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CERTAIN CONTACT CR-SUBMANIFOLDS OF AN ODD-DIMENSIONAL UNIT SPHERE

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ABSTRACT. We study an $(n+1)(n \geq 3)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in a $(2m+1)$-unit sphere S^{2m+1}, and to determine such submanifolds under conditions concerning the second fundamental form and the induced almost contact structure.

1. Introduction

Let S^{2m+1} be a $(2m+1)$-unit sphere in the complex $(m+1)$-space $\mathbb{C}^{m+1}$. For any point $z \in S^{2m+1}$ we put $\xi = Jz$, where J denotes the complex structure of $\mathbb{C}^{m+1}$. Denoting by π the orthogonal projection : $T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$ and putting $\phi = \pi \circ J$, we can see that the aggregate (ϕ, ξ, η, g) is a Sasakian structure on S^{2m+1}, where g is the standard metric on S^{2m+1} induced from that of $\mathbb{C}^{m+1}$ and η is a 1-form dual to ξ. Hence S^{2m+1} can be considered as a Sasakian manifold of constant curvature 1 (cf. [1, 2, 4, 5, 6, 8]).

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of S^{2m+1} and denote by $\mathcal{D}_x$ the ϕ-invariant subspace $T_xM \cap \phi T_xM$ of the tangent space T_xM of M at x in M. Then ξ cannot be contained in $\mathcal{D}_x$ at any point x in M (cf. [5]). Thus the assumption $\dim \mathcal{D}_x^{\perp}$ being constant and equal to 2 at each point x in M yields that M can be dealt with a contact CR-submanifold in the sense of Yano-Kon (cf. [5, 6, 8]), where $\mathcal{D}_x^{\perp}$ denotes the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. In fact, if there exists a non-zero vector U which is orthogonal to ξ and contained in $\mathcal{D}_x^{\perp}$, then $N := \phi U$ must be normal to M.

In this point of view, the present authors, Kwon and Kim ([6]) studied $(n+1)$-dimensional contact CR-submanifolds of maximal contact CR-dimension in S^{2m+1}, namely, those with $\dim \mathcal{D}_x = n-1$ at each point x in M and proved

Theorem P-K. *Let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a $(2m+1)$-unit sphere S^{2m+1}. If*

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the distinguished normal vector field N is parallel with respect to the normal connection and the equality appeared in (3.1) *holds on M, then M is locally isometric to*

$$S^{2n_1+1}(r_1) \times S^{2n_2+1}(r_2) \quad (r_1^2 + r_2^2 = 1)$$

for some integers n_1, n_2 with $n_1 + n_2 = (n-1)/2$.

In this paper we study contact CR-submanifolds of maximal contact CR-dimension in S^{2m+1} under the assumption only that the equality given in (3.1) holds on M, and improve Theorem P-K.

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞.

2. Fundamental properties of contact CR-submanifolds

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g). Then by definition it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to $\overline{M}$.

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of $\overline{M}$. If the ϕ-invariant subspace $\mathcal{D}_x$ has constant dimension for any $x \in M$, then M is called a *contact CR-submanifold* and the constant is called *contact CR-dimension* of M (cf. [1, 5, 8]).

From now on we assume that M is a contact CR-submanifold of $(n-1)$ contact CR-dimension in $\overline{M}$, where $n-1$ must be even. Then, as already mentioned in section 1, the structure vector ξ is always contained in $\mathcal{D}_x^\perp$ and $\phi\mathcal{D}_x^\perp \subset T_xM^\perp$ at any point $x \in M$. Further, by definition $\dim \mathcal{D}_x^\perp = 2$ at any point $x \in M$, and so there exists a unit vector field U contained in $\mathcal{D}^\perp$ which is orthogonal to ξ. Since $\phi\mathcal{D}^\perp \subset TM^\perp$, ϕU is a unit normal vector field to M, which will be denoted by N, that is,

$$N := \phi U. \tag{2.2}$$

Moreover, it is clear that $\phi TM \subset TM \oplus \mathrm{Span}\{N\}$. Hence we have, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 = N,\ p = 2m - n)$ of normal vectors to M, the following decomposition in tangential and normal components :

$$\phi X = FX + u^1(X)N, \tag{2.3}$$

$$\phi N_\alpha = -U_\alpha + PN_\alpha, \quad \alpha = 1, \ldots, p. \tag{2.4}$$

