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POLYGONAL PRODUCTS OF RESIDUALLY FINITE GROUPS

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Abstract. A group G is called cyclic subgroup separable for the cyclic subgroup H if for each $x \in G\backslash H$, there exists a normal subgroup N of finite index in G such that $x \notin HN$. Clearly a cyclic subgroup separable group is residually finite. In this note we show that certain polygonal products of cyclic subgroup separable groups amalgamating normal subgroups are again cyclic subgroup separable. We then apply our results to polygonal products of polycyclic-by-finite groups and free-by-finite groups.

1. Introduction

The polygonal products of groups were introduced by Karrass, Pietrowski and Solitar [6] in their study of the subgroup structure of the Picard group PSL(2, Z[i]). By using their results, Brunner, Frame, Lee and Wielenberg [3] characterized all the torson-free subgroups of finite index in the Picard group. Polygonal products also form a large subclass in the class of one-relator products of cyclic groups. For certain groups in the above class, Fine, Howie and Rosenberger [4] had proved a Freiheitssatz but the word problem and residual finiteness are still unknown.

Unlike the generalized free products of groups, the residual finiteness of polygonal products are little known even when the amalgamated subgroups are cyclic. In addition, polygonal products do not have many residual properties. In [1] Allenby and Tang showed that the polygonal products of finitely generated free abelian groups amalgamating cyclic subgroups with trivial intersections are residually finite. But in the same paper, Allenby and Tang also gave an example of a polygonal product of finitely generated nilpotent groups of class 2 amalgamating cyclic subgroups which is not residually finite.

In [2], G. Baumslag proved that the generalized free products of two polycyclic-by-finite groups amalgamating a central subgroup are residually finite.

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More recently, Kim [7] has shown that the polygonal products of polycyclic-by-finite groups amalgamating central subgroups are cyclic subgroup separable (or π_c for short) and hence residually finite. In this note we will prove that the polygonal products of polycyclic-by-finite groups amalgamating normal subgroups are π_c and hence residually finite. More precisely, we shall show that the polygonal products of subgroup separable groups amalgamating finitely generated normal subgroups are π_c. Thus the polygonal products of polycyclic-by-finite groups and free-by-finite groups amalgamating normal subgroups are π_c.

The notations used here are standard. In addition the following notations will be used for any group G:

$N \triangleleft_f G$ means N is a normal subgroup of finite index in G.

$\|x\|$ means the usual generalized free product length of x.

$A \underset{H}{*} B$ denotes the generalized free product of A and B amalgamating a subgroup H.

2. Preliminaries

We begin with the following definition and theorems.

Definition 2.1. A group G is called H-separable for the subgroup H if for each $x \in G\backslash H$, there exists $N \triangleleft_f G$ such that $x \notin HN$.

G is called HK-separable for the subgroups H, K if for each $x \in G\backslash HK$, there exists $N \triangleleft_f G$ such that $x \notin HKN$.

G is termed subgroup separable if G is H-separable for every finitely generated subgroup H.

G is termed cyclic subgroup separable (or π_c for short) if G is H-separable for every cyclic subgroup H.

It is well known that polycyclic groups and free groups are subgroup separable (Mal'cev [8], M. Hall [5]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and free-by-finite groups are subgroup separable.

The following theorems will be used in the proof of several theorems.

Theorem 2.2. (Baumslag [2]) *Let $G = A \underset{H}{*} B$ where A and B are finite. Then G is subgroup separable and hence π_c.*

Theorem 2.3. (Kim [7]) *Let $G = A \underset{H}{*} B$. Suppose that*

(a) *A and B are π_c and H-separable,*
(b) *for each $N \triangleleft_f H$, there exist $N_A \triangleleft_f A$ and $N_B \triangleleft_f B$ such that $N_A \cap H = N_B \cap H \subset N$.*

Then G is π_c.

3. Free products amalgamating normal subgroups

In this section we shall show that the generalized free products of finitely many subgroup separable groups amalgamating normal subgroups with trivial intersections are π_c. We begin with the generalized free products of two groups.

The followings two lemmas (Lemma 3.1, Lemma 3.2) are well known to researchers in this area. However we provide the proof for expository purposes.

Lemma 3.1. *Let H be a finitely generated group and $S \lhd_f H$. Then there exists $f_H(S) \subseteq S$ such that $f_H(S)$ is a characteristic subgroup of finite index in H.*

Proof. We define $f_H(S)$ as follow: If S is a characteristic subgroup of H then $f_H(S) = S$. Suppose S is not a characteristic subgroup of H. Let $[H : S] = m$ where m is a positive integer. Since H is finitely generated, the number of subgroups of index m in H is finite. Let N be the intersection of all these subgroups. Then N is a characteristic subgroup of finite index in H and $N \subseteq S$. We then define $f_H(S) = N$. □

Lemma 3.2. *Let A be a subgroup separable group and H be a finitely generated normal subgroup of A. If $S \lhd_f H$ and S is normal in A, then there exists $N \lhd_f A$ such that $N \cap H = S$.*

