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IMPLICITIZATION OF RATIONAL CURVES AND POLYNOMIAL SURFACES

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ABSTRACT. In this paper, we first present a method for finding the implicit equation of the curve given by rational parametric equations. The method is based on the computation of Gröbner bases. Then, another method for implicitization of curve and surface is given. In the case of rational curves, the method proceeds via giving the implicit polynomial f with indeterminate coefficients, substituting the rational expressions for the given curve and surface into the implicit polynomial to yield a rational expression $\frac{g}{h}$ in the parameters. Equating coefficients of g in terms of parameters to 0 to get a system of linear equations in the indeterminate coefficients of polynomial f, and finally solving the linear system, we get all the coefficients of f, and thus we obtain the corresponding implicit equation. In the case of polynomial surfaces, we can similarly as in the case of rational curves obtain its implicit equation. This method is based on characteristic set theory. Some examples will show that our methods are efficient.

1. Introduction

For curves and surfaces, to transform their parametric equations to the implicit forms is of fundamental importance in geometric modeling, computer graphics and many methods have been given to do this, see e.g. Sederberg [17], Sederberg et al. [18], Chung and Hoffman [5], Li [13], Busé [3], Busé et al. [4], Cox [7, 8], Corless et al. [6], Marco and Martínez [14, 15], Gao and Chou [12], Wang [23], Sun [21], Yu [25] and references therein. But the different methods of implicitization belong to three main classes.

The first class of methods is based on classical elimination theory. Resultants (in one or several variables) are used to compute the implicit equation, the computation is not a trivial task (Cox et al. [11]) and leads to expressions spoiled by extraneous factors (Manocha and Canny [16]).

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The second class of methods relies on Gröbner bases (Becker and Weispfenning [2], Gao and Chou [12]). The algorithm gives a basis of the ideal spanned by the system of parametric equations. It is proven that elements of this basis independent of parameters are the corresponding implicit equations with no extraneous factor (Cox et al. [10]). Although this method is generally recognized that techniques using Gröbner bases are very time and space consuming in implicitization, we must accept that this method is very important from theoretical point of view.

The third class of methods for implicitization is based on Mathematics Mechanization theory, founded by professor Wu (Wu [24]). This method has been generally proposed and studied in implicitization (Gao and Chou [12], Li [13]) and other problems (Shi and Sun [19]). Mathematics Mechanization theory, especially the characteristic set theory, is much more important and useful in some areas.

In this paper, we assume that the parametrization is proper, i.e., there is a one to one relation between the points and the parameter values. We first present an improved method for implicitization of rational curves, this method is based on Gröbner bases. Then we give another method for implicitization of rational curves and polynomial surfaces which is defined by polynomials in $\mathbb{R}[x, y, z]$. The basic idea underlying this method is the characteristic set theory and the usage of principle of indeterminate coefficients of the desired implicit polynomial to reduce the implicitization problem to solving the system of linear equations with constant coefficients. Some examples are given to illustrate the efficiency of our methods.

2. Description of the problem

A parametrization of a geometric object in a space of dimension n can be described by the following set of parametric equations:

$$(*) \qquad x_1 = \frac{P_1(t_1, \ldots, t_k)}{Q_1(t_1, \ldots, t_k)}, \ldots, x_n = \frac{P_n(t_1, \ldots, t_k)}{Q_n(t_1, \ldots, t_k)}$$

where $t_1, \ldots, t_k$ are parameters and functions $P_i, Q_i, i = 1, \ldots, n$ are all polynomials. The case $n = 2, k = 1$ corresponds to plane curves; the case $n = 3, k = 1$ corresponds to space curves and the case $n = 3, k = 2$ corresponds to surfaces.

Our aim is to compute the implicit polynomial equation

$$(**) \qquad F(x_1, \ldots, x_n) = 0$$

of the geometric object described by the parametric equations $(*)$, which satisfies

$$F\left(\frac{P_1(t_1, \ldots, t_k)}{Q_1(t_1, \ldots, t_k)}, \ldots, \frac{P_n(t_1, \ldots, t_k)}{Q_n(t_1, \ldots, t_k)}\right) = 0$$

for all values of parameters $t_1, \ldots, t_k$.

3. Improved method for implicitization of curves

3.1. Gröbner bases

Gröbner bases is very useful to devise alternative generators for an ideal, and several excellent books have been written on Gröbner bases (W. Adams et al. [1], D. Cox et al. [9]).

Let k be any field, e.g., the rational numbers field, $\mathbb{Q}$, the real numbers field, $\mathbb{R}$, or the complex numbers field, $\mathbb{C}$. We will denote the set of power products by

$$\mathbb{T}^n = \left\{x_1^{\beta_1} \cdots x_n^{\beta_n} | \beta_i \in \mathbb{N}, i = 1, \dots, n\right\}$$

and denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by x^α, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We choose a term order "$>$" on $k[x_1, \dots, x_n]$, then for all $f \in k[x_1, \dots, x_n]$, with $f \neq 0$, we may write

$$f = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_r x^{\alpha_r}$$

where $0 \neq a_i \in k, x^{\alpha_i} \in \mathbb{T}^n$, and $x^{\alpha_1} > x^{\alpha_2} > \cdots > x^{\alpha_r}$. We will always try to write our polynomial in this way. We define:

- $lp(f) = x^{\alpha_1}$, the leading power product of f;
- $lc(f) = a_1$, the leading coefficient of f;
- $lt(f) = a_1 x^{\alpha_1}$, the leading term of f.

