} \frac{t + \log q}{qte^t - 1} = \sum_{k=0}^{\infty} B_k(q, 1) \frac{t^k}{k!},
\]

where \(|t|_p < q^{-1/(p-1)}\) (cf. [1], [3], [4], [9], [10], [12], [16], [20]).

**Corollary 2.3.** Let \(\omega \in \mathbb{T}_p\) with \(\omega^N = 1, \omega \neq 1\) for \(N > 1\). Set

\[
\lim_{q \to 1} I_q(e^{tx} \phi_\omega) = \sum_{k=0}^{\infty} B_k(1, \omega) \frac{t^k}{k!}.
\]

Then

\[
B_k(1, \omega) = N^{k-1} \sum_{i=0}^{N-1} \omega^i B_k(i/N),
\]

where \(B_k(\cdot)\) is the usual Bernoulli polynomials and \(k \geq 1\).

**Corollary 2.4.** Let \(k \geq 0\). Then

1. \(\frac{q - 1}{\log q} q^x x^k = \sum_{\omega \in \mathbb{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) B_k(q, \omega) + B_k(q, 1).\)
2. \(\frac{q - 1}{\log q} q^{(k+1)x} = \sum_{\omega \in \mathbb{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) \frac{B_{k+1}(q, \omega)}{[k+1]_{q, \omega}} + \frac{k+1}{[k+1]_{q, 1}}.\)

**Proof.** From (2.3) and Corollary 2.2, the series

\[
\sum_{\omega \in \mathbb{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) I_q(x^k \phi_\omega)
\]

converges uniformly to \(\frac{q - 1}{\log q} q^x x^k\) as \(N \to \infty\). So Part 1 follows directly. Part 2 follows by a similar method of Part 1. □

In Part 1 and Part 2 of Corollary 2.4, putting \(k = 0\), we obtain the integral series expansion

\[
\frac{q^x}{\log q} = \sum_{\omega \in \mathbb{T}_p} \phi_{\omega^{-1}}(x) \frac{1}{\omega q - 1},
\]
whence, for \( x = 0 \) in the above, we have \( \frac{1}{\log q} = \sum_{\omega \in \mathbb{T}_p} \frac{1}{\omega q - 1} \) for \( q \neq 1 \). This formula gives an explicit expression for \( \frac{1}{\log q} \) in terms of \( \frac{1}{\omega q - 1} \) (see [23, p. 692]).

Now we consider the recursion formula for the sequence of numbers \( \{B_k(q, \omega)\} \). From Proposition 2.1 we obtain the difference formula

\[
I_q(f_1 \phi_\omega) - \frac{1}{\omega q} I_q(f \phi_\omega) = \frac{q - 1}{\omega q \log q} (f'(0) + f(0) \log q),
\]

where \( f_1(x) = f(x + 1) \). From this expression, when \( f(x) = x^k \) for \( k \geq 0 \), we easily deduce that

\[
(2.6) \quad I_q((x + 1)^k \phi_\omega) - \frac{1}{\omega q} I_q(x^k \phi_\omega) = \begin{cases} 
\frac{q - 1}{\omega q}, & k = 0, \\
\frac{q - 1}{\omega q \log q}, & k = 1, \\
0, & k \geq 2.
\end{cases}
\]

We expand the left-hand side of Equation (2.6) by the binomial theorem. It may be stated as

\[
(2.7) \quad I_q((x + 1)^k \phi_\omega) = \sum_{i=0}^{k} \binom{k}{i} \int_{\mathbb{Z}_p} \phi_\omega(x) x^i d\mu_q(x).
\]

From (2.5), (2.7) and Part 1 of Corollary 2.2, we derive

\[
(2.8) \quad I_q((x + 1)^k \phi_\omega) - \frac{1}{\omega q} I_q(x^k \phi_\omega)
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} B_i(q, \omega) - \frac{1}{\omega q} B_k(q, \omega)
\]

\[
= \begin{cases} 
\frac{\omega q - 1}{\omega q} B_0(q, \omega), & k = 0, \\
\frac{q - 1}{\omega q} B_1(q, \omega) + \sum_{i=0}^{k-1} \binom{k}{i} B_i(q, \omega), & k \geq 1.
\end{cases}
\]

