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A LOWER BOUND FOR AREA OF COMPACT SINGULAR SURFACES OF NONPOSITIVE CURVATURE

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Abstract. In this paper, we obtain some lower bounds for area of non-simply connected compact singular surfaces of nonpositive curvature. One inequality involves systole and area of the surface.

1. Introduction

A CAT(k)-space is a singular metric space of curvature bounded above by k in the sense of Alexandrov [1]. The systole is, by definition, the infimum of the lengths of closed curves, which are not homotopic to zero, in a CAT(k)-space X, and we denote it by $sys(X)$. $sys(X)$ is realized by a closed geodesic, and this closed geodesic of minimal length is called a systolic in X. In this paper, we obtain geometric inequalities giving lower bound for the area of nonsimply connected compact singular surface of nonpositive curvature by terms of $sys(X)$.

Related with $sys(X)$ to area of a 2-dimensional regular surface X, the first result is the Loewner's inequality which states for the torus T^2, $Area(T^2) \geq \frac{\sqrt{3}}{2} sys^2(T^2)$, for every metric on T^2; equality holds if and only if T^2 is a flat torus obtained from a hexagonal lattice. For the projective plane $\mathbb{R}P^2$, Pu [8] proved $Area(\mathbb{R}P^2) \geq \frac{2}{\pi} sys^2(\mathbb{R}P^2)$, for every metric on $\mathbb{R}P^2$; equality holds if and only if $\mathbb{R}P^2$ has the elliptic metric. For the Klein bottle KB, Bavard [2] obtained the optimal inequality that $Area(KB) \geq \frac{2\sqrt{2}}{\pi} sys^2(KB)$. Also it is known that for an oriented regular surface X, $Area(X) \geq \frac{1}{2} sys^2(X)$ and for a nonoriented surface X, $Area(X) \geq \frac{1}{4} sys^2(X)$ [3], [6]. In [5], Gromov gave a result that the area of a nonsimply connected regular surface X is not less than the area of a ball of radius $\frac{1}{2} sys(X)$. In this paper, we first show that Gromov's result still holds for nonsimply connected compact singular surfaces of nonpositive curvature, that is to say,

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Theorem 1.1. *For a nonsimply connected compact singular surface X of nonpositive curvature, we have*

$$H(X) \geq \frac{\pi}{4} sys^2(X),$$

where H denotes 2-dimensional Hausdorff measure.

We also obtain another lower bound for area of such a surface as follows:

Theorem 1.2. *Let X be a nonsimply connected compact singular surface of nonpositive curvature, and let γ be a unit speed closed geodesic in X. We assume that an ε-neighborhood $N(\gamma, \varepsilon)$ of γ has a property that the metric projection $\pi_\gamma : N(\gamma, \varepsilon) \to \gamma$ is well-defined. Then we have*

$$H(X) \geq H(N(\gamma, \varepsilon)) \geq 2\varepsilon \ell(\gamma),$$

where $\ell(\gamma)$ denotes the length of γ.

2. Singular surfaces of nonpositive curvature

Let (X, d) be an intrinsic metric space, that is to say, for any p, q in the metric space X,

$$d(p,q) = \inf_{\gamma} \ell(\gamma)\,,$$

where the infimum is taken over all continuous curves γ joining p and q, and the length $\ell(\gamma)$ of a continuous curve $\gamma : [a, b] \to X$ is

$$\ell(\gamma) = \sup \sum_{i=1}^{n-1} d(p_i, p_{i+1})\,,$$

where $p_1, p_2, \ldots, p_n$ is an arbitrary sequence of points of γ numbered in the order of their position on the curve, and the supremum is taken over all such sequence of points. A continuous curve $\gamma : [a, b] \to X$ is called a unit speed geodesic in X if each $t \in [a, b]$ has an open neighborhood $V \subset [a, b]$ such that $d(\gamma(t_1), \gamma(t_2)) = |t_1 \, t_2|$ for all $t_1, t_2 \in V$. If $d(\gamma(t_1), \gamma(t_2)) = |t_1 \, t_2|$ for all $t_1, t_2 \in [a, b]$, then a geodesic $\gamma : [a, b] \to X$ is called a minimizer or a shortest curve joining $\gamma(a)$ and $\gamma(b)$. A subset $U \subset X$ is said to be convex if any two points in U are joined by a minimizer of X lying inside U. A triangle $\triangle = (\sigma_1, \sigma_2, \sigma_3)$ in X is a set consisting of three minimizers $\sigma_1, \sigma_2, \sigma_3$ called the sides, which are pairwisely joining three points called the vertices.

