A CERTAIN PROPERTY OF POLYNOMIALS AND
THE CI-STABILITY OF TANGENT BUNDLE
OVER PROJECTIVE SPACES

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Abstract. We determine the largest integer \( i \) such that \( 0 < i \leq n \) and the coefficient of \( t^i \) is odd in the polynomial \( (1 + t + t^2 + \cdots + t^n)^{n+1} \). We apply this to prove that the co-index of the tangent bundle over \( FP^n \) is stable if \( 2^r \leq n < 2^r + \frac{1}{3}(2^r - 2) \) for some integer \( r \).

1. Introduction

Let \( \alpha \) be a finite-dimensional real vector bundle over a finite complex \( B \), and let \( S(\alpha) \) be its sphere bundle equipped with a \( \mathbb{Z}_2 \)-action by the antipodal map on each fibre. The co-index of \( \alpha \), denoted \( \text{co-ind} \alpha \), is defined to be the smallest integer \( k \) for which there exists a \( \mathbb{Z}_2 \)-map from \( S(\alpha) \) to \( S^{k-1} \) [1, 2, 4]. Here, \( S^{k-1} \) also is equipped with a \( \mathbb{Z}_2 \)-action by the antipodal map. By the Borsuk-Ulam theorem, \( \text{co-ind} \alpha \) is equal to \( \dim \alpha \) if \( \alpha \) is a trivial bundle. We have the inequality \( \text{co-ind} \alpha \leq \text{co-ind}(\alpha \oplus 1) \leq \text{co-ind} \alpha + 1 \). We describe \( \alpha \) as CI-stable if the equality \( \text{co-ind}(\alpha \oplus k) = \text{co-ind} \alpha + k \) holds for any positive integer \( k \). Here, we abuse notation and denote the \( k \)-dimensional trivial bundle simply by \( k \). Our definition of the stability is slightly different from that in [1] in the sense that we consider the fibrewise suspension. It is obvious that a trivial bundle is CI-stable.

In this paper, we prove the following theorem.

Theorem 1.1. Let \( \tau_n \) be the tangent bundle over the projective space \( FP^n \) \((F = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H})\) and let \( n = 2^r + k \) with \( 0 \leq k < 2^r \). If \( 0 \leq k < \frac{1}{3}(2^r - 2) \), then \( \tau_n \) is CI-stable.

The proof of this theorem is given by using the Stiefel-Whitney classes and the following theorem.

Theorem 1.2. In \( \mathbb{Z}_2[t]/(t^{n+1}) \), the truncated polynomial algebra over \( \mathbb{Z}_2 \), the degree of \((1 + t + t^2 + \cdots + t^n)^{n+1}\) is equal to \( 2^r - k - 1 \) where \( n = 2^r + k \) and \( 0 \leq k < 2^r \).

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2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, for \( i = 0, 1, 2, \ldots \), we define power series \( f_i \in \mathbb{Z}_2[[t]] \) as follows.

\[
f_0 \equiv f_0(t) := 1 + t + t^2 + \cdots + t^n + \cdots \\
f_i \equiv f_i(t) := f_0(t^{2^i}) \quad (i = 1, 2, 3, \ldots).
\]

**Lemma 2.1.** We have the following equalities.

1. \( f_{i+1} = f_i^2 \).
2. \( f_i = f_0^{2^i} \).

**Proof.** In \( \mathbb{Z}_2[[t]] \), we have the following relation in general.

\[
(1 + u + u^2 + \cdots + u^n + \cdots)^2 = 1 + u^2 + u^4 + \cdots + u^{2n} + \cdots.
\]

Putting \( u = t^2 \), we obtain \( f_i^2 = f_{i+1} \), which is the equality (1). The equality (2) follows immediately from (1). \( \square \)

Next, we define polynomials \( G(i, j) \in \mathbb{Z}_2[t] \) by

\[
G(i, j) := 1 + t^{2^i} + (t^{2^i})^2 + (t^{2^i})^3 + \cdots + (t^{2^i})^{2^{i-1}-1}
\]

for non-negative integers \( i \) and \( j \) with \( i \leq j \). We note that \( \deg G(i, j) = 2^j - 2^i \) and also note that \( G(i, i) = 1 \).