We also define $lp(0) = lc(0) = lt(0) = 0$.

Definition 3.1. A set of non-zero polynomials $G = \{g_1, \dots, g_t\}$ contained in an ideal $I \subset k[x_1, \dots, x_n]$, is called a Gröbner basis for I if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in \{1, \dots, t\}$ such that $lp(g_i)$ divides $lp(f)$.

Definition 3.2. Let $PS = \{p_1, \dots, p_m\}$ be a set of polynomials, where $p_i \in \mathbb{R}[x_1, \dots, x_n]$. Then $V(PS)$ denote

$$V(PS) = \{(x_1, \dots, x_n) \in \mathbb{R}^n |\ p_i(x_1, \dots, x_n) = 0,\ i = 1, \dots, m\}.$$

3.2. Method

In this section, we will mainly discuss the implicitization for rational parametric plane curve and space curve. We first deal with the rational parametric plane curve.

In the general situation of rational parametric plane curve, with the term order $t > x > y$, we have

$$x = \frac{P_1(t)}{Q_1(t)},\ y = \frac{P_2(t)}{Q_2(t)} \tag{1}$$

where $P_i(t), Q_i(t)$ are polynomials in $\mathbb{R}[t],\ i = 1, 2$.

Theorem 3.3. *For the above parametrization* (1) *of plane curve, if P_i and Q_i are coprime $i = 1, 2$, then we have that*

(i) $V\left(x - \dfrac{P_1(t)}{Q_1(t)}, y - \dfrac{P_2(t)}{Q_2(t)}\right) = V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right);$

(ii) *Let* $I = \langle Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\rangle$ *be an ideal. Then we have that* $V(I_1)$ *is the smallest variety containing*

$$\pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right)\right),$$

where $I_1 = I \bigcap \mathbb{R}[x, y]$, π_1 *is the projection map*

$$\begin{array}{cccc} \pi_1: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (a_1, a_2, a_3) & \longmapsto & (a_2, a_3) \end{array}$$

Proof. (i) Clearly we have that

$$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right) \subseteq V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right).$$

For any $(t_0, x_0, y_0) \in V(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t))$, we have

$$Q_1(t_0)x_0 - P_1(t_0) = 0,\ Q_2(t_0)y_0 - P_2(t_0) = 0. \tag{2}$$

So $Q_1(t_0) \neq 0$. Otherwise, $Q_1(t_0) = 0$, from (2) we have $P_1(t_0) = 0$, thus $Q_1(t)$ and $P_1(t)$ have the common divisor $(t - t_0)$, which is in contradiction with that $P_1(t)$ and $Q_1(t)$ are coprime. Therefore $Q_1(t_0) \neq 0$. Similarly, $Q_2(t_0) \neq 0$. So that

$$(t_0, x_0, y_0) \in V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right).$$

For any (t_0, x_0, y_0), we then have

$$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right) \supseteq V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right).$$

Thus

$$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right) = V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right).$$

(ii) By the Closure Theorem (see Cox et al. [10]), we know that $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right)\right).$$

From (i) we have that

$$\pi_1\left(V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t)\right)\right) = \pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right)\right).$$

Thus $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right)\right).$$

If

$$x = \frac{P_1(t)}{Q_1(t)},\ y = \frac{P_2(t)}{Q_2(t)} \tag{3}$$

but $P_i(t)$ and $Q_i(t)$ are not coprime $i = 1, 2$. We can first find the greatest common divisor $d_i(t)$ of $P_i(t)$ and $Q_i(t)$, so that $x = \dfrac{P_1(t)}{Q_1(t)}$, $y = \dfrac{P_2(t)}{Q_2(t)}$ can be simplified to $x = \dfrac{p_1(t)}{q_1(t)}$, $y = \dfrac{p_2(t)}{q2(t)}$, where $p_i(t)$ and $q_i(t)$ are coprime and satisfy

$$p_i(t)d_i(t) = P_i(t),\ q_i(t)d_i(t) = Q_i(t), i = 1, 2.$$

Let

$$I = \langle q_1(t)x - p_1(t), q_2(t)y - p_2(t)\rangle$$

be the ideal generated by polynomials $q_1(t)x - p_1(t)$, $q_2(t)y - p_2(t)$, and

$$I_1 = I \bigcap \mathbb{R}[x, y],$$

$$ES = \{(x(t_1), y(t_1))|\ t_1 \in V\left(d_1(t)d_2(t)\right)\},$$

which contains finite points in $\mathbb{R}[x, y]$. □

Definition 3.4. Let V_0 be a variety. If V_0 is the smallest variety satisfying

$$\pi_1\left(V\left(x - \frac{p_1(t)}{q_1(t)},\ y - \frac{p_2(t)}{q2(t)}\right)\right) \subseteq V_0,$$

and

$$\pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)},\ y - \frac{P_2(t)}{Q_2(t)}\right)\right) \subseteq V_0 - ES.$$

Then we call V_0 the implicitization variety with respect to parametrization (3).

