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SOME RESULTS ON NON-ASSOCIATIVE ALGEBRAS

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Abstract. We define the non-associative algebra $\overline{W(n,m,m+s)}$ and we show that it is simple. We find the non-associative algebra automorphism group $\mathrm{Aut}_{non}(\overline{W(1,0,0)})$ of $\overline{W(1,0,0)}$. Also we find that any derivation of $\overline{W(1,0,0)}$ is a scalar derivation in this paper.

1. Preliminaries

Let $\mathbf{N}$ be the set of all non-negative integers and $\mathbf{Z}$ be the set of all integers. Let $\mathbf{F}$ be a field of characteristic zero. Let $\mathbf{F}^\bullet$ be the multiplicative group of non-zero elements of $\mathbf{F}$. The non-associative algebra $\overline{W(n,m,m+s)}$ is the vector space spanned by

$$\{e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_m^{i_m}x_{m+1}^{i_{m+1}}\cdots x_{m+s}^{i_{m+s}}\partial_u|a_1,\ldots,a_n,i_1,\ldots,i_m\in\mathbf{Z},$$
$$i_{m+1},\ldots,i_{m+s}\in\mathbf{N}\}$$

with the obvious addition and the multiplication $*$ where ∂_u is the usual partial derivative with respect to x_u, $1\le u\le\max\{n,m+s\}$ [2], [8], [9]. The non-associative algebra $\overline{W(n,m,m+s)}$ is a subalgebra of the algebra in the papers [2], [3], [8]. For an element l in an algebra A, l is full, if an ideal containing l is A. The matrix ring $M_{m+s}(\mathbf{F})$ is imbedded in the non-associative algebra $\overline{W(n,m,m+s)}$. The matrix ring $M_n(\mathbf{F})$ is not imbedded in $\overline{W(n,0,0)}$. The non-associative algebra $\overline{W(n,0,0)}$ has neither a right nor a left multiplicative identity element. Note that the definition of a non-associative algebra in this paper is a little different from the definition of the non-associative algebras in the papers [2], [8], [9], because of some results. Similarly to the non-associative algebra $\overline{W(n,m,m+s)}$, we can define the non-associative algebra $\overline{W(n^+,0,s)}$ spanned by $\{e^{a_1x_1}\cdots e^{a_nx_n}x_{n+1}^{i_{n+1}}\cdots x_{n+s}^{i_{n+s}}\partial_u|a_1,\ldots,a_n\in\mathbf{Z},i_{n+1},\ldots,i_{n+s}\in$

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$\mathbf{N}, 1 \leq u \leq n+m\}$. The non-associative algebra $\overline{W(n,m,m+s)}$ is Lie-admissible, since $\overline{W(n,m,m+s)}_{[,]}$ is a Lie algebra with respect to the commutator $[,]$ of $\overline{W(n,m,m+s)}$. The non-associative algebra $\overline{W(n,m,m+s)}$ has idempotents.

2. Simplicity of $\overline{W(n,m,m+s)}$

Even if the non-associative algebra $\overline{W(n,m,m+s)}$ has right annihilators, we have the following results.

Remark 1. An (non-associative, Lie, or associative) algebra A is simple if and only if every element of the (non-associative, Lie, or associative) algebra A is full.

Lemma 1. *For any ∂_u, $1 \leq u \leq m+s$, in the non-associative algebra $\overline{W(n,m,m+s)}$, ∂_u is full.*

Proof. Let I be a non-zero ideal of the non-associative algebra $\overline{W(n,m,m+s)}$ which contains ∂_u in the lemma. For any basis element $e^{a_1x_1}\cdots e^{a_nx_n}\partial_v$ of $\overline{W(n,m,m+s)}$ with $a_u \neq 0$,

$$\partial_u * e^{a_1x_1}\cdots e^{a_nx_n}\partial_v = a_u e^{a_1x_1}\cdots e^{a_nx_n}\partial_v \in I$$

This implies that by appropriate inductions on $i_1, \ldots, i_{m+s}$ of $e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{m+s}^{i_{m+s}}\partial_v$, we can prove that $e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{m+s}^{i_{m+s}}\partial_v \in I$. This implies that $\overline{W(n,m,m+s)} \subset I$, i.e., $\overline{W(n,m,m+s)} = I$. This implies that ∂_u is full. Therefore we have proven the lemma. □