As a consequence of the above formulae (2.6) and (2.8) we deduce the recurrence relation for the sequence of numbers \( \{B_k(q, \omega)\} \) as follows

**Proposition 2.5.** The numbers \( \{B_k(q, \omega)\} \) satisfies

\[
B_0(q, \omega) = \frac{1}{[1]_{q, \omega}}, \quad B_1(q, \omega) = \frac{1}{[1]_{q, \omega} \log q} - \frac{\omega q}{[1]_{q, \omega} (\omega q - 1)}
\]

and

\[
B_k(q, \omega) = \frac{\omega q}{1 - \omega q} \sum_{i=0}^{k-1} \binom{k}{i} B_i(q, \omega) \quad \text{for} \ k \geq 2.
\]
3. Twisted $p$-adic $q$-L-functions

Let $d$ be a fixed integer and $p$ be a fixed prime number. We set

$$X = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{0 < a < dp \atop (a,p)=1} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N$. Let $D = \{ q \in \mathbb{C}_p \mid |q - 1|_p < 1 \}$, and let $\overline{D} = \mathbb{C}_p \setminus D$ be the complement of the open unit disc around 1. Note that if $q \in \overline{D}$ and $\text{ord}_p(1-q) \neq -\infty$, then $\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$ is the measure (cf. [7]). Hereafter, we assume that $q \in \overline{D}$ and $\text{ord}_p(1-q) \neq -\infty$.

Let $\chi$ be a primitive Dirichlet character with conductor $d$. Defining the generalized numbers of $B_k(q, \omega)$ by the formula

$$B_{k,\chi}(q, \omega) = \frac{k!}{\text{coefficient of } t^k \text{ in } \frac{q-1}{\log q} \sum_{a=1}^{d} \chi(a)\phi_{\omega}(a)q^a(t + \log q)e^{ta} - \phi_{\omega}(d)q^d e^{dt} - 1}.$$

Thus we deduce the integral of the generalized numbers

$$B_{k,\chi}(q, \omega) = \int_X \phi_{\omega}(x)\chi(x)x^k d\mu_q(x) \quad \text{for } k \geq 0.$$

To see that (3.3) follows from

$$\int_X \phi_{\omega}(x)\chi(x)e^{tx} d\mu_q(x) = \sum_{k=0}^{\infty} \int_X \phi_{\omega}(x)\chi(x)x^k d\mu_q(x)^{\frac{t^k}{k!}},$$

we note

$$\int_X \phi_{\omega}(x)\chi(x)e^{tx} d\mu_q(x)$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N} \chi(x)\phi_{\omega}(x)e^{tx} q^x$$

$$= [d]_q^{-1} \sum_{a=1}^{d} \chi(a)\phi_{\omega}(a)q^a e^{ta} \frac{q^d - 1}{\log q^d} \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_{\omega}(x)e^{tx} q^d q^x$$

$$= [d]_q^{-1} \sum_{a=1}^{d} \chi(a)\phi_{\omega}(a)q^a e^{ta} \int_{\mathbb{Z}_p} \phi_{\omega}(x)e^{tx} d\mu_q(x)$$

$$= \frac{q-1}{\log q} \sum_{a=1}^{d} \chi(a)\phi_{\omega}(a)q^a(t + \log q)e^{ta} - \phi_{\omega}(d)q^d e^{dt} - 1 \quad \text{(by (2.5))}.$$
Proposition 3.1. Let $\chi$ be a primitive Dirichlet character with conductor $d$ and $x \in \mathbb{Z}_p$. Then
\[
\frac{q - 1}{\log q} x^k \chi(x) = \sum_{\omega \in \mathbf{T}_p, \omega \neq 1} \phi_{\omega^1}(x) B_{k,\chi}(q, \omega) + B_{k,\chi}(q, 1)
\]
for $k \geq 0$.