For a triangle $\triangle = (\sigma_1, \sigma_2, \sigma_3)$ in X, a triangle $\bar{\triangle} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ in 2-dimensional Euclidean plane $\mathbb{R}^2$ with the usual metric d_0 is called a comparison triangle for $\triangle$ if $\ell(\bar{\sigma}_i) = \ell(\sigma_i)$, $1 \leq i \leq 3$. A triangle $\triangle$ in X is said to be 0-thin if

$$d(x, y) \leq d_0(\bar{x}, \bar{y})\,,$$

for all points x, y on sides of $\triangle$ and the corresponding points $\bar{x}, \bar{y}$ on the sides of the comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$. An intrinsic metric space X is called a CAT(0)-space, or a space of nonpositive curvature, if each point $p \in X$ has a convex domain R_0 such that every triangle $\triangle$ in R_0 is 0-thin. From now on, a singular surface of nonpositive curvature means a 2-dimensional topological manifold endowed with an intrinsic metric of nonpositive curvature in the sense of Alexandrov.

For $\delta > 0$, let $\gamma_1, \gamma_2 : [0, \delta] \to X$ be a pair of unit speed geodesics emanating from a point p in a complete CAT(0)-space X. For $s, t \in (0, \delta]$ let $\overline{\triangle}_{st} \subset \mathbb{R}^2$ be the comparison triangle for the triangle $\triangle_{st} = (p, \gamma_1(s), \gamma_2(t))$. Then the angle between γ_1 and γ_2 is defined by

$$\angle(\gamma_1, \gamma_2) = \lim_{s,t \to 0} \alpha(s,t),$$

where $\alpha(s,t)$ is the angle of $\overline{\triangle}_{st}$ at $\overline{p}$ in $\mathbb{R}^2$.

Let C be a convex closed subset in a complete simply connected singular surface X of nonpositive curvature. Then the metric projection $\pi_C : X \to C$ defined by the closest point $\pi_C(p) \in C$ to the point $p \in X$ is well defined in X, and the unique point $\pi_C(p) \in C$ is called the footpoint of p on C. Also, $\pi_C : X \to C$ is a 1-Lipschitz retraction on X, and for the projective geodesic segment $\overline{x\,\pi_C(x)}$ and a geodesic segment $\overline{\pi_C(x)\,y}$ contained in C, we have

$$\angle(\overline{\pi_C(x)x}, \overline{\pi_C(x)y}) \geq \frac{\pi}{2},$$

for any $y \in C$ [4].

In what follows, we denote by H 2-dimensional Hausdorff measure in a metric space and by L Lebesgue measure on $\mathbb{R}^2$, respectively.

3. A lower bound for area of singular spaces of nonpositive curvature

Let r_p be the injectivity radius of $p \in X$, and suppose that a metric ball $B(p, r_p)$ is contained in a convex domain of X. Then it is known the following area comparison theorem for balls in a singular surface of nonpositive curvature [7].

Proposition 3.1. *Let (M^2, d) be a singular surface of nonpositive curvature. Then*

$$H(B(p, r_p)) \geq L(B_0(\overline{p}, r_p)),$$

where $B(p, r_p)$ is an open metric ball centered at p of the injectivity radius r_p in M^2 and $B_0(\overline{p}, r_p)$ is an open ball centered at $\overline{p} \in \mathbb{R}^2$ of radius r_p in $\mathbb{R}^2$.

From now on, we shall identify a unit speed curve γ with its image $\gamma([0, \ell(\gamma)]) \subset X$. Now we apply Proposition 3.1 to obtain a lower bound of area of a nonsimply connected singular surfaces of nonpositive curvature.

Lemma 3.1. *Let γ be a shortest closed geodesic in a singular surface X of nonpositive curvature. For $p \in \gamma$, any point q in the metric ball $B(p, \ell(\gamma)/2)$ is joined to p by the unique minimizer.*

Proof. Suppose two points p and q are joined by two minimizers σ_1 and σ_2. Since a minimizer is a convex subset of X, any point a in σ_1 has the unique footpoint $\pi_{\sigma_2}(a) \in \sigma_2$. If $\pi_{\sigma_2}(a)$ is either p or q for all $a \in \sigma_1$, then the concatenation $\sigma_1 \star \sigma_2^{-1}$ is a closed geodesic with length less than $\ell(\gamma)$. This is contrary to the assumption.

On the other hand, if $\pi_{\sigma_2}(a)$ is neither p nor q for some $a \in \sigma_1$, then the angle $\angle(\overline{a\,\pi_{\sigma_2}(a)}, \overline{p\,\pi_{\sigma_2}(a)})$ between two geodesics $\overline{a\,\pi_{\sigma_2}(a)}$ and $\overline{p\,\pi_{\sigma_2}(a)}$ is not less than $\frac{\pi}{2}$. Similarly, we have

$$\angle(\overline{a\,\pi_{\sigma_2}(a)}, \overline{q\,\pi_{\sigma_2}(a)}) \geq \frac{\pi}{2}.$$

This implies that either the triangle $\triangle\, p\, a\, \pi_{\sigma_2}(a)$ or the triangle $\triangle\, q\, a\, \pi_{\sigma_2}(a)$ has an interior angle more than π, which is a contradiction. Hence, we prove the lemma. □