Theorem 1. *The non-associative algebra $\overline{W(n,m,m+s)}$ is simple.*

Proof. Let I be a non-zero ideal of the non-associative algebra

$$\overline{W(n,m,m+s)}.$$

Without loss of generality, we can assume that $n \leq m+s$. By Lemma 1, we know that ∂_u, $1 \leq u \leq m+n$, is full. It is standard to prove that $\partial_u \in I$. By Remark 1 and Lemma 1, this completes the proof of the theorem. □

Corollary 1. *The non-associative algebra $\overline{W(n,m,m+s)}$ is simple.*

Proof. The proof of the corollary is straightforward by Theorem 1. Thus the proof is omitted. □

Theorem 2. *The Lie algebra $\overline{W(n,m,m+s)}_{[,]}$ is simple.*

Proof. Since every element of the Lie algebra $\overline{W(n,m,m+s)}_{[,]}$ is full, the proof of the theorem is straightforward by Theorem 1. So let omit it. □

Corollary 2. *The Lie algebras $\overline{W(n^+,0,s)}_{[,]}$ and $\overline{W(0,m,m+s)}_{[,]}$ are simple.*

Proof. The proof of the corollary is straightforward by Theorem 2, so omitted. □

The Lie algebra $\overline{W(n,m,m+s)}_{[,]}$ is called the Witt type Lie algebra [12]. The Lie algebra $\overline{W(1,0,0)}_{[,]}$ is the well known centerless Virasoro algebra [7]. It is easy to prove that the non-associative algebra $\overline{W(n^+,0,s)}$ is simple.

3. Automorphism group $\mathrm{Aut}_{non}(\overline{W(1,0,0)})$

Note that by Corollary 1, the non-associative algebra $\overline{W(1,0,0)}$ span-ned by $\{e^{ax}\partial | a \in \mathbf{Z}\}$ is simple.

Example 1. The Lie algebra $sl_2(\mathbf{F})$ is isomorphic to the Lie subalgebra of $\overline{W(0,1,0)}_{[,]}$ (resp.$\overline{W(1,0,0)}_{[,]}$) spanned by $\{x^{k+2}\partial, x\partial, x^{-k}\partial\}$ (resp. $\{e^{-ax}\partial$, ∂, $e^{ax}\partial\}$) where $k, a \in \mathbf{N}$.

Proposition 1. *For any non-associative algebra endomorphism θ of $\overline{W(1,0,0)}$, if θ is non-zero, then θ is injective.*

Proof. Let θ be a non-associative algebra endomorphism θ of

$$\overline{W(1,0,0)}.$$

$\mathrm{Ker}(\theta)$ is an ideal of $\overline{W(1,0,0)}$. By Corollary 1, either $\mathrm{Ker}(\theta) = 0$ holds or $\mathrm{Ker}(\theta) = \overline{W(1,0,0)}$ holds. Since θ is not the zero map, $\mathrm{Ker}(\theta) = 0$. This implies that θ is injective. So we have proven the proposition. □

Note 1. For any basis element $e^{ax}\partial$ of $\overline{W(1,0,0)}$, if we define **F**-linear maps θ_{+,d_1} and $\theta_{-,d_2}$of $\overline{W(1,0,0)}$), as follows:

$$\theta_{+,d_1}(e^{(k)x}\partial) = d_1^k e^{(k)x}\partial$$

and

$$\theta_{-,d_2}(e^{kx}\partial) = d_2^k e^{-kx}\partial$$

then θ_{+,d_1} and θ_{-,d_2} can be linearly extended to non-associative algebra automorphisms of $\overline{W(1,0,0)}$ where $d_1, d_2 \in \mathbf{F}^{\bullet}$.

Lemma 2. *For any non-associative algebra automorphism θ of*

$$\overline{W(1,0,0)},$$

$\theta(\partial) = c\partial$ holds where c is a non-zero scalar.

Proof. Let θ be the non-associative algebra automorphism θ of

$$\overline{W(1,0,0)}$$

in the lemma. Since ∂ is a basis element of the right annihilator of $\overline{W(1,0,0)}$, ∂ is invariant under any automorphism of $\overline{W(1,0,0)}$. This implies that $\theta(\partial) = c\partial$ holds where c is a non-zero scalar. □

Lemma 3. *For any θ in the non-associative algebra automorphism group* $\mathrm{Aut}_{non}(\overline{W(1,0,0)})$ *of* $\overline{W(1,0,0)}$, *θ is either θ_{+,d_1} or θ_{-,d_2} in Note 1 where* $d_1, d_2 \in \mathbf{F}^{\bullet}$.