Proof. Since H is finitely generated and $S \lhd_f H$, then S is also finitely generated. Since A is subgroup separable, then $\overline{A} = A/S$ is residually finite. Now $\overline{H} = H/S$ is finite. Therefore there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \overline{H} = \{1\}$. Let N be the preimage of $\overline{N}$. Then $N \cap H = S$. □

Lemma 3.3. *Let A be a subgroup separable group and H be a finitely generated normal subgroup of A. Then for each $S \lhd_f H$, there exists $f_H(S) \subseteq S$ such that $f_H(S)$ is a characteristic subgroup of finite index in H and there exists $N \lhd_f A$ such that $N \cap H = f_H(S)$.*

Proof. Follows from Lemmas 3.1 and 3.2. □

Theorem 3.4. *Let $G = A \underset{H}{*} B$ where A and B are subgroup separable and H is a finitely generated normal subgroup of A and B. Then G is π_c.*

Proof. We shall use Theorem 2.3. Since A, B are π_c and H-separable, it is sufficient to show that given any $N_H \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Now given any $N_H \lhd_f H$, then by Lemma 3.3, there exists a characteristic subgroup $f_H(N_H)$ of H such that $f_H(N_H) \subseteq N_H$ and there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = f_H(N_H) = N_B \cap H$. The result follows from Theorem 2.3. □

Next we extend Theorem 3.4 to generalized free products of more than two groups.

Lemma 3.5. *Let A be a subgroup separable group and H_1, H_2 be finitely generated normal subgroups of A and $H_1 \cap H_2 = \{1\}$. If $S_1 \lhd_f H_1, S_2 \lhd_f H_2$ and S_1, S_2*

are normal in A, then there exists $N \lhd_f A$ such that $N \cap H_1 = S_1, N \cap H_2 = S_2$ and $NH_1 \cap NH_2 = N$.

Proof. Since H_1, H_2 are finitely generated and $S_1 \lhd_f H_1, S_2 \lhd_f H_2$, then S_1 and S_2 are also finitely generated. Therefore S_1S_2 is finitely generated and hence $\overline{A} = A/S_1S_2$ is residually finite. Since $\overline{H_1H_2} = H_1H_2/S_1S_2$ is finite, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \overline{H_1H_2} = \{1\}$. Let N be the preimage of $\overline{N}$. Then $N \cap H_1 = S_1, N \cap H_2 = S_2$ and $NH_1 \cap NH_2 = N$. □

Lemma 3.6. *Let A be a subgroup separable group and H_1, H_2 be finitely generated normal subgroups of A and $H_1 \cap H_2 = \{1\}$. Then for each i and each $S_i \lhd_f H_i$, there exists $f_{H_i}(S_i) \subseteq S_i$ such that $f_{H_i}(S_i)$ is a characteristic subgroup of finite index in H_i. Furthermore A has the property*

(a) *for each $S_1 \lhd_f H_1, S_2 \lhd_f H_2$, there exists $N \lhd_f A$ such that $N \cap H_1 = f_{H_1}(S_1), N \cap H_2 = f_{H_2}(S_2)$ and $NH_1 \cap NH_2 = N$.*

Proof. Follows from Lemma 3.1 and 3.5. □

Lemma 3.7. *Let $G = A \underset{H}{*} B$ and M, K be subgroups of A, B respectively with $M \cap H = \{1\} = K \cap H$. Let $X = H, M$ or K. Suppose for each $S_X \lhd_f X$ there exists $f_X(S_X) \subseteq X$ such that $f_X(S_X)$ is a characteristic subgroup of X. Further suppose that A and B have the following properties:*

(a) *for each $S_H \lhd_f H$, $S_M \lhd_f M$, there exists $N_A \lhd_f A$ such that $N_A \cap H = f_H(S_H)$, $N_A \cap M = f_M(S_M)$ and $N_AM \cap N_AH = N_A$,*
(b) *for each $S_H \lhd_f H$, $S_K \lhd_f K$, there exists $N_B \lhd_f B$ such that $N_B \cap H = f_H(S_H)$, $N_B \cap K = f_K(S_K)$ and $N_BK \cap N_BH = N_B$.*

Then for each $S_M \lhd_f M$ and $S_K \lhd_f K$, there exists $N \lhd_f G$ such that $N \cap M = f_M(S_M)$, $N \cap K = f_K(S_K)$ and $NM \cap NK = N$.