**Lemma 2.2.** In \( \mathbb{Z}_2[[t]] \), we have the following formulas for all \( i \leq j \).

1. \( f_i = G(i, j) f_j \).
2. \( f_i f_j = G(i, j) f_{j+1} \).

**Proof.** In \( \mathbb{Z}_2[[t]] \), we have the following relation in general.

\[
(1 + u + u^2 + \cdots + u^\ell - 1)(1 + u^\ell + u^{2\ell} + \cdots + u^{n\ell} + \cdots) = 1 + u + u^2 + \cdots + u^n + \cdots.
\]

Putting \( u = t^2 \) and \( \ell = 2^j - i \), we obtain \( G(i, j) f_j = f_i \), which is the formula (1). Using this and Lemma 2.1, we have \( f_i \cdot f_j = G(i, j) f_j \cdot f_j = G(i, j) f_j^2 = G(i, j) f_{j+1} \) and the formula (2) follows. \( \square \)

**Proof of Theorem 1.2.** Let \( n = 2^r + k \) with \( 0 \leq k < 2^r \). We express \( k + 1 \) as \( k + 1 = 2^{i_1} + 2^{i_2} + 2^{i_3} + \cdots + 2^{i_s} \) where \( 0 \leq i_1 < i_2 < i_3 < \cdots < i_s \leq r \). Then, in \( \mathbb{Z}_2[[t]] \), we have

\[
(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} = f_0^{2^{i_1} + 2^{i_2} + 2^{i_3} + \cdots + 2^{i_s} + 2^r} = f_{i_1} f_{i_2} f_{i_3} \cdots f_{i_s} f_r
\]

from the formula (2) of Lemma 2.1. This is equal to \( G(i_1, i_2) f_{i_2+1} f_{i_3} \cdots f_i f_r \) from the formula (2) of Lemma 2.2. And so, the repeated use of the formula (2) of Lemma 2.2 yields:

\[
(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} = G(i_1, i_2) G(i_2 + 1, i_3) G(i_3 + 1, i_4) \cdots G(i_s + 1, r) f_{r+1}.
\]
Here, the degree of \( G(i_1, i_2)G(i_2 + 1, i_3)G(i_3 + 1, i_4) \cdots G(i_s + 1, r) \) is equal to:
\[
(2^{i_2} - 2^{i_1}) + (2^{i_3} - 2^{i_2 + 1}) + (2^{i_4} - 2^{i_3 + 1}) + \cdots + (2^r - 2^{i_s + 1})
\]
\[
= -2^{i_1} - 2^{i_2} - 2^{i_3} - \cdots - 2^{i_s} + 2^r
\]
\[
= 2^r - k - 1.
\]

On the other hand, \( f_{r+1} \) is of the form
\[
f_{r+1} = 1 + t^{2^{r+1}} + \text{higher terms}
\]
and \( n < 2^{r+1} \). Hence, in \( \mathbb{Z}_2[t]/(t^{n+1}) \), we have
\[
(1 + t + t^2 + \cdots + t^n)^{n+1}
\]
\[
= G(i_1, i_2)G(i_2 + 1, i_3)G(i_3 + 1, i_4) \cdots G(i_s + 1, r),
\]
the degree of which is equal to \( 2^r - k - 1 \). This completes the proof. \( \square \)

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let \( \alpha \) be a finite-dimensional real vector bundle over a finite complex \( B \). For a virtual vector bundle \( \gamma \), we write \( \text{g-dim} \gamma \) for its geometric dimension, that is, the smallest integer \( \ell \) such that there exists an \( \ell \)-dimensional vector bundle \( \beta \) which represents the same stable class as \( \gamma \). Now, let \( \text{dim} \alpha = m \) and suppose \( m \geq \text{dim} B \).