Proof. Let θ be the non-associative algebra automorphism of

$$\overline{W(1,0,0)}$$

in the lemma. By Lemma 2, $\theta(\partial) = c\partial$ holds where c is a non-zero scalar. By Lemma 2 and since ∂ is a left identity of $e^x\partial$, we have that

$$c\partial * \theta(e^x\partial) = \theta(e^x\partial). \tag{1}$$

This implies that $\theta(e^x\partial)$ can be written as follows:

$$\theta(e^x\partial) = C(b_1)e^{b_1x}\partial + \cdots + C(b_t)e^{b_tx}\partial, \tag{2}$$

where $C(b_1), \ldots, C(b_t) \in \mathbf{F}$ and $b_1 > \cdots > b_t$. By (1) and (2), we have that $cb_1 = 1$. This implies that either $c = b_1 = 1$ holds or $c = b_1 = -1$ holds.

Case I. Let us assume that $c = b_1 = 1$ holds. Let us put $\theta(\partial) = \partial$ and $\theta(e^x\partial) = d_1e^x\partial$ where $d_1 \in \mathbf{F}^{\bullet}$. By $\theta(e^{-x}\partial * e^x\partial) = \partial$, we have that $\theta(e^{-x}\partial) * d_1e^x\partial = \partial$. This implies that

$$\theta(e^{-x}\partial) = d_1^{-1}e^{-x}\partial. \tag{3}$$

By $\theta(e^x\partial * e^x\partial) = e^{2x}\partial$, we have that

$$\theta(e^{2x}\partial) = d_1^2e^{2x}\partial. \tag{4}$$

By (3) and (4), we may assume that $\theta(e^{kx}\partial) = d_1^ke^{kx}\partial$ holds by induction on $k \in \mathbf{N}$ of $e^{kx}\partial$. By $\theta(e^x\partial * e^{kx}\partial) = ke^{(k+1)x}\partial$, we have that $\theta(e^{(k+1)x}\partial) = d_1^{k+1}e^{(k+1)x}\partial$. This proves that $\theta(e^{kx}\partial) = d_1^ke^{kx}\partial$ holds for any $k \in \mathbf{N}$. Symmetrically, we can prove that

$$\theta(e^{kx}\partial) = d_1^ke^{kx}\partial \tag{5}$$

holds for any negative integer k by (3). This implies that θ is the non-associative algebra automorphism θ_{+,d_1} which is defined in Note 1.

Case II. Let us assume that $c = -1$ and $b_1 = -1$ hold. Let us put $\theta(\partial) = -\partial$ and $\theta(e^x\partial) = d_2e^{-x}\partial$ where $d_2 \in \mathbf{F}^{\bullet}$. By $\theta(e^x\partial * e^x\partial) = e^{2x}\partial$, we have that

$$\theta(e^{2x}\partial) = d_2^2e^{-2x}\partial. \tag{6}$$

By induction on $k \in \mathbf{N}$ of $e^{kx}\partial$, we can prove that

$$\theta(e^{kx}\partial) = d_2^ke^{-kx}\partial. \tag{7}$$

By $\theta(e^{-x}\partial * e^x\partial) = \partial$, we have that $\theta(e^{-x}\partial) * d_2e^{-x}\partial) = -\partial$. This implies that $\theta(e^{-x}\partial) = d_2^{-1}e^x\partial$. By induction on $k \in \mathbf{N}$ of $e^{kx}\partial$, we can prove that

$$\theta(e^{-kx}\partial) = d_2^{-k}e^{kx}\partial. \tag{8}$$

This implies that θ is the non-associative algebra automorphism θ_{-,d_2} which is defined in Note 1. By Case I and Case II, we have proven the lemma. $\square$

Theorem 3. *The non-associative algebra automorphism group*

$$\mathrm{Aut}_{non}(\overline{W(1,0,0)})$$

of $\overline{W(1,0,0)}$ *is generated by* θ_{+,d_1} *and* θ_{-,d_2} *which are defined in Note 1 where* $d_1, d_2 \in \mathbf{F}^{\bullet}$.