Proof. Let $S_M \lhd_f M, S_K \lhd_f K$ be given. Next we let $S_H = H$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = f_H(S_H)$, $N_A \cap M = f_M(S_M)$, $N_AM \cap N_AH = N_A$ and $N_B \cap H = f_H(S_H)$, $N_B \cap K = f_K(S_K)$, $N_BK \cap N_BH = N_B$. Let $\overline{G} = A/N_A \underset{\overline{H}}{*} B/N_B$ where $\overline{H} = HN_A/N_A = HN_B/N_B$. Since $\overline{G}$ is residually finite by Theorem 2.2 and $\overline{MK}$ is a finite set, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{N} \cap \overline{MK} = \{1\}$. Let N be the preimage of $\overline{N}$. Then $N \cap M = f_M(S_M), N \cap K = f_K(S_K)$ and $NM \cap NK = N$. □

Lemma 3.8. *Let $\{A_i\}$, $i = 1, 2, \ldots, n$, be groups and H_{i-1}, H_i be subgroups of A_i with $H_{i-1} \cap H_i = \{1\}$. Suppose for each i and each $S_i \lhd_f H_i$, there exists $f_{H_i}(S_i) \subseteq S_i$ such that $f_{H_i}(S_i)$ is a characteristic subgroup of H_i. Further suppose that each A_i has the property*

(a) *for each $S_{i-1} \lhd_f H_{i-1}, S_i \lhd_f H_i$, there exists $N_{A_i} \lhd_f A_i$ such that $N_{A_i} \cap H_{i-1} = f_{H_{i-1}}(S_{i-1}), N_{A_i} \cap H_i = f_{H_i}(S_i)$ and $N_{A_i}H_{i-1} \cap N_{A_i}H_i = N_{A_i}$.*

Let $E_n = A_1 \underset{H_1}{*} A_2 \underset{H_2}{*} \cdots \underset{H_{n-1}}{*} A_n$. Then E_n has the property (a') *for each* $S_0 \lhd_f H_0, S_n \lhd_f H_n$, *there exists* $N \lhd_f E_n$ *such that* $N \cap H_0 = f_{H_0}(S_0), N \cap H_n = f_{H_n}(S_n)$ *and* $NH_0 \cap NH_n = N$

Proof. We prove by induction on n. The case $n = 2$ follows from Lemma 3 7. Let $n \geq 3$. Then $E_n = E_{n-1} \underset{H_{n-1}}{*} A_n$ where $E_{n-1} = A_1 \underset{H_1}{*} A_2 \underset{H_2}{*} \cdots \underset{H_{n-2}}{*} A_{n-1}$. By the induction hypothesis, for each $S_0 \lhd_f H_0, S_{n-1} \lhd_f H_{n-1}$, there exists $N_{E_{n-1}} \lhd_f E_{n-1}$ such that $N_{E_{n-1}} \cap H_0 = f_{H_0}(S_0), N_{E_{n-1}} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{E_{n-1}}H_0 \cap N_{E_{n-1}}H_{n-1} = N_{E_{n-1}}$. By (a), for each $S_{n-1} \lhd_f H_{n-1}, S_n \lhd_f H_n$, there exists $N_{A_n} \lhd_f A_n$ such that $N_{A_n} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1}), N_{A_n} \cap H_n = f_{H_n}(S_n)$ and $N_{A_n}H_{n-1} \cap N_{A_n}H_n = N_{A_n}$. The result now follows from Lemma 3.7. □

Lemma 3.9. (Kim [7]) *Let* $G = A \underset{H}{*} B$ *where* A, B *are* H*-separable. Suppose for each* $N_H \lhd_f H$, *there exist* $N_A \lhd_f A, N_B \lhd_f B$ *such that* $N_A \cap H = N_B \cap H \subseteq N_H$. *Let* S *be any subgroup of* B. *If* B *is* S*-separable, then* G *is* S*-separable.*

Lemma 3.10. *Let* $\{A_i\}$, $i = 1, 2, \ldots, n$, *be groups and* H_{i-1}, H_i *be subgroups of* A_i *with* $H_{i-1} \cap H_i = \{1\}$. *Suppose each* A_i *and* H_i *satisfy the hypothesis of Lemma 3.8. Further suppose each* A_i *is* H_{i-1}*-separable and* H_i*-separable. Let* $E_n = A_1 \underset{H_1}{*} \cdots \underset{H_{n-1}}{*} A_n$. *Then* E_n *is* H_0*-separable and* H_n*-separable.*

Proof. We prove by induction on n. The case $n = 2$ follows from Lemma 3.9. Let $n \geq 3$. Then $E_n = E_{n-1} \underset{H_{n-1}}{*} A_n$ as in Lemma 3.8. By induction E_{n-1} is H_0-separable and H_{n-1}-separable. By assumption, A_n is H_{n-1}-separable and H_n-separable. By Lemma 3.8, for each $S_0 \lhd_f H_0, S_{n-1} \lhd_f H_{n-1}$, there exists $N_{E_{n-1}} \lhd_f E_{n-1}$ such that $N_{E_{n-1}} \cap H_0 = f_{H_0}(S_0), N_{E_{n-1}} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{E_{n-1}}H_0 \cap N_{E_{n-1}}H_{n-1} = N_{E_{n-1}}$. By assumption, for each $S_{n-1} \lhd_f H_{n-1}$, there exists $N_{A_n} \lhd_f A_n$ such that $N_{A_n} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$. Hence by Lemma 3.9, E_n is H_0-separable and H_n-separable. □

Theorem 3.11. *Let* $\{A_i\}$, $i = 1, 2, \ldots, n$, *be subgroup separable groups and* H_{i-1}, H_i *be finitely generated normal subgroups of* A_i *with* $H_{i-1} \cap H_i = \{1\}$. *Let* $E_n = A_1 \underset{H_1}{*} \cdots \underset{H_{n-1}}{*} A_n$. *Then* E_n *is* π_c.