Let $p$ be odd rational prime and let $\omega_p : X^* \to X$ be the function defined by (see [2], [15], [19], [22])
\[
\omega_p(x) = \lim_{n \to \infty} x^{p^n}.
\]
The function $\omega_p$ is called the Teichmüller character, and it appears quite frequently in many different guises. For $s \in \mathbb{Z}_p$ and $\omega \in \mathbf{T}_p$, we define
\[
L_{p,q}(s, \chi, \omega) = \lim_{N \to \infty} \frac{1}{[dp]^N q^{0 \leq \xi \leq dp^{N-1}} \sum_{(p, x) = 1} q^x \phi_{\omega}(x) \chi(x) \left( \frac{x}{\omega_p(x)} \right)^{1-s}}.
\]
For $k \geq 0$, we set $\chi_k = \chi \omega_p^{-k}$. Since $\mu_q(pU) = [p]^{-1} \mu_q(U)$ for $U \subset X$, the value of the function $L_{p,q}(s, \chi, \omega)$ at non-positive integers are given by
\[
L_{p,q}(1 - k, \chi, \omega) = B_{k,\chi_k}(q, \omega) - p^k [p]^{-1} \chi_k(p) B_{k,\chi_k}(q^p, \omega^p)
\]
for $k \geq 1$. We thus obtain the following

Theorem 3.2. Let $\chi$ be a primitive Dirichlet character with conductor $d$ and $s \in \mathbb{Z}_p, \omega \in \mathbf{T}_p$. Then the function $L_{p,q}(s, \chi, \omega)$ interpolates the values $B_{k,\chi_k}(q, \omega) - p^k [p]^{-1} \chi_k(p) B_{k,\chi_k}(q^p, \omega^p)$ when $s = 1 - k$ for $k \geq 1$.

For $q \in \overline{D}$, we have
\[
\left| \frac{\mu_q(a + dp^N \mathbb{Z}_p)}{1-q} \right|_p = \left| \frac{q^a}{(1-q)[dp^N]} \right|_p = \left| \frac{q^a}{1 - q^{dp^N}} \right|_p \leq 1.
\]
By [13, p. 31, Eq. (3.4)], if $k \equiv k' \pmod{(p-1)p^N}$, then we obtain the assertion that
\[
\left| x^k - x^{k'} \right|_p \leq \frac{1}{p^{N+1}} \quad \text{for } x \in X^*.
\]
Using the corollary at the end of [14, Chapter II, §5] and (3.5), their integrals over the compact set $X^*$ are also close together, and in fact, it is easy to see
that for $k \geq 1$

$$
(1 - q)^{-1} L_{p, q}(1 - k, \chi_{-k}, \omega) \\
= \lim_{N \to \infty} \frac{1}{1 - q^{dp^N}} \sum_{0 \leq x \leq dp^{N-1}} q^x \phi_\omega(x) \chi_{\omega_p}(x) \left( \frac{x}{\omega_p(x)} \right)^k \\
= \int_{X'} \chi(x) \phi_\omega(x) x^k \frac{d\mu_q(x)}{1 - q} \\
\equiv \int_{X'} \chi(x) \phi_\omega(x) x^{k'} \frac{d\mu_q(x)}{1 - q} \pmod{p^{N+1}} \\
= (1 - q)^{-1} L_{p, q}(1 - k', \chi_{-k'}, \omega).
$$

Hence we can prove the following congruence.

**Theorem 3.3.** Let $\chi$ be a primitive Dirichlet character with conductor $d$, and let $k \equiv k' \pmod{(p - 1)p^N}$ and $\omega \in T_p$. Then

$$(1 - q)^{-1} L_{p, q}(1 - k, \chi_{-k}, \omega) \equiv (1 - q)^{-1} L_{p, q}(1 - k', \chi_{-k'}, \omega) \pmod{p^{N+1}}.$$