Proof. Let θ be the non-associative algebra automorphism of

$$\overline{W(1,0,0)}.$$

By Lemma 3, θ is either θ_{+,d_1} or θ_{-,d_2} where $d_1, d_2 \in \mathbf{F}^{\bullet}$. So

$$\mathrm{Aut}_{non}(\overline{W(1,0,0)})$$

of $\overline{W(1,0,0)}$ is generated by θ_{+,d_1} and θ_{-,d_2}. Therefore we have proven the theorem. □

Corollary 3. *The non-associative algebra automorphism group*

$$\mathrm{Aut}_{non}(\overline{W(1,0,0)})$$

of the non-associative algebra $\overline{W(1,0,0)}$ *is a non-abelian group.*

Proof. By Theorem 3, the non-associative algebra automorphism group

$$\mathrm{Aut}_{non}(\overline{W(1,0,0)})$$

of the non-associative algebra $\overline{W(1,0,0)}$ is generated by θ_{+,d_1} and θ_{-,d_2} where $d_1, d_2 \in \mathbf{F}^{\bullet}$. Thus it is enough to check that $\theta_{+,d_1} \circ \theta_{-,d_2} \neq \theta_{-,d_2} \circ \theta_{+,d_1}$ where $\circ$ is the composition of the non-associative algebra automorphisms θ_{+,d_1} and θ_{-,d_2}. But it is trivial to check the inequality by taking some basis element of the non-associative algebra $\overline{W(1,0,0)}$. So let omit the remaining steps of its proof. □

Proposition 2. *The non-associative algebra* $\overline{W(1,0,0)}$ *is not isomorphic to the non-associative algebra* $\overline{W(0,1,0)}$ *as non-associative algebras.*

Proof. Since the non-associative algebra $\overline{W(0,1,0)}$ has a right identity and the non-associative algebra $\overline{W(1,0,0)}$ does not have a right identity, the proof of the proposition is straightforward. So it is omitted. □

Proposition 3. *The Lie algebra* $\overline{W(n^+,0,n+s)}_{[,]}$ *is isomorphic to the Lie algebra* $\overline{W(0,n,n+s)}_{[,]}$ *as Lie algebras. The non-associative algebra*

$$\overline{W(n^+,0,n+s)}$$

is not isomorphic to the non-associative algebra $\overline{W(0,n,n+s)}$ *as non-associative algebras.*

Proof. It is standard to find isomorphisms between appropriate algebras, so the proof of the proposition is omitted. □

Proposition 4. *The Lie algebra $\overline{W(1,0,0)}_{[,]}$ (resp. the non-associa-tive algebra $\overline{W(1,1,0)}_{[,]}$) does not hold its Jacobian conjecture.*

Proof. It is easy to define a non-zero endomorphism θ of $\overline{W(1,0,0)}_{[,]}$ (resp. $\overline{W(1,1,0)}_{[,]}$) which is not surjective. This completes its proof. □

Proposition 3 shows that there are non-isomorphic two non-associa-tive algebras whose corresponding Lie algebras (i.e., using the commutators of them) are isomorphic. This fact is one of the reasons to study non-associative algebras.

4. Derivations of $\overline{W(1,0,0)}$

Note that the $\mathbf{F}$-algebra $\mathbf{F}[x,x^{-1}]$ is isomorphic to the $\mathbf{F}$-algebra $\mathbf{F}[e^{\pm x}]$ as $\mathbf{F}$-algebras. Let A be an $\mathbf{F}$-algebra. An additive $\mathbf{F}$-map D from A to itself is a derivation if $D(l_1 * l_2) = D(l_1) * l_2 + l_1 * D(l_2)$ for any $l_1, l_2 \in A$.

Note 2. For any basis element $e^{kx}\partial$ of the non-associative algebra $\overline{W(1,0,0)}$, if we define an $\mathbf{F}$-additive linear map D_c of the non-associative algebra $\overline{W(1,0,0)}$ as follows:

$$D_c(e^{kx}\partial) = cke^{kx}\partial$$

then D_c can be linearly extended to a derivation of the non-associative algebra $\overline{W(1,0,0)}$ where $c \in \mathbf{F}$.