Theorem 3.1. *Let X be a nonsimply connected compact singular surface of nonpositive curvature. Then we have*

$$H(X) \geq \frac{\pi}{4} sys^2(X).$$

Proof. Let γ be a systolic in X. Then it is obvious that, for any point $p \in \gamma$, the metric ball $B(p, \frac{\ell(\gamma)}{2})$ does not contain γ. Also, the lemma above implies that $B(p, \frac{\ell(\gamma)}{2})$ is contractible. Hence, by the generalized Hadamard-Cartan theorem for singular space, the ball $B(p, \frac{\ell(\gamma)}{2})$ does not contain any closed geodesic, and a pair of two points in $B(p, \frac{\ell(\gamma)}{2})$ is joined by the unique minimizer. This implies that the ball $B(p, \frac{\ell(\gamma)}{2})$ is homeomorphic a topological disc, and it is of nonpositive curvature. Hence

$$H(X) \geq H(B(p, \frac{\ell(\gamma)}{2})).$$

Since the Hausdorff measure $H(B(p, \frac{\ell(\gamma)}{2}))$ of the ball $B(p, \frac{\ell(\gamma)}{2})$ is not less than the Lebesgue measure of a corresponding ball in 2-dimensional Euclidean space by Proposition 3.1, we obtain that

$$H(B(p, \frac{\ell(\gamma)}{2})) \geq \frac{\pi}{4}\ell(\gamma)^2.$$

□

The following is another lower bound for area of a nonsimply connected compact singular surface of nonpositive curvature.

Theorem 3.2. *Let (X, d_X) be a nonsimply connected compact singular surface of nonpositive curvature. Let γ be a unit speed closed geodesic in X, and we denote the length of γ by $\ell(\gamma)$. We assume that an ε-neighborhood $N(\gamma, \varepsilon)$ of γ has a property that the metric projection $\pi_\gamma : N(\gamma, \varepsilon) \to \gamma$ is well-defined. Then we have*

$$H(X) \geq H(N(\gamma, \varepsilon)) \geq 2\varepsilon\ell(\gamma).$$

Proof. Divide the closed geodesic γ by two parts $\gamma_1 \equiv \gamma|_{[0,\frac{\ell(\gamma)}{2})}$ and $\gamma_2 \equiv \gamma|_{[\frac{\ell(\gamma)}{2},\ell(\gamma))}$. Then ε-neighborhood $N(\gamma,\varepsilon)$ of γ is the union of ε-neighborhood $N(\gamma_1,\varepsilon)$ of γ_1 and ε-neighborhood $N(\gamma_2,\varepsilon)$ of γ_2. Also, for each $i = 1,2$, the ε-neighborhood $N(\gamma_i,\varepsilon)$ of γ_i consists of two connected domains $N^+(\gamma_i,\varepsilon)$ and $N^-(\gamma_i,\varepsilon)$ such that $N^+(\gamma_i,\varepsilon)\cap N^-(\gamma_i,\varepsilon) = \gamma_i$. The four domains are simply connected singular surfaces of nonpositive curvature. From the assumption, for $x \in N(\gamma_i,\varepsilon)$, there exists a unique footpoint $\pi_{\gamma_i}(x) \in \gamma_i$.

Define four mappings

$$F_i^+ : N^+(\gamma_i,\varepsilon) \to [0, \frac{\ell(\gamma)}{2}] \times [0,\varepsilon)(\subset \mathbb{R}^2)$$

and

$$F_i^- : N^-(\gamma_i,\varepsilon) \to [0, \frac{\ell(\gamma)}{2}] \times [0,\varepsilon)(\subset \mathbb{R}^2)$$

by

$$F_i^+(x) = (\, d_X(\gamma(0)\,,\pi_{\gamma_i}(x))\,,\, d_X(x,\pi_{\gamma_i}(x))\,),$$

and

$$F_i^-(x) = (\, d_X(\gamma(0)\,,\pi_{\gamma_i}(x))\,,\, d_X(x,\pi_{\gamma_i}(x))\,),$$

which are well-defined by the assumption.

Choose $x, y \in N^+(\gamma_1,\varepsilon)$. Then both x and y have their unique footpoints $\pi_{\gamma_1}(x)$ and $\pi_{\gamma_1}(y)$ in γ_1, respectively. By the convexity of distance function between geodesics, $d_X(x,y) \geq d_0(F_1^+(x), F_1^+(y))$. Therefore, F_1^+ is nonexpanding, and by Kolmogorov's principle, we obtain that

$$H(N^+(\gamma_1,\varepsilon)) \geq L([0, \frac{\ell(\gamma)}{2}] \times [0,\varepsilon)).$$

In a similar way, we obtain that

$$H(N^-(\gamma_1,\varepsilon)) \geq L([0, \frac{\ell(\gamma)}{2}] \times [0,\varepsilon)),$$

and they induce the desired inequality. □

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