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A CERTAIN PROPERTY OF POLYNOMIALS AND THE CI-STABILITY OF TANGENT BUNDLE OVER PROJECTIVE SPACES

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Abstract. We determine the largest integer i such that $0 < i \leq n$ and the coefficient of t^i is odd in the polynomial $(1 + t + t^2 + \cdots + t^n)^{n+1}$. We apply this to prove that the co-index of the tangent bundle over FP^n is stable if $2^r \leq n < 2^r + \frac{1}{3}(2^r - 2)$ for some integer r.

1. Introduction

Let α be a finite-dimensional real vector bundle over a finite complex B, and let $S(\alpha)$ be its sphere bundle equipped with a $\mathbb{Z}_2$-action by the antipodal map on each fibre. The co-index of α, denoted co-ind α, is defined to be the smallest integer k for which there exists a $\mathbb{Z}_2$-map from $S(\alpha)$ to S^{k-1} [1, 2, 4]. Here, S^{k-1} also is equipped with a $\mathbb{Z}_2$-action by the antipodal map. By the Borsuk-Ulam theorem, co-ind α is equal to $\dim\alpha$ if α is a trivial bundle. We have the inequality co-ind $\alpha \leq$ co-ind$(\alpha \oplus 1) \leq$ co-ind $\alpha + 1$. We describe α as *CI-stable* if the equality co-ind$(\alpha \oplus k) =$ co-ind $\alpha + k$ holds for any positive integer k. Here, we abuse notation and denote the k-dimensional trivial bundle simply by k. Our definition of the stability is slightly different from that in [1] in the sense that we consider the fibrewise suspension. It is obvious that a trivial bundle is CI-stable.

In this paper, we prove the following theorem.

Theorem 1.1. *Let τ_n be the tangent bundle over the projective space FP^n ($F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) and let $n = 2^r + k$ with $0 \leq k < 2^r$. If $0 \leq k < \frac{1}{3}(2^r - 2)$, then τ_n is CI-stable.*

The proof of this theorem is given by using the Stiefel-Whitney classes and the following theorem.

Theorem 1.2. *In $\mathbb{Z}_2[t]/(t^{n+1})$, the truncated polynomial algebra over $\mathbb{Z}_2$, the degree of $(1 + t + t^2 + \cdots + t^n)^{n+1}$ is equal to $2^r - k - 1$ where $n = 2^r + k$ and $0 \leq k < 2^r$.*

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2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, for $i = 0, 1, 2, \dots,$ we define power series $f_i \in \mathbb{Z}_2[[t]]$ as follows.

$$f_0 \equiv f_0(t) := 1 + t + t^2 + \cdots + t^n + \cdots$$
$$f_i \equiv f_i(t) := f_0(t^{2^i}) \quad (i = 1, 2, 3, \dots).$$

Lemma 2.1. *We have the following equalities.*

(1) $\quad f_{i+1} = {f_i}^2.$

(2) $\quad f_i = {f_0}^{2^i}.$

Proof. In $\mathbb{Z}_2[[t]]$, we have the following relation in general.

$$(1 + u + u^2 + \cdots + u^n + \cdots)^2 = 1 + u^2 + u^4 + \cdots + u^{2n} + \cdots .$$

Putting $u = t^{2^i}$, we obtain ${f_i}^2 = f_{i+1}$, which is the equality (1). The equality (2) follows immediately from (1). □

Next, we define polynomials $G(i,j) \in \mathbb{Z}_2[t]$ by

$$G(i,j) := 1 + t^{2^i} + (t^{2^i})^2 + (t^{2^i})^3 + \cdots + (t^{2^i})^{2^{j-i}-1}$$

for non-negative integers i and j with $i \le j$. We note that $\deg G(i,j) = 2^j - 2^i$ and also note that $G(i,i) = 1$.

Lemma 2.2. *In $\mathbb{Z}_2[[t]]$, we have the following formulas for all $i \le j$.*

(1) $\quad f_i = G(i,j) f_j.$

(2) $\quad f_i f_j = G(i,j) f_{j+1}.$

Proof. In $\mathbb{Z}_2[[t]]$, we have the following relation in general.

$$(1+u+u^2+\cdots+u^{\ell-1})(1+u^\ell+u^{2\ell}+\cdots+u^{n\ell}+\cdots) = 1+u+u^2+\cdots+u^n+\cdots .$$

Putting $u = t^{2^i}$ and $\ell = 2^{j-i}$, we obtain $G(i,j)f_j = f_i$, which is the formula (1). Using this and Lemma 2.1, we have $f_i \cdot f_j = G(i,j)f_j \cdot f_j = G(i,j){f_j}^2 = G(i,j)f_{j+1}$ and the formula (2) follows. □

Proof of Theorem 1.2. Let $n = 2^r + k$ with $0 \le k < 2^r$. We express $k+1$ as $k + 1 = 2^{i_1} + 2^{i_2} + 2^{i_3} + \cdots + 2^{i_s}$ where $0 \le i_1 < i_2 < i_3 < \cdots < i_s \le r$. Then, in $\mathbb{Z}_2[[t]]$, we have

$$\begin{aligned}(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} &= {f_0}^{2^{i_1}+2^{i_2}+2^{i_3}+\cdots+2^{i_s}+2^r} \\ &= f_{i_1} f_{i_2} f_{i_3} \cdots f_{i_s} f_r\end{aligned}$$

from the formula (2) of Lemma 2.1. This is equal to $G(i_1, i_2) f_{i_2+1} f_{i_3} \cdots f_{i_s} f_r$ from the formula (2) of Lemma 2.2. And so, the repeated use of the formula (2) of Lemma 2.2 yields :

$$\begin{aligned}&(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} \\ &= G(i_1, i_2) G(i_2 + 1, i_3) G(i_3 + 1, i_4) \cdots G(i_s + 1, r) f_{r+1}.\end{aligned}$$