Corollary 3.5. *For the above parametrization* (3) *of plane curve. If $P_i(t)$ and $Q_i(t)$ are not coprime $i = 1, 2$, then we have that $V(I_1)$ is the implicitization variety with respect to* (3).

Proof. Since $d_i(t)$ is the greatest common divisor of $P_i(t)$ and $Q_i(t)$, then

$$p_i(t)d_i(t) = P_i(t),\ q_i(t)d_i(t) = Q_i(t),\ i = 1, 2.$$

We have

$$\begin{aligned} x &= \frac{P_1(t)}{Q_1(t)} = \frac{p_1(t)}{q_1(t)}, \\ y &= \frac{P_2(t)}{Q_2(t)} = \frac{p_2(t)}{q_2(t)}. \end{aligned}$$

So that

$$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right) = V\left(x - \frac{p_1(t)}{q_1(t)}, y - \frac{p_2(t)}{q_2(t)}\right) - ES.$$

By Theorem 3.3, we get that $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(x - \frac{p_1(t)}{q_1(t)}, y - \frac{p_2(t)}{q_2(t)}\right)\right),$$

thus $V(I_1) - ES$ is the corresponding set containing

$$\pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}\right)\right).$$

In the above, we mainly deal with the implicitization for rational parametric plane curves, the following is the case of rational parametric space curves. With term order $t > x > y > z$, let

$$x = \frac{P_1(t)}{Q_1(t)},\ y = \frac{P_2(t)}{Q_2(t)},\ z = \frac{P_3(t)}{Q_3(t)} \tag{4}$$

where $P_i(t), Q_i(t)$ are polynomials in $\mathbb{R}[t]$, $i = 1, 2, 3$. □

Theorem 3.6. *For the above parametrization* (4) *of space curve, if P_i and Q_i are coprime $i = 1, 2, 3$, then we have that*

(i) $$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}, z - \frac{P_3(t)}{Q_3(t)}\right) = V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t), Q_3(t)y - P_3(t)\right);$$

(ii) *Let $I = \langle Q_1(t)x - P_1(t), Q_2(t)y - P_2(t), Q_3(t)y - P_3(t)\rangle$. Then we have that $V(I_1)$ is the smallest variety containing*

$$\pi_1\left(V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}, z - \frac{P_3(t)}{Q_3(t)}\right)\right),$$

where $I_1 = I \bigcap \mathbb{R}[x, y, z]$, π_1 is the projection map

$$\begin{array}{rrcl} \pi_1 : & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^3 \\ & (a_1, a_2, a_3, a_4) & \longmapsto & (a_2, a_3, a_4) \end{array}$$

Proof. (i) Clearly we have that

$$V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}, z - \frac{P_3(t)}{Q_3(t)}\right) \subseteq V\left(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t), Q_3(t)z - P_3(t)\right).$$

For any

$$(t_0, x_0, y_0, z_0) \in V(Q_1(t)x - P_1(t), Q_2(t)y - P_2(t), Q_3z(t) - P_3(t)),$$

we have

$$Q_1(t_0)x_0 - P_1(t_0) = 0,\ Q_2(t_0)y_0 - P_2(t_0) = 0,\ Q_3(t_0)z_0 - P_3(t). \tag{5}$$

So $Q_1(t_0) \neq 0$. Otherwise, $Q_1(t_0) = 0$, from (5) we have $P_1(t_0) = 0$, thus $Q_1(t)$ and $P_1(t)$ have the common divisor $(t - t_0)$, which is in contradiction with the fact that $P_1(t)$ and $Q_1(t)$ are coprime. Similarly, $Q_2(t_0) \neq 0$, $Q_3(t_0) \neq 0$. So that

$$(t_0, x_0, y_0, z_0) \in V\left(x - \frac{P_1(t)}{Q_1(t)}, y - \frac{P_2(t)}{Q_2(t)}, z - \frac{P_3(t)}{Q_3(t)}\right).$$

For any (t_0, x_0, y_0, z_0), we have

$$\begin{aligned}&V\left(x-\frac{P_1(t)}{Q_1(t)}, y-\frac{P_2(t)}{Q_2(t)}, z-\frac{P_3(t)}{Q_3(t)}\right)\\ &\supseteq V\left(Q_1(t)x-P_1(t), Q_2(t)y-P_2(t), Q_3(t)z-P_3(t)\right).\end{aligned}$$

Thus

$$\begin{aligned}&V\left(x-\frac{P_1(t)}{Q_1(t)}, y-\frac{P_2(t)}{Q_2(t)}, z-\frac{P_3(t)}{Q_3(t)}\right)\\ &= V\left(Q_1(t)x-P_1(t), Q_2(t)y-P_2(t), Q_3(t)z-P_3(t)\right).\end{aligned}$$