Proof. We use induction on n. The case $n = 2$ follows from Theorem 3.4. Suppose $n \geq 3$. Then $E_n = E_{n-1} \underset{H_{n-1}}{*} A_n$ as in Lemma 3.8. By induction, E_{n-1} is π_c. By assumption, A_n is subgroup separable and hence π_c. By Lemma 3.10, E_{n-1} is H_0-separable and H_{n-1}-separable. Since A_n is subgroup separable, then A_n is H_{n-1}-separable. By Lemma 3.8, for each $S_0 \lhd_f H_0, S_{n-1} \lhd_f H_{n-1}$, there exists $N_{E_{n-1}} \lhd_f E_{n-1}$ such that $N_{E_{n-1}} \cap H_0 = f_{H_0}(S_0), N_{E_{n-1}} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{E_{n-1}}H_0 \cap N_{E_{n-1}}H_{n-1} = N_{E_{n-1}}$. By Lemma 3.3, for each $S_{n-1} \lhd_f H_{n-1}$, there exists $N_{A_n} \lhd_f A_n$ such that $N_{A_n} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$. Hence by Theorem 2.3, E_n is π_c. □

4. Polygonal products amalgamating normal subgroups

In this section we extend Theorem 3.11 to polygonal products of finitely many subgroup separable groups amalgamating normal subgroups with trivial intersections. As a consequence polygonal products of polycyclic-by-finite groups and free-by-finite groups are π_c. A polygonal product can be described as follows: (Kim [7]) Let P be a polygon. To each vertex v of P assign a vertex group G_v and to each edge e of P assign an edge group G_e together with monomorphisms α_e and β_e embedding G_e into the two vertex groups at the end of e. The polygonal product G is defined to be the group generated by all the generators of the vertex groups with defining relations given by the defining relations of all the vertex groups together with the relations $g_e\alpha_e = g_e\beta_e$ for each g_e of G_e. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_0, G_1, \ldots, G_n$ amalgamating the (edge) groups $H_0, H_1, \ldots, H_n$ with trivial intersections if $G_i \cap G_{i+1} = H_i$ and $H_{i-1} \cap H_i = 1$ where $0 \leq i \leq n$ and $H_{-1} = H_{n+1}$.

Lemma 4.1. *Let $G = A \ast_H B$ and M, K be subgroups of A, B respectively with $M \cap H = \{1\} = K \cap H$. Suppose for each $N_H \lhd_f H$ there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$, $N_B H \cap N_B K = N_B$. If A is H, M, MH-separable and B is H, K, HK-separable then G is MK-separable.*

Proof. Let $g \in G - MK$.

Case 1. $g \in B - K$, $g \notin H$. Since B is H, K-separable, there exists $M_B \lhd_f B$ such that $g \notin M_B K$, $g \notin M_B H$. Let $N_H = M_B \cap H$. By assumption there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$. Now let $R_A = N_A, R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Clearly $\overline{G}$ is a homorphic image of G and $\overline{H} \cap \overline{M} = 1$. Then $\overline{g} \notin \overline{MK}$. Since $\overline{G}$ is residually finite by Theorem 2.2 and $\overline{MK}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{NMK}$. Let N be the preimage of $\overline{N}$. Then $g \notin NMK$.

Case 2. $g \in A - M$, $g \notin H$. The proof for this case is similar to the proof of Case 1.

Case 3. $g \in H$. Since $g \notin MK$, we have $g \notin M$, $g \notin K$. Since A is M-separable, B is K-separable, there exist $M_A \lhd_f A, M_B \lhd_f B$ such that $g \notin M_A M$, $g \notin M_B K$. Let $N_H = M_A \cap M_B$. By assumption, there exists $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$. Now let $R_A = N_A \cap M_A$ and $R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Then $\overline{H} \cap \overline{M} = 1$. Clearly $\overline{MK}$ is finite and $\overline{g} \notin \overline{MK}$. We can now proceed as in Case 1.

Case 4. $g \notin A \cup B$.

Subcase 1. $\|g\| \geq 3$ or $\|g\| = 2$ and $g = b_1 a_1$. We will only consider the case $g = b_1 a_1 b_2 a_2 \cdots b_k a_k$ where $a_i \in A - H$, $b_i \in B - H$, $i = 1, 2, \ldots, k$. The other cases are similar. Since A is H-separable, B is H-separable, there exist

$M_A \lhd_f A$, $M_B \lhd_f B$ such that $a_i \notin M_A H$, $b_i \notin M_B H$, $i = 1, 2, \ldots, k$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Now let $R_A = N_A \cap M_A, R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Clearly $\|\overline{g}\| = \|g\|$ and hence $\overline{g} \notin \overline{MK}$. Furthermore $\overline{MK}$ is finite. We can now proceed as in Case 1.