Finally, we shall also want to consider modified twisted $L$-functions in the complex field $\mathbb{C}$. Let $q \in \mathbb{C}$ with $0 < |q| < 1$, and let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. We set

$$(3.7) \quad L_q(s, \chi, \omega) = \frac{q - 1}{1 - s} \sum_{n=1}^\infty \frac{\omega^n q^n \chi(n)}{n^{s-1}} + \frac{q - 1}{\log q} \sum_{n=1}^\infty \frac{\omega^n q^n \chi(n)}{n^s},$$

the series being absolutely convergent (cf. [3], [4], [6], [9], [12], [17], [18], [21]). In particular, if we replace $s$ by $1 - k$, one then sees easily that

$$
L_q(1 - k, \chi, \omega) \\
= \frac{q - 1}{k} \sum_{n=1}^\infty \omega^n q^n \chi(n)n^k + \frac{q - 1}{\log q} \sum_{n=1}^\infty \omega^n q^n \chi(n)n^{k-1} \\
= \frac{q - 1}{k} \left( \frac{d}{dt} \right)^k \left( \sum_{n=1}^\infty \omega^n q^n \chi(n)e^{nt} + \frac{t}{\log q} \sum_{n=1}^\infty \omega^n q^n \chi(n)e^{nt} \right) \bigg|_{t=0}.
$$

We consider the function

$$
\Psi_q(t) = (1 - q) \sum_{n=1}^\infty \omega^n q^n \chi(n)e^{nt} + \frac{(1 - q)t}{\log q} \sum_{n=1}^\infty \omega^n q^n \chi(n)e^{nt}.
$$
Since $\chi$ is a character mod $d$ we rearrange the terms in the series for $\Psi_q(t)$ according to the residue classes mod $d$. Then we have
\[
\Psi_q(t) = \left(1 - q + \frac{(1 - q)t}{\log q}\right) \sum_{a=1}^{d} \sum_{b=0}^{\infty} \omega^{a+bd} q^{a+bd} \chi(a + bd) e^{(a+bd)t} = q - 1 - \frac{1}{\log q} \sum_{a=1}^{d} \frac{\chi(a) \omega^a q^a (t + \log q) e^{ta}}{\omega^d q^d e^{dt} - 1},
\]
which is equal to the formula in (3.2). We will apply the recipe above. Then we see that for $k \geq 1$
\[
(3.8) \quad L_q(1 - k, \chi, \omega) = -\frac{1}{k} \left. \left(\frac{d}{dt}\right)^k \Psi_q(t) \right|_{t=0}.
\]
Comparing (3.2) and (3.8), we arrive at the following

**Proposition 3.4.** Let $\chi$ be a primitive Dirichlet character with conductor $d$, and let $q \in \mathbb{C}$ with $0 < |q| < 1$. Then
\[
L_q(1 - k, \chi, \omega) = -\frac{1}{k} B_{k, \chi}(q, \omega) \quad \text{for } k \geq 1.
\]

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Using (3.6) and Proposition 3.4 we have
\[
-\frac{1}{k} L_{p,q}(1 - k, \chi, \omega) = L_q(1 - k, \chi_k, \omega) - p^k[p]_q^{-1} \chi_k(p) L_{q^p}(1 - k, \chi_k, \omega^p)
\]
for $k \geq 1$. Here the right-hand side is the value of the complex $L$-function which the left-hand side is the values of the $p$-adic $L$-function and the value are equal in the field $\overline{\mathbb{Q}}$ common to $\mathbb{C}_p$ and $\mathbb{C}$.

**Theorem 3.5.** Let $\omega \in \mathbf{T}_p$ and $q \in \overline{\mathbb{D}}$ with $\text{ord}_p(1 - q) \neq -\infty$. Let $\omega_p$ be the Teichmüller character. For $\chi$ a primitive Dirichlet character with conductor $d$, the function from $\mathbb{Z}_p\setminus\{1\}$ to $\mathbb{C}_p$
\[
\frac{L_{p,q}(s, \chi, \omega)}{s - 1} = \frac{1}{s - 1} \lim_{N \to \infty} \frac{1}{[d^p N]_q} \sum_{0 \leq \sigma \leq d^p N - 1} q^{\sigma} \phi_\omega(x)(x) \left(\frac{x}{\omega(x)}\right)^{1-s}
\]
interpolates the values $L_q(1 - k, \chi_k, \omega) - p^k[p]_q^{-1} \chi_k(p) L_{q^p}(1 - k, \chi_k, \omega^p)$ when $s = 1 - k$.

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