(ii) By the Closure Theorem [Cox et al., [10]], we know that $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(Q_1(t)x-P_1(t), Q_2(t)y-P_2(t), Q_3(t)z-P_3(t)\right)\right).$$

From (i) we have that

$$\begin{aligned}&\pi_1\left(V\left(Q_1(t)x-P_1(t), Q_2(t)y-P_2(t), Q_3(t)z-P_3(t)\right)\right)\\ =\ &\pi_1\left(V\left(x-\frac{P_1(t)}{Q_1(t)}, y-\frac{P_2(t)}{Q_2(t)}, z-\frac{P_3(t)}{Q_3(t)}\right)\right).\end{aligned}$$

Thus $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(x-\frac{P_1(t)}{Q_1(t)}, y-\frac{P_2(t)}{Q_2(t)}, z-\frac{P_3(t)}{Q_3(t)}\right)\right).$$

If

$$x=\frac{P_1(t)}{Q_1(t)},\ y=\frac{P_2(t)}{Q_2(t)},\ z=\frac{P_3(t)}{Q_3(t)} \tag{6}$$

$P_i(t)$ and $Q_i(t)$ are not coprime $i=1,2,3$. We can first find the greatest common divisor $d_i(t)$ of $P_i(t)$ and $Q_i(t)$, so that $x=\frac{P_1(t)}{Q_1(t)}$, $y=\frac{P_2(t)}{Q_2(t)}$, $z=\frac{P_3(t)}{Q_3(t)}$ can be simplified to $x=\frac{p_1(t)}{q_1(t)}$, $y=\frac{p_2(t)}{q2(t)}$, $z=\frac{p_3(t)}{q_3(t)}$, where $p_i(t)$ and $q_i(t)$ are coprime and satisfy

$$p_i(t)d_i(t)=P_i(t),\ q_i(t)d_i(t)=Q_i(t), i=1,2,3.$$

Let

$$I=\langle q_1(t)x-p_1(t), q_2(t)y-p_2(t), q_3(t)z-p_3(t)\rangle$$

be the ideal generated by polynomials $q_1(t)x-p_1(t)$, $q_2(t)y-p_2(t)$, $q_3(t)z-p_3(t)$, and

$$I_1=I\bigcap\mathbb{R}[x,y,z],$$

$$ES=\left\{(x(t_1), y(t_1), z(t_1))|\ t_1\in V(d_1(t)d_2(t)d_3(t))\right\}.$$

which contains finite points in $\mathbb{R}[x,y,z]$. □

Definition 3.7. Let V_0 be a variety. If V_0 is the smallest variety satisfying

$$\pi_1\left(V\left(x-\frac{p_1(t)}{q_1(t)},\ y-\frac{p_2(t)}{q2(t)},\ z-\frac{p_3(t)}{q_3(t)}\right)\right)\subseteq V_0,$$

and

$$\pi_1\left(V\left(x-\frac{P_1(t)}{Q_1(t)},\ y-\frac{P_2(t)}{Q_2(t)},\ z-\frac{P_3(t)}{Q_3(t)}\right)\right)\subseteq V_0-ES.$$

Then we call V_0 the implicitization variety with respect to parametrization (6).

Corollary 3.8. *For the above parametrization* (6) *of space curve. If $P_i(t)$ and $Q_i(t)$ are not coprime $i=1,2,3$, then we have that $V(I_1)$ is the implicitization variety with respect to parametrization* (6).

Proof. For $d_i(t)$ is the greatest common divisor of $P_i(t)$ and $Q_i(t)$, and

$$p_i(t)d_i(t)=P_i(t),\ q_i(t)d_i(t)=Q_i(t), i=1,2,3.$$

We have

$$\begin{aligned}
x &= \frac{P_1(t)}{Q_1(t)}=\frac{p_1(t)}{q_1(t)},\\
y &= \frac{P_2(t)}{Q_2(t)}=\frac{p_2(t)}{q_2(t)},\\
z &= \frac{P_3(t)}{Q_3(t)}=\frac{p_3(t)}{q_3(t)}.
\end{aligned}$$

So that

$$\begin{aligned}
&V\left(x-\frac{P_1(t)}{Q_1(t)},y-\frac{P_2(t)}{Q_2(t)},z-\frac{P_3(t)}{Q_3(t)}\right)\\
&=V\left(x-\frac{p_1(t)}{q_1(t)},y-\frac{p_2(t)}{q_2(t)},z-\frac{p_3(t)}{q_3(t)}\right)-ES.
\end{aligned}$$

By Theorem 3.6, we get that $V(I_1)$ is the smallest variety containing

$$\pi_1\left(V\left(x-\frac{p_1(t)}{q_1(t)},y-\frac{p_2(t)}{q_2(t)},z-\frac{p_3(t)}{q_3(t)}\right)\right),$$

and thus $V(I_1)-ES$ is the corresponding set containing

$$\pi_1\left(V\left(x-\frac{P_1(t)}{Q_1(t)},y-\frac{P_2(t)}{Q_2(t)},z-\frac{P_3(t)}{Q_3(t)}\right)\right).$$

$\square$

3.3. Examples

In this section, we will present some examples to illustrate our method.