Subcase 2. $\|g\| = 2$ and $g = ab$ where $a \in A - H$, $b \in B - H$. Now $g \notin MK$ if and only if $a \notin MH$ or $a = mh$ and $hb \notin K$ where $m \in M$, $h \in H$. Suppose $a \notin MH$. Since A is MH-separable, B is H-separable, there exist $M_A \lhd_f A$, $M_B \lhd_f B$ such that $a \notin M_A MH$ and $b \notin M_B H$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Let $R_A = N_A \cap M_A, R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Now $\overline{a} \notin \overline{MH}$ implies $\overline{g} \notin \overline{MK}$. Furthermore $\overline{MK}$ is finite. We can now proceed as in Case 1.

Suppose $a = mh$ and $hb \notin K$. Since A is H-separable, B is H, K-separable, there exist $M_A \lhd_f A$, $M_B \lhd_f B$ such that $a \notin M_A H$, $b \notin M_B H$ and $hb \notin M_B K$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$, $N_B H \cap N_B K = N_B$. Let $R_A = N_A \cap M_A$, $R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$ and $R_A H \cap R_A M = R_A, R_B H \cap R_B K = R_B$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Now $\overline{g} \notin \overline{MK}$, for otherwise, $\overline{a} = \overline{m}_1 \overline{h}_1$ and $\overline{h}_1 \overline{b} \in \overline{K}$ where $\overline{m}_1 \in \overline{M}$, $\overline{h}_1 \in \overline{H}$. Since $\overline{M} \cap \overline{H} = \{1\}$ and $\overline{a} = \overline{m}\overline{h}$, we have $\overline{m}_1 = \overline{m}$, $\overline{h}_1 = \overline{h}$. But this implies that $\overline{hb} \in \overline{K}$, a contradiction. Furthermore $\overline{MK}$ is finite. We can now proceed as in Case 1. This completes the proof of the lemma. □

Lemma 4.2. *Let $G = A \ast_H B$ and M, K be subgroups of A, B respectively with $M \cap H = \{1\} = K \cap H$. Suppose A, B, M, K satisfy the hypothesis of Lemma 4.1. Then G is $M \ast K$-separable.*

Proof. Let $g \in G - (M \ast K)$.

Case 1. $g \in H$. Since A is M-separable, B is K-separable, there exist $M_A \lhd_f A$, $M_B \lhd_f B$ such that $g \notin M_A M$, $g \notin M_B K$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$ and $N_B H \cap N_B K = N_B$. Let $R_A = N_A \cap M_A$ and $R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Clearly $g \notin R_A M$ and $g \notin R_B K$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Then $\overline{H} \cap \overline{M} = 1$ and $\overline{H} \cap \overline{K} = 1$. Hence $\overline{g} \notin \overline{M} \ast \overline{K}$. Since $\overline{G}$ is subgroup separable by Theorem 2.2 and $\overline{M} \ast \overline{K}$ is finitely generated, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{N}(\overline{M} \ast \overline{K})$. Let N be the preimage of $\overline{N}$. Then $g \notin N(M \ast K)$.

Case 2. $\|g\| \geq 1$. We will only consider the case $g = a_1 b_1 a_2 b_2 \cdots a_n b_n$ where $a_i \in A - H$, $b_i \in B - H$, $i = 1, 2, \ldots, n$. The other cases are similar. Now $g \notin M \ast K$ if and only if there exist m_i, k_i, h_i and h_i' where $m_i \in M, k_i \in K$

and $h_i, h'_i \in H$ such that one of the following is true:

(1) $a_1 \notin MH$

or (1′) $a_1 = m_1h_1$ but $h_1b_1 \notin KH$

or (2) $a_1 = m_1h_1, h_1b_1 = k_1h'_1$ but $h'_1a_2 \notin MH$

or (2′) $a_1 = m_1h_1, h_1b_1 = k_1h'_1, h'_1a_2 = m_2h_2$ but $h_2b_2 \notin KH$

$\vdots$

or (n) $a_1 = m_1h_1, h_1b_1 = k_1h'_1, \dots, h_{n-1}b_{n-1} = k_{n-1}h'_{n-1}$ but $h'_{n-1}a_n \notin MH$

or (n') $a_1 = m_1h_1, h_1b_1 = k_1h'_1, \dots, h_{n-1}b_{n-1} = k_{n-1}h'_{n-1}, h'_{n-1}a_n = m_nh_n$ but $h_nb_n \notin K$