Lemma 4. *For any derivation D of the non-associative algebra*

$$\overline{W(1,0,0)},$$

if $D(\partial) = 0$, then D is the derivation D_c which is defined in Note 2.

Proof. Let D be the derivation of the non-associative algebra

$$\overline{W(1,0,0)}$$

in the lemma. Since ∂ is a left identity of $e^x\partial$, we have that $D(\partial) * e^x\partial + \partial * D(e^x\partial) = D(e^x\partial)$, i.e., $\partial * D(e^x\partial) = D(e^x\partial)$ by assumption. This implies that

$$D(e^x\partial) = ce^x\partial \tag{9}$$

for $c \in \mathbf{F}$. We have two cases $c = 0$ or $c \neq 0$.

Case I. Let us assume that $c = 0$. By (9), we have that $D(e^x\partial) = 0$. By $D(e^x\partial * e^x\partial) = D(e^{2x}\partial)$, we have that $D(e^x\partial) * e^x\partial + e^x\partial * D(e^x\partial) = D(e^{2x}\partial)$. This implies that $D(e^{2x}\partial) = 0$. By induction on $k \in \mathbf{N}$ of $e^{kx}\partial$, we can prove that

$$D(e^{kx}\partial) = 0. \tag{10}$$

For any $k \in \mathbf{N}$, we have that $D(e^{-kx}\partial) * e^{(k+1)x}\partial = 0$ by (10). Since the left annihilator of $e^{(k+1)x}\partial$ is zero, this implies that $D(e^{kx}\partial) = 0$ holds for any negative integers. This implies that D is the zero map of the non-associative algebra $\overline{W(1,0,0)}$.

Case II. Let us assume that $c \neq 0$. By (9), we have that $D(e^x\partial) = ce^x\partial$. By $D(e^x\partial * e^x\partial) = D(e^{2x}\partial)$, we also have that $ce^x\partial * e^x\partial + e^x\partial * ce^x\partial = D(e^{2x}\partial)$. This implies that $D(e^{2x}\partial) = 2ce^{2x}\partial$. By induction on $k \in \mathbf{N}$ of $e^{kx}\partial$, we can prove that

$$D(e^{kx}\partial) = kce^{kx}\partial. \tag{11}$$

By $D(e^{-x}\partial*e^x\partial) = D(\partial)$, we have that $D(e^{-x}\partial)*e^x\partial+e^{-x}\partial*D(e^x\partial) = 0$. This implies that $D(e^{-x}\partial) = -ce^{-x}\partial$. Similarly to (11), by induction on $-k \in \mathbf{N}$ of $e^{kx}\partial$, we can also prove that

$$D(e^{kx}\partial) = kce^{kx}\partial. \tag{12}$$

This implies that D is the derivation D_c in Note 2. Therefore we have proven the lemma. □

Theorem 4. *For any derivation D of the non-associative algebra $\overline{W(1,0,0)}$, $D = \sum_{c\in\mathbf{F}} D_c$ where D_c is the derivation which is defined in Note 2.*

Proof. Let D be the derivation of the non-associative algebra

$$\overline{W(1,0,0)}$$

in the theorem. Since ∂ annihilates itself, we have that $D(\partial)*\partial+\partial*D(\partial) = 0$. This implies that $D(\partial) = c_1\partial$ for $c_1 \in \mathbf{F}$. It is easy to prove that $c_1 = 0$. So by Lemma 4, D is D_c for $c \in \mathbf{F}$. This implies that $D = \sum_{c\in\mathbf{F}} D_c$ where D_c is the derivation which is defined in Note 2. Therefore we have proven theorem. □

By Theorem 4, we know that every derivation the non-associative algebra $\overline{W(1,0,0)}$ is a scalar derivation. All the derivations of the non-associative algebras $\overline{WN_{0,0,1_r}}$ are found in the papers [1], [10], [11] and please refer to the definitions of the algebras $\overline{WN_{0,0,1_r}}$ and $\overline{WN_{n,0,0_r}}$ in the papers [2], [3]. Thus it is an interesting problem to find all the derivations of the non-associative algebras $\overline{WN_{n,0,0_r}}$. Also it is an interesting problem to find the non-associative algebra automorphism group $\mathrm{Aut}_{non}(\overline{WN_{n,0,0_r}})$ of the non-associative algebras $\overline{WN_{n,0,0_r}}$ [1], [6], [9].

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