Example 3.1. For an example of how our method works, let us look at the circle in $\mathbb{R}^2$, which is given by the rational parametrization $x = \dfrac{1-t^2}{1+t^2}$, $y = \dfrac{2t}{1+t^2}$.

Because $1-t^2$ and $2t$ are coprime with $1+t^2$ respectively, then we need to determine the ideal

$$I = \langle (1+t^2)x - (1-t^2), (1+t^2)y - 2t \rangle.$$

Using lex order with $t > x > y$, a Gröbner basis for I is given by

$$\begin{aligned} g_1 &= y^2 + x^2 - 1, \\ g_2 &= x - 1 + yt, \\ g_3 &= -y + xt + t. \end{aligned}$$

The Elimination Theorem tells us that $I_1 = I \bigcap \mathbb{R}[x, y] = \langle g_1 \rangle$, and thus by Theorem 3.3, $V(g_1)$ is the smallest variety containing the above rational parametric circles, then $g_1 = 0$ is exactly the implicit equation of the circle.

Example 3.2. $x = \dfrac{1+2t^2}{1+t^2}$, $y = \dfrac{(1+t)(1+t^2)}{1+2t^2}$.

For ideal

$$I = \langle (1+t^2)x - (1+2t^2), (1+2t^2)y - (1+t)(1+t^2) \rangle,$$

using lex order with $t > x > y$, we have its Gröbner basis given by

$$\begin{aligned} g_1 &= 2x - 3 + x^3y^2 - 2x^2y - 2y^2x^2 + 4yx, \\ g_2 &= t - yx + 1. \end{aligned}$$

Then the corresponding implicit equation is $g_1 = 0$.

Example 3.3. $x = t^4 + 1$, $y = t^3 + 1$, $z = t^2 + 2$.

For ideal

$$I = \langle x - t^4 - 1, y - t^3 - 1, z - t^2 - 2 \rangle,$$

using lex order with $t > x > y > z$, we have its Gröbner basis given by

$$\begin{aligned} g_1 &= y^2 - 2y - z^3 + 6z^2 - 12z + 9, \\ g_2 &= x - 5 - z^2 + 4z, \\ g_3 &= zt - 2t - y + 1, \\ g_4 &= -z^2 + 4z + yt - t - 4, \\ g_5 &= -z + t^2 + 2. \end{aligned}$$

By Theorem 3.6, the corresponding implicit equations are $g_1 = 0$, $g_2 = 0$.

Example 3.4. $x = \dfrac{(t-1)(t+1)}{t-1}$, $y = t$.

For the ideal $\langle x-(t+1), y-t\rangle$, using lex order with $t > x > y$, we have its Gröbner basis given by

$$g_1 = -y + x - 1,$$
$$g_2 = -y + t.$$

Then we have $I_1 = \langle g_1 \rangle$. For corresponding d_1, d_2 defined as the above are $t-1$ and 1 respectively, from Corollary 3.8, we know that $V(I_1) - \{(x=2, y=1)\}$ is the smallest set containing

$$\pi_1\left(V\left(x - \frac{(t-1)(t+1)}{t-1},\ y-t\right)\right).$$

Thus the corresponding implicit equation is $g_1 = 0$ except the special point set $\{(2,1)\}$.

4. Implicitization via characteristic set

4.1. Characteristic set

In this section, we will first present some fundamental concepts and theories of characteristic set. All the details can be found in the related book (Wu [24]).

Let k be the basic field of characteristic 0, $k[x_1, \dots, x_n]$ is the polynomial ring over k with independent indeterminate $x_1, \dots, x_n$, and $x_1 < x_2 < \cdots < x_n$.

For the set $AS = \{A_j |\ j = 1, \dots, r\}$, let

$$A_j = a_{j0}x_{i_j}^{m_j} + a_{j1}x_{i_j}^{m_j-1} + \cdots + a_{jm_j}.$$

If $0 < i_1 < \cdots < i_r$, let $y_j = x_{i_j}$, other x_i can be denoted by $u_1, \dots, u_d, d+r = n$. Then

$$A_j = a_{j0}y_j^{m_j} + a_{j1}y_j^{m_j-1} + \cdots + a_{jm_j},$$

where $a_{jk} \in k[u_1, \dots, u_d, y_1, \dots, y_{j-1}]$, and a_{j0} is called the initial of polynomial A_j. For any $j > i$, if $\deg_{y_i}(A_j) < \deg_{y_i}(A_i)$, then we call AS an ascending set.