Here, the degree of $G(i_1, i_2)G(i_2+1, i_3)G(i_3+1, i_4)\cdots G(i_s+1, r)$ is equal to :

$$\begin{aligned} & (2^{i_2} - 2^{i_1}) + (2^{i_3} - 2^{i_2+1}) + (2^{i_4} - 2^{i_3+1}) + \cdots + (2^r - 2^{i_s+1}) \\ = & -2^{i_1} - 2^{i_2} - 2^{i_3} - \cdots - 2^{i_s} + 2^r \\ = & 2^r - k - 1. \end{aligned}$$

On the other hand, f_{r+1} is of the form

$$f_{r+1} = 1 + t^{2^{r+1}} + \text{higher terms}$$

and $n < 2^{r+1}$. Hence, in $\mathbb{Z}_2[t]/(t^{n+1})$, we have

$$\begin{aligned} & (1 + t + t^2 + \cdots + t^n)^{n+1} \\ = & G(i_1, i_2)G(i_2+1, i_3)G(i_3+1, i_4)\cdots G(i_s+1, r), \end{aligned}$$

the degree of which is equal to $2^r - k - 1$. This completes the proof. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let α be a finite-dimensional real vector bundle over a finite complex B. For a virtual vector bundle γ, we write g-dim γ for its geometric dimension, that is, the smallest integer ℓ such that there exists an ℓ-dimensional vector bundle β which represents the same stable class as γ. Now, let $\dim \alpha = m$ and suppose $m \geq \dim B$.

First, we put g-dim$(-\alpha) = \ell$ and assume $\ell > 0$. By the definition of geometric dimension, there exists an ℓ-dimensional vector bundle β such that $\alpha \oplus \beta$ is stably trivial. Then, by the stability theorem, we have $\alpha \oplus \beta \cong m \oplus \ell$ since $m + \ell \geq \dim B + 1$. The composite of the inclusion $\alpha \hookrightarrow \alpha \oplus \beta$ with this isomorphism gives rise to a $\mathbb{Z}_2$-map $S(\alpha) \to S^{m+\ell-1}$. Hence, we have co-ind $\alpha \leq m + \ell$.

Next, we put co-ind $\alpha = k$. By the definition of co-index, there exists a $\mathbb{Z}_2$-map $f : S(\alpha) \to S^{k-1}$. If $\dim B \leq 2(k-m) - 1$, that is, if $k > m + \frac{1}{2}\dim B$, the map f becomes fibrewise-homotopic to a restriction of some fibrewise-monomorphism $g : \alpha \to \mathbb{R}^k$ [3, Theorem 1.2]. We consider g as the bundle map $g : \alpha \to B \times \mathbb{R}^k$ and let β be the cokernel of g. Then, we have $\alpha \oplus \beta \cong k$. Hence, we have g-dim$(-\alpha) \equiv \ell \leq \dim \beta = k - m$, that is, $k \geq m + \ell$.

From the above two paragraphs, we obtain the following.

Lemma 3.1. [4, Proposition 3.3] *Let α be a real vector bundle over B with $\dim \alpha \geq \dim B$ and suppose that α is not stably trivial. Then, if* co-ind $\alpha >$ $\dim \alpha + \frac{1}{2}\dim B$, *we have* co-ind $\alpha = m +$ g-dim$(-\alpha)$.

For a virtual vector bundle α, we denote by ω-dim α the largest integer k for which the kth Stiefel-Whitney class $w_k(\alpha)$ is not zero. It is clear that ω-dim $\alpha \leq$ g-dim α. As shown in the proof of [5, Theorem 2.5], we have the following.

Lemma 3.2. co-ind $\alpha \geq \dim \alpha + \omega$-dim$(-\alpha)$.

Combining these two lemmas, we obtain the following proposition.

Proposition 3.3. *Let α be a vector bundle over a finite complex B with $\dim\alpha \geq \dim B$, and suppose that the inequality $\omega\text{-dim}(-\alpha) > \frac{1}{2}\dim B$ holds. Then:*

(1) $\text{co-ind}\,\alpha = \dim\alpha + \text{g-dim}(-\alpha)$.
(2) *α is CI-stable.*

In fact, (2) follows from (1) since $\omega\text{-dim}(-\alpha\oplus k) = \omega\text{-dim}(-\alpha)$ and $\text{g-dim}(-\alpha \oplus k) = \text{g-dim}(-\alpha)$ for any positive integer k.

Proof of Theorem 1.1. Let τ_n be the tangent bundle over the projective space FP^n with $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The total Stiefel-Whitney class of τ_n is given by $(1+t)^{n+1}$, where t is the generator of $H^*(FP^n;\mathbb{Z}_2) = \mathbb{Z}_2[t]/(t^{n+1})$. Let $n = 2^r + k$ with $0 \leq k < 2^r$. Then, Theorem 1.2 states that $\omega\text{-dim}(-\tau_n)$ is equal to $d(2^r - k - 1)$, where $d = 1, 2, 4$ according as $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Therefore, by Proposition 3.3, τ_n is CI-stable if $d(2^r - k - 1) > \frac{1}{2}\dim FP^n$. This last inequality is equivalent to $0 \leq k < \frac{1}{3}(2^r - 2)$ and the proof is completed. □

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