Let s be the smallest integer such that (s) (or (s')) is true. Now the m_j, k_j, h_j, h'_j are uniquely determined because $M \cap H = \{1\} = K \cap H$. Since A is H, MH-separable, B is H-separable, there exist $M_A \triangleleft_f A$, $M_B \triangleleft_f B$ such that $a_i \notin M_AH$, $b_i \notin M_BH$ and $h'_{s-1}a_s \notin M_AMH$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \triangleleft_f A, N_B \triangleleft_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_AH \cap N_AM = N_A$, $N_BH \cap N_BK = N_B$. Let $R_A = N_A \cap M_A, R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$ and $R_AH \cap R_AM = R_A, R_BH \cap R_BK = R_B$. Let $\overline{G} = A/R_A \underset{\overline{H}}{*} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Clearly $\|g\| = \|\overline{g}\|$. Suppose $\overline{g} \in \overline{M} * \overline{K}$. Then there exist $\overline{t}_i \in \overline{M}, \overline{u}_i \in \overline{K}$ and $\overline{v}_i, \overline{v}'_i \in \overline{H}$ such that $\overline{a}_1 = \overline{t}_1\overline{v}_1, \overline{v}_1\overline{b}_1 = \overline{u}_1\overline{v}'_1, \dots, \overline{v}_{s-1}\overline{a}_s = \overline{t}_s\overline{v}_s, \dots, \overline{v}_{n-1}\overline{a}_n = \overline{t}_n\overline{v}_n$ and $\overline{v}_n\overline{b}_n = \overline{u}_n$. Since $\overline{M} \cap \overline{H} = \{1\} = \overline{K} \cap \overline{H}$, we have $\overline{t}_i = \overline{m}_i, \overline{v}_i = \overline{h}_i, \overline{u}_i = \overline{k}_i, \overline{v}'_i = \overline{h}'_i$ for $i = 1, 2, \dots, s-1$. But then $\overline{h}'_{s-1}\overline{a}_s \in \overline{MH}$ a contradiction. Therefore $\overline{g} \notin \overline{M} * \overline{K}$. Furthermore $\overline{G}$ is subgroup separable by Theorem 2.2 and $\overline{M} * \overline{K}$ is finitely generated. We can now proceed as in Case 1. This completes the proof of the lemma. □

Lemma 4.3. *Let* $\{A_i\}$, $i = 1, 2, \dots, n$, *be groups and* H_{i-1}, H_i *be subgroups of* A_i *with* $H_{i-1} \cap H_i = \{1\}$. *Suppose each* A_i *and* H_i *satisfy the hypothesis of Lemma 3.8. Further suppose each* A_i *is* H_{i-1}*-separable,* H_i*-separable,* $H_{i-1}H_i$*-separable and* H_iH_{i-1}*-separable. Let* $E_n = A_1 \underset{H_1}{*} \cdots \underset{H_{n-1}}{*} A_n$. *Then* E_n *is* H_0H_n*-separable and* $H_0 * H_n$*-separable.*

Proof. We prove by induction on n. The case $n = 2$ follows from Lemmas 4.1 and 4.2. Let $n \geq 3$. Then $E_n = E_{n-1} \underset{H_{n-1}}{*} A_n$ as in Lemma 3.8. By Lemma 3.10, E_{n-1} is H_0-separable and H_{n-1}-separable. By induction E_{n-1} is H_0H_{n-1}-separable. By assumption, A_n is H_{n-1}-separable, H_n-separable and $H_{n-1}H_n$-separable. By Lemma 3.8, for each $S_0 \triangleleft_f H_0, S_{n-1} \triangleleft_f H_{n-1}$, there exists $N_{E_{n-1}} \triangleleft_f E_{n-1}$ such that $N_{E_{n-1}} \cap H_0 = f_{H_0}(S_0), N_{E_{n-1}} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{E_{n-1}}H_0 \cap N_{E_{n-1}}H_{n-1} = N_{E_{n-1}}$. By assumption, for each $S_{n-1} \triangleleft_f H_{n-1}, S_n \triangleleft_f H_n$, there exists $N_{A_n} \triangleleft_f A_n$ such that $N_{A_n} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1}), N_{A_n} \cap H_n = f_{H_n}(S_n)$ and $N_{A_n}H_{n-1} \cap N_{A_n}H_n = N_{A_n}$. Hence by Lemma 4.1, E_n is H_0H_n-separable and by Lemma 4.2, E_n is $H_0 * H_n$-separable. □

Lemma 4.4. *Let $G = A \underset{H}{*} B$ and M, K be subgroups of A, B respectively with $M \cap H = \{1\} = K \cap H$ and H, M, K be finitely generated. Suppose further that A and B satisfy the hypothesis of Lemma 3.7. Then for each $S \lhd_f (M * K)$, there exists $N \lhd_f G$ such that $N \cap (M * K) = f_{M*K}(S)$.*