We suppose that $k_0 = k(u_1, \dots, u_d)$ an extension field of field k by adjoining transcendental elements $u_1, \dots, u_d$. For the ascending set $AS = \{A_j |\ j = 1, \dots, r\}$, if $\tilde{A}_1 = A_1$ is irreducible when it is considered as a polynomial in $k_0[y_1]$, and η_1 is a zero point of $\tilde{A}_1$, i.e., $\tilde{A}_1(\eta_1) = 0$. We denote $k_0(\eta_1)$ by k_1, we let η_{i-1} be a zero point of polynomial $\tilde{A}_{i-1}$, i.e., $\tilde{A}_{i-1}(\eta_{i-1}) = 0$, similarly, we also denote $k_{i-2}(\eta_{i-1})$ by k_{i-1}. For $i \le r$, we have

$$\tilde{A}_i = A_i|_{y_1=\eta_1, \dots,\ y_{i-1}=\eta_{i-1}}$$

is irreducible when it is considered as a polynomial in $k_{i-1}[y_i]$, then we call the ascending set AS an irreducible ascending set.

Definition 4.1. For a set PS of polynomials, if an ascending set AS satisfies

(1) $\mathrm{Remdr}(PS/AS) = 0$, which is the remainder set of PS with respect to AS;

(2) $V(PS) \subset V(AS)$, where $V(PS)$ and $V(AS)$ are zero sets of PS and AS respectively.

Then we call AS a characteristic set of the set PS.

Lemma 4.2. *Let PS be a finite set of polynomials. Then*

$$V(PS) = \bigcup_i V(AS_i/IP_i),$$

where IP_i is the initial product of the corresponding characteristic set AS_i, and $V(AS_i/IP_i) = V(AS_i) - V(IP_i)$.

4.2. Characteristic set method for implicitization of rational curves

In this section, all the discussions are over the field $\mathbb{R}$, which is the set of all real numbers. Our method works for rational plane curves.

Let a rational plane curve be defined by the parametric equations

$$x = \frac{P(t)}{A(t)}, \ y = \frac{Q(t)}{B(t)}, \tag{7}$$

where $P(t), A(t); Q(t), B(t)$ belong to the polynomial ring $\mathbb{R}[t]$, and

$$\gcd(P(t), A(t)) = \gcd(Q(t), B(t)) = 1.$$

Let the implicit polynomial of the above curve be f in x, y, and $n_x = \deg(f, x)$, $n_y = \deg(f, y)$. By Theorem 2 in Marco and Martinez [14], we know

$$n_x = \max\{\deg(Q(t), t), \deg(B(t), t)\}, \quad n_y = \max\{\deg(P(t), t), \deg(A(t), t)\},$$

and the implicit polynomial f has the form

$$f = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} u_{ij} x^i y^j.$$

Therefore, we has formed the implicit polynomial f of correct degree n_x in x, and n_y in y with $(n_x+1)(n_y+1)$ indeterminate coefficients, i.e., the coefficients u_{ij} are to be determined. Substituting the expression (7) into f and expanding the result, we get the following rational expression

$$\frac{g}{h} = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} u_{ij} \frac{P(t)^i Q(t)^j}{A(t)^i B(t)^j}, \tag{8}$$

where both g and h are polynomials in the parameter t. Let L be the set of all the nonzero coefficients of g considered as a polynomial in t. Then every element of L is a homogeneous linear polynomial in the indeterminate u_{ij} with rational coefficients.

Now, we can solve the system of linear equations

$$E = 0, \quad E \in L. \tag{9}$$

For u_{ij} as unknowns can be obtained by Lemma 4.2. Clearly this homogeneous system always has the trivial solution $u_{ij} = 0$ for all i, j. But the curve defined by (7) has an implicit equation if and only if the linear system (9) has a nontrivial solution for u_{ij}, i.e., when the general solution $u_{ij} = \bar{u}_{ij}$ of (9) is found and nontrivial, then an implicit polynomial can easily be obtained by substituting $\bar{u}_{ij}$ into f. Combined with theories in D. M. Wang [23], we have the following algorithm.

Algorithm 4.2.3.

Input: Rational parametric equations $x = \dfrac{P(t)}{A(t)}$, $y = \dfrac{Q(t)}{B(t)}$, where

$$\gcd(P(t), A(t)) = \gcd(Q(t), B(t)) = 1$$

Output: Implicit polynomial F in x, y

Step 1. Substituting (7) into f and simplify it to the expression (8)

Step 2. Get the set L, which is the set of all coefficients of g in t

Step 3. Solve the linear system (9) by Lemma 4.2 to get the general solution $u_{ij} = \bar{u}_{ij}$

Step 4. If all the $\bar{u}_{ij}$ are zero
Then the curve defined by (7) has no implicit polynomial and exit
Else if $\bar{u}_{ij}$ are not all zero
Then we obtain $f|_{u_{ij}=\bar{u}_{ij}}$

Step 5. If $f|_{u_{ij}=\bar{u}_{ij}}$ has a nonconstant factor $F \in \mathbb{R}[x, y]$
Then $F = 0$ is the corresponding implicit equation
Else (7) does not define a proper rational curve

4.3. Characteristic set method for implicitization of polynomial surfaces

Our method not only works efficiently for rational curve but also for polynomial surface in $\mathbb{R}^3$ space, which is defined by a polynomial in $\mathbb{R}[x, y, z]$. A brief discussion of the case of surfaces will be given in this section.