First, we put \( \text{g-dim}(-\alpha) = \ell \) and assume \( \ell > 0 \). By the definition of geometric dimension, there exists an \( \ell \)-dimensional vector bundle \( \beta \) such that \( \alpha \oplus \beta \) is stably trivial. Then, by the stability theorem, we have \( \alpha \oplus \beta \cong m \oplus \ell \) since \( m + \ell \geq \text{dim} B + 1 \). The composite of the inclusion \( \alpha \hookrightarrow \alpha \oplus \beta \) with this isomorphism gives rise to a \( \mathbb{Z}_2 \)-map \( S(\alpha) \to S^{m+\ell-1} \). Hence, we have \( \text{co-ind} \alpha \leq m + \ell \).

Next, we put \( \text{co-ind} \alpha = k \). By the definition of co-index, there exists a \( \mathbb{Z}_2 \)-map \( f : S(\alpha) \to S^{k-1} \). If \( \text{dim} B \leq 2(k-m) - 1 \), that is, if \( k > m + \frac{1}{2} \text{dim} B \), the map \( f \) becomes fibrewise-homotopic to a restriction of some fibrewise-monomorphism \( g : \alpha \to \mathbb{R}^k \) [3, Theorem 1.2]. We consider \( g \) as the bundle map \( g : \alpha \to B \times \mathbb{R}^k \) and let \( \beta \) be the cokernel of \( g \). Then, we have \( \alpha \oplus \beta \cong k \).

Hence, we have \( \text{g-dim}(-\alpha) \equiv \ell \leq \text{dim} \beta = k - m \), that is, \( k \geq m + \ell \).

From the above two paragraphs, we obtain the following.

**Lemma 3.1.** [4, Proposition 3.3] Let \( \alpha \) be a real vector bundle over \( B \) with \( \text{dim} \alpha \geq \text{dim} B \) and suppose that \( \alpha \) is not stably trivial. Then, if \( \text{co-ind} \alpha > \text{dim} \alpha + \frac{1}{2} \text{dim} B \), we have \( \text{co-ind} \alpha = m + \text{g-dim}(-\alpha) \).

For a virtual vector bundle \( \alpha \), we denote by \( \omega \)-dim \( \alpha \) the largest integer \( k \) for which the \( k \)th Stiefel-Whitney class \( w_k(\alpha) \) is not zero. It is clear that \( \omega \)-dim \( \alpha \leq \text{g-dim} \alpha \). As shown in the proof of [5, Theorem 2.5], we have the following.

**Lemma 3.2.** \( \text{co-ind} \alpha \geq \text{dim} \alpha + \omega \text{-dim}(-\alpha) \).

Combining these two lemmas, we obtain the following proposition.
Proposition 3.3. Let $\alpha$ be a vector bundle over a finite complex $B$ with \( \dim \alpha \geq \dim B \), and suppose that the inequality \( \omega \dim (-\alpha) > \frac{1}{2} \dim B \) holds. Then:

1. \( \coind \alpha = \dim \alpha + g \dim (-\alpha) \).
2. $\alpha$ is CI-stable.

In fact, (2) follows from (1) since $\omega \dim (-\alpha \oplus k) = \omega \dim (-\alpha)$ and $g \dim (-\alpha \oplus k) = g \dim (-\alpha)$ for any positive integer $k$.

Proof of Theorem 1.1. Let $\tau_n$ be the tangent bundle over the projective space $FP^n$ with $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The total Stiefel-Whitney class of $\tau_n$ is given by $(1 + t)^{n+1}$, where $t$ is the generator of $H^*(FP^n; \mathbb{Z}_2) = \mathbb{Z}_2[t]/(t^{n+1})$. Let $n = 2^r + k$ with $0 \leq k < 2^r$. Then, Theorem 1.2 states that $\omega \dim (-\tau_n)$ is equal to $d(2^r - k - 1)$, where $d = 1, 2, 4$ according as $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Therefore, by Proposition 3.3, $\tau_n$ is CI-stable if $d(2^r - k - 1) > \frac{1}{2} \dim FP^n$. This last inequality is equivalent to $0 \leq k < \frac{1}{3}(2^r - 2)$ and the proof is completed. $\square$

References


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