Proof. Let $S \lhd_f (M * K)$ be given. Since $M * K$ is finitely generated, then by Lemma 3.1, there exists a subgroup $f_{M*K}(S) \subseteq S$ such that $f_{M*K}(S)$ is characteristic and of finite index in $M * K$. Let $S_M = f_{M*K}(S) \cap M$ and $S_K = f_{M*K}(S) \cap K$. Then S_M and S_K are characteristic subgroups of M and K respectively. Next we let $S_H = H$. Since H is finitely generated, then by Lemma 3.1, we have $f_H(S_H) = S_H = H$. By assumption, there exists $N_A \lhd_f A$ such that $N_A \cap M = f_M(S_M) = S_M$, $N_A \cap H = f_H(S_H) = H$ and $N_AM \cap N_AH = N_A$. Similarly there exists $N_B \lhd_f B$ such that $N_B \cap K = f_K(S_K) = S_K$, $N_B \cap H = f_H(S_H) = H$ and $N_BH \cap N_BK = N_B$. Since $N_A \cap H = H = N_B \cap H$, we can form $\overline{G} = A/N_A * B/N_B$. Clearly $\overline{G}$ is a homomorphic image of G. Let $\psi : G \to \overline{G}$ be the natural map from G to $\overline{G}$.

We first show that Ker $\psi \cap (M * K) \subseteq f_{M*K}(S)$. Let g be any element with the smallest length $\|g\|$ such that $g \in$ Ker $\psi \cap (M * K)$ but $g \notin f_{M*K}(S)$. Without loss of generality let $g = m_1k_1 \cdots m_nk_n$ where $m_i \in M, k_i \in K$. Then $\overline{g} = \overline{m_1k_1 \cdots m_nk_n} = 1$. Hence $\overline{m_i} = 1$ or $\overline{k_i} = 1$ for some i. Without loss of generality let $\overline{m_i} = 1$. Then $m_i \in N_A$ since $\overline{M} = MN_A/N_A$. Hence $m_i \in N_A \cap M = S_M \subseteq f_{M*K}(S)$. Now $g = m_1k_1 \cdots m_ik_i \cdots m_nk_n = (m_1k_1 \cdots m_{i-1}k_{i-1})m_i(m_1k_1 \cdots m_{i-1}k_{i-1})^{-1}(m_1k_1 \cdots m_{i-1}k_{i-1})k_i \cdots m_nk_n$. But $(m_1k_1 \cdots m_{i-1}k_{i-1})m_i(m_1k_1 \cdots m_{i-1}k_{i-1})^{-1} \in f_{M*K}(S)$. This implies that $g_1 = (m_1k_1 \cdots m_{i-1}k_{i-1})k_i \cdots m_nk_n \notin f_{M*K}(S)$. But $\overline{g_1} = \overline{g} = 1$ since $\overline{m_i} = 1$. Therefore $g_1 \in$ Ker $\psi \cap (M * K)$. But $\|g_1\| < \|g\|$, which is a contradiction. Thus Ker $\psi \cap (M * K) \subseteq f_{M*K}(S)$.

Next we show $\overline{f_{M*K}(S)} \cap \overline{M} = 1$ and $\overline{f_{M*K}(S)} \cap \overline{K} = 1$. Let $\overline{y} \in \overline{f_{M*K}(S)} \cap \overline{M}$. Then $\overline{y} = \overline{s} = \overline{m}$ where $s \in f_{M*K}(S)$ and $m \in M$. Hence $\overline{sm^{-1}} = 1$ which implies that $sm^{-1} \in$ Ker $\psi \cap (M * K) \subseteq f_{M*K}(S)$. Therefore $m \in f_{M*K}(S) \cap M \subseteq N_A$. This implies that $\overline{m} = 1$ and hence $\overline{y} = 1$. Hence $\overline{f_{M*K}(S)} \cap \overline{M} = 1$. Similarly $\overline{f_{M*K}(S)} \cap \overline{K} = 1$.

Now $\overline{G} = A/N_A * B/N_B = A/N_A \underset{\overline{M}}{*} (\overline{M} * \overline{K}) \underset{\overline{K}}{*} B/N_B$. Since $\overline{f_{M*K}(S)} \cap \overline{M} = 1$, then $\overline{\overline{M}} = \overline{M}\,\overline{f_{M*K}(S)}/\overline{f_{M*K}(S)} \simeq \overline{M}/\overline{f_{M*K}(S)} \cap \overline{M} \simeq \overline{M}$. Similarly $\overline{f_{M*K}(S)} \cap \overline{K} = 1$ implies that $\overline{\overline{K}} = \overline{K}\,\overline{f_{M*K}(S)}/\overline{f_{M*K}(S)} \simeq \overline{K}/\overline{f_{M*K}(S)} \cap \overline{K} \simeq \overline{K}$. Thus we can form $\overline{\overline{G}} = A/N_A \underset{\overline{\overline{M}}}{*} (\overline{M} * \overline{K})/\overline{f_{M*K}(S)} \underset{\overline{\overline{K}}}{*} B/N_B$. Clearly $\overline{\overline{G}}$ is a homomorphic image of $\overline{G}$ and hence $\overline{\overline{G}}$ is a homomorphic image of G.