A polynomial surfaces can be defined by the parametric equations

$$x = P(s,t),\ y = Q(s,t),\ z = R(s,t), \tag{10}$$

where $P(s,t), Q(s,t), R(s,t)$ belong to the polynomial ring $\mathbb{R}[s,t]$, and d_1, d_2, d_3 are the total degree of them respectively. Let the implicit polynomial of the above curve be f in x, y, z, and $n_x = \deg(f, x), n_y = \deg(f, y), n_z = \deg(f, z)$. By Proposition 3 in Marco and Martinez [15], we have that

$$n_x = d_2 d_3, \quad n_y = d_1 d_3, \quad n_z = d_1 d_2.$$

The implicit polynomial f has the form

$$f = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \sum_{k=0}^{n_z} u_{ijk} x^i y^j z^k.$$

Therefore, we has formed the implicit polynomial f of correct degree n_x in x, n_y in y and n_z in z with $(n_x + 1)(n_y + 1)(n_z + 1)$ indeterminate coefficients, i.e., the coefficients u_{ijk} are to be determined. Substituting the expression (10) into f and expanding the result, we get

$$g = \sum_{i=0}^{n_x}\sum_{j=0}^{n_y}\sum_{k=0}^{n_z} u_{ijk}P(s,t)^iQ(s,t)^jR(s,t)^k, \tag{11}$$

where both g is a polynomial in the parameters s, t. Let L be the set of all the nonzero coefficients of g considered as a polynomial in s, t. Then every element of L is a homogeneous linear polynomial in the indeterminate u_{ijk} with rational coefficients.

Now, we can solve the system of linear equations

$$E = 0, \quad E \in L. \tag{12}$$

For u_{ijk} as unknowns can be obtained by Lemma 4.2. Clearly this homogeneous system always has the trivial solution $u_{ijk} = 0$ for all i, j, k. But the curve defined by (10) has an implicit equation if and only if the linear system (12) has a nontrivial solution for u_{ijk}. i.e., when the general solution $u_{ijk} = \bar{u}_{ijk}$ of (12) is found and nontrivial, then an implicit polynomial can easily be obtained by substituting $\bar{u}_{ijk}$ into f. Combined with theories in D. M. Wang [23], we have the following algorithm.

Algorithm 4.3.4.

Input: Rational parametric equations $x = P(s,t)$, $y = Q(s,t)$, $z = R(s,t)$

Output: Implicit polynomial F in x, y, z

Step 1. Substituting (10) into f and simplify it to the expression (11)

Step 2. Get the set L, which is the set of all coefficients of g in s, t

Step 3. Solve the linear system (12) by Lemma 4.2 to get the general solution $u_{ijk} = \bar{u}_{ijk}$

Step 4. If all the $\bar{u}_{ijk}$ are zero

Then the curve defined by (10) has no implicit polynomial

Else if $\bar{u}_{ijk}$ are not all zero

Then we obtain the implicit polynomial $f|_{u_{ijk}=\bar{u}_{ijk}}$

Step 5. If $f|_{u_{ijk}=\bar{u}_{ijk}}$ has a nonconstant factor $F \in \mathbb{R}[x, y, z]$

Then $F = 0$ is the corresponding implicit equation

Else (10) does not define a proper rational curve

4.4. Examples

Example 4.1. $x = \dfrac{5t^2}{1+t^5}$, $y = \dfrac{5t^3}{1+t^5}$.

Let

$$f = \sum_{i=0}^{5}\sum_{j=0}^{5} u_{ij}x^iy^j. \tag{13}$$

According to the algorithm 4.3.3 and using the package *wsolve* of D. K. Wang, we can easily get

$$AS = \{u_{00}, u_{53}, u_{54}, u_{55}, u_{33}, u_{35}, u_{40}, u_{41}, u_{42}, u_{43}, u_{44}, u_{45}, 5u_{50} + u_{22}, u_{51}, u_{52}, u_{34}, u_{01}, u_{02}, u_{03}, u_{04}, 5u_{05} + u_{22}, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{20}, u_{21}, u_{23}, u_{24}, u_{25}, u_{30}, u_{31}, u_{32}\}.$$

For the coefficients of every element of the linear system are all nonzero rational numbers, we get

$$\{u_{00} = 0,\ u_{53} = 0,\ u_{54} = 0,\ u_{55} = 0,\ u_{33} = 0,\ u_{35} = 0,\ u_{40} = 0,\ u_{41} = 0, u_{42} = 0,\ u_{43} = 0,\ u_{44} = 0,\ u_{45} = 0,\ u_{22} = -5u_{50},\ u_{51} = 0,\ u_{52} = 0, u_{34} = 0,\ u_{01} = 0, u_{02} = 0,\ u_{03} = 0,\ u_{04} = 0,\ u_{22} = -5u_{05},\ u_{10} = 0, u_{11} = 0, u_{12} = 0,\ u_{13} = 0,\ u_{14} = 0,\ u_{15} = 0,\ u_{20} = 0,\ u_{21} = 0,\ u_{23} = 0,\ u_{24} = 0, u_{25} = 0,\ u_{30} = 0,\ u_{31} = 0, u_{32} = 0\}.$$

Substituting them into (13), we get that

$$f = u_{05}(x^5 - 5x^2y^2 + y^5).$$

Thus from D. M. Wang [23], we know that the corresponding implicit equation is

$$x^5 - 5x^2y^2 + y^5 = 0.$$

Example 4.2. $x = t + s,\ y = t^2 + 2st,\ z = t^3 + 3st^2.$

Let

$$f = \sum_{i=0}^{6}\sum_{j=0}^{3}\sum_{k=0}^{2} x^i y^j z^k. \tag{14}$$

According to algorithm 4.3.4 and using the package *wsolve* of D. K. Wang, characteristic set AS can be readily obtained

$$AS = \{4u_{002} - u_{301}, u_{010}, u_{011}, u_{012}, u_{020}, u_{021}, u_{022}, -u_{301} + u_{030}, u_{031}, u_{032}, u_{100}, u_{101}, -u_{401} + 4u_{102}, u_{110}, u_{001}a1, u_{600}, 4u_{520} + 3u_{601}, u_{602}, u_{610}, u_{611}, u_{612}, u_{620}, u_{621}, u_{622}, u_{630}, u_{631}, u_{632}, u_{000}, 2u_{111} + 3u_{301}, u_{112}, u_{120}, u_{121}, u_{122}, u_{130} - u_{401}, u_{131}, u_{132}, u_{200}, u_{201}, -u_{501} + 4u_{202}, u_{210}, 3u_{401} + 2u_{211}, u_{212}, 3u_{301} + 4u_{220}, u_{221}, u_{222}, u_{230} - u_{501}, u_{231}, u_{232}, u_{300}, u_{520} + 3u_{302}, u_{310}, 2u_{311} + 3u_{501}, u_{312}, 4u_{320} + 3u_{401}, u_{321}, u_{322}, 4u_{520} + 3u_{330}, u_{331}, u_{332}, u_{400}, u_{402}, u_{410}, -2u_{520} + u_{411}, u_{412}, 4u_{420} + 3u_{501}, u_{421}, u_{422}, u_{430}, u_{431}, u_{432}, u_{500}, u_{502}, u_{510}, u_{511}, u_{512}, u_{521}, u_{522}, u_{530}, u_{531}, u_{532}\}.$$

Similarly as in Example 1, we have

$$\{u_{020} = 0, u_{021} = 0, u_{100} = 0, u_{101} = 0, u_{321} = 0, u_{031} = 0, u_{022} = 0, u_{101} = 0, u_{322} = 0, u_{110} = 0, u_{001} = 0, u_{600} = 0, u_{331} = 0, u_{602} = 0, u_{610} = 0, u_{532} = 0,$$

$u_{332}=0, u_{611}=0, u_{501}=4u_{202}, u_{612}=0, u_{102}=u_{102}, u_{202}=u_{202},$
$u_{302}=u_{302}, u_{220}=-3u_{002}, u_{111}=-6u_{002}, u_{230}=4u_{202}, u_{130}=4u_{102},$
$u_{401}=4u_{102}, u_{420}=-3u_{202}, u_{520}=-3u_{302}, u_{320}=-3u_{102}, u_{211}=-6u_{102},$
$u_{411}=-6u_{302}, u_{601}=4u_{302}, u_{330}=4u_{302}, u_{311}=-6u_{202}, u_{030}=4u_{002},$
$u_{301}=4u_{002}, u_{002}=u_{002}, u_{620}=0, u_{400}=0, u_{621}=0, u_{402}=0, u_{232}=0,$
$u_{631}=0, u_{630}=0, u_{622}=0, u_{632}=0, u_{000}=0, u_{300}=0, u_{410}=0, u_{121}=0,$
$u_{120}=0, u_{112}=0, u_{122}=0, u_{131}=0, u_{412}=0, u_{132}=0, u_{200}=0, u_{201}=0,$
$u_{421}=0, u_{310}=0, u_{422}=0, u_{210}=0, u_{212}=0, u_{430}=0, u_{431}=0, u_{312}=0,$
$u_{432}=0, u_{221}=0, u_{500}=0, u_{222}=0, u_{502}=0, u_{231}=0, u_{510}=0, u_{511}=0,$
$u_{011}=0, u_{010}=0, u_{512}=0, u_{012}=0, u_{521}=0, u_{531}=0, u_{522}=0, u_{530}=0\}.$

Substituting them into (14), we have that

$$f=(z^2+4y^3-6xyz-3x^2y^2+4x^3z)(u_{302}x^3+u_{202}x^2+u_{102}x+u_{002}).$$

Thus from D. M. Wang [23], we obtain the corresponding implicit equation

$$z^2+4y^3-6xyz-3x^2y^2+4x^3z=0.$$

5. Conclusion

In this paper, we introduce an improved method for implicitization based on Gröbner bases and another method for implicitization via Characteristic Set. At the same time, we give some examples to illustrate our methods' efficiency. For the future work, we will mainly focus on the speed and significant improvement of our algorithms.

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