Since $(\overline{M} * \overline{K})/\overline{f_{M*K}(S)}$ is finite and $\overline{\overline{G}}$ is residually finite by Theorem 2.2, then there exists $\overline{\overline{N}} \lhd_f \overline{\overline{G}}$ such that $\overline{\overline{N}} \cap ((\overline{M} * \overline{K})/\overline{f_{M*K}(S)}) = \{1\}$. Let N be the preimage of $\overline{\overline{N}}$. Clearly $f_{M*K}(S) \subseteq N \cap (M * K)$. Let $g_2 \in N \cap (M * K)$.

Since $\overline{\overline{g_2}} = 1$, we have $\overline{g_2} \in \overline{f_{M*K}(S)}$. Let $\overline{g_2} = \overline{t}$ where $t \in f_{M*K}(S)$. Hence $\overline{g_2}\overline{t}^{-1} = 1$ which implies that $g_2t^{-1} \in \text{Ker } \psi \cap (M * K) \subseteq f_{M*K}(S)$. Thus $g_2 \in f_{M*K}(S)$ and $N \cap (M * K) = f_{M*K}(S)$. □

Lemma 4.5. *Let $\{A_i\}$, $i = 1, 2, \ldots, n$, be groups and H_{i-1}, H_i be finitely generated subgroups of A_i with $H_{i-1} \cap H_i = \{1\}$. Suppose each A_i and H_i satisfy the hypothesis of Lemma 4.3. Let $E_n = A_1 \underset{H_1}{*} \cdots \underset{H_{n-1}}{*} A_n$. Then for each $S \lhd_f (H_0 * H_n)$, there exists $N_{E_n} \lhd_f E_n$ such that $N_{E_n} \cap (H_0 * H_n) = f_{H_0 * H_n}(S)$.*

Proof. We prove by induction on n. The case $n = 2$ follows from Lemma 4.4. Let $n \geq 3$. Then $E_n = E_{n-1} \underset{H_{n-1}}{*} A_n$ as in Lemma 3.8. Let $S \lhd_f (H_0 * H_n)$ be given. Since $H_0 * H_n$ is finitely generated, then by Lemma 3.1, there exists a subgroup $f_{H_0*H_n}(S) \subseteq S$ such that $f_{H_0*H_n}(S)$ is characteristic in $H_0 * H_n$. Let $S_0 = f_{H_0*H_n}(S) \cap H_0$ and $S_n = f_{H_0*H_n}(S) \cap H_n$. Then S_0 and S_n are characteristic subgroups of H_0 and H_n respectively. Next we let $S_{n-1} = H_{n-1}$. By Lemma 3.8, there exists $N_{E_{n-1}} \lhd_f E_{n-1}$ such that $N_{E_{n-1}} \cap H_0 = f_{H_0}(S_0)$, $N_{E_{n-1}} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{E_{n-1}}H_0 \cap N_{E_{n-1}}H_{n-1} = N_{E_{n-1}}$. By assumption, there exists $N_{A_n} \lhd_f A_n$ such that $N_{A_n} \cap H_n = f_{H_n}(S_n)$, $N_{A_n} \cap H_{n-1} = f_{H_{n-1}}(S_{n-1})$ and $N_{A_n}H_{n-1} \cap N_{A_n}H_n = N_{A_n}$. Hence by Lemma 4.4, there exists $N_{E_n} \lhd_f E_n$ such that $N_{E_n} \cap (H_0 * H_n) = f_{H_0*H_n}(S)$. This completes the proof. □

Theorem 4.6. *Let $\{A_i\}$, $i = 0, 1, \ldots, n$, be subgroup separable groups and H_{i-1}, H_i be finitely generated normal subgroups of A_i with $H_{i-1} \cap H_i = \{1\}$ and $H_{-1} = H_n$. Let G be the polygonal product of $A_1, A_2, \ldots, A_n$ amalgamating the subgroups $H_0, H_1, \ldots, H_n$. Then G is π_c.*

Proof. Let $E = A_1 \underset{H_1}{*} A_2 \underset{H_2}{*} \cdots \underset{H_{n-2}}{*} A_{n-1}$ and $F = A_0 \underset{H_n}{*} A_n$. By Theorem 3.11, E and F are π_c. By Lemma 4.3, E and F are $H_0 * H_n$-separable. By Lemma 4.5, for any $S \lhd_f (H_0 * H_n)$, there exist $N_F \lhd_f F, N_E \lhd_f E$ such that $N_F \cap (H_0 * H_n) = N_E \cap (H_0 * H_n) \subseteq S$. The result now follows from Theorem 2.3. □

Since polycyclic-by-finite groups and free-by-finite groups are subgroup separable, from Theorem 4.6, we have the following two corollaries.

Corollary 4.7. *Let G be a polygonal product of polycyclic-by-finite or free-by-finite groups amalgamating finitely generated normal subgroups with trivial intersections. Then G is π_c.*

Corollary 4.8. (Kim [7]) *Let G be a polygonal product of finitely generated abelian groups amalgamating subgroups with trivial intersections. Then G is π_c.*

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