It is easily shown that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Since the structure vector field ξ is tangent to M, (2.1) implies

$$g(FU_\alpha, X) = -u^1(X)g(N_1, PN_\alpha), \tag{2.5}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(PN_\alpha, PN_\beta). \tag{2.6}$$

We also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha} \tag{2.7}$$

and consequently

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \ldots, p. \tag{2.8}$$

Moreover it is clear from (2.3) that

$$F\xi = 0, \quad u^1(\xi) = 0, \quad FU = 0, \quad u^1(U) = 1. \tag{2.9}$$

Next, applying ϕ to (2.2) and using (2.1) and (2.4), we have

$$U_1 = U, \quad PN_1 = PN = 0. \tag{2.10}$$

From now on, in the sense of (2.8) and (2.10), we denote by u instead of u^1.

Applying ϕ to (2.3) and using (2.1), (2.3), (2.4) and (2.10), we also have

$$F^2X = -X + \eta(X)\xi + u(X)U, \quad u(FX) = 0. \tag{2.11}$$

On the other hand, it follows from (2.4), (2.8) and (2.10) that

$$\phi N = -U, \quad \phi N_\alpha = PN_\alpha, \quad \alpha = 2, \ldots, p, \tag{2.12}$$

and consequently we can take a local orthonormal basis $\{N, N_a, N_{a^*}\}_{a=1,\ldots,q}$ of normal vectors to M such that

$$N_{a^*} := \phi N_a, \quad a = 1, \ldots, q := (2m - n)/2. \tag{2.13}$$

We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}$ and M, respectively, and by $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.14}$$

$$\overline{\nabla}_X N = -AX + \nabla^\perp_X N = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*}\}, \tag{2.15$_1$}$$

$$\overline{\nabla}_X N_a = -A_aX - s_a(X)N + \sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*}\}, \tag{2.15$_2$}$$

$$\overline{\nabla}_X N_{a^*} = -A_{a^*}X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*}\}, \tag{2.15$_3$}$$

for any tangent vector fields X, Y to M, where $s's$ are coefficients of the normal connection $\nabla^\perp$, namely,

$$\nabla^\perp_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)N_\beta, \quad \alpha = 1, \ldots, p,$$

the matrix $(s_{\alpha\beta})$ being skew-symmetric. Here h denotes the second fundamental form and A, A_a, A_{a^*} the shape operators corresponding to the normals N, N_a, N_{a^*}, respectively. They are related by

$$h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*}\}. \tag{2.16}$$

On the other hand, by definition the structure vector ξ is tangent to M. Hence, from (2.1), (2.3), (2.12), (2.13) and $(2.15)_2 - (2.15)_3$, it can be easily verified that

$$A_aX = -FA_{a^*}X + s_{a^*}(X)U, \quad A_{a^*}X = FA_aX - s_a(X)U, \tag{2.17}$$

$$s_a(X) = -u(A_{a^*}X), \quad s_{a^*}(X) = u(A_aX). \tag{2.18}$$

Since F is skew-symmetric, (2.17) implies

$$g(FA_a + A_aF)X,Y) = s_a(X)u(Y) - s_a(Y)u(X), \tag*{$(2.19)_1$}$$

$$g(FA_{a^*} + A_{a^*}F)X,Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X). \tag*{$(2.19)_2$}$$

From now on we specialize to the case of an ambient Sasakian manifold $\overline{M}$, that is,

$$\overline{\nabla}_X\xi = \phi X, \tag{2.20}$$

$$(\overline{\nabla}_X\phi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.21}$$

Differentiating (2.3) and (2.12) covariantly and comparing the tangential and normal parts, we have

$$(\nabla_YF)X = -g(Y,X)\xi + \eta(X)Y - g(AY,X)U + u(X)AY, \tag{2.22}$$

$$(\nabla_Yu)X = g(FAY,X), \tag{2.23}$$

$$\nabla_XU = FAX. \tag{2.24}$$

On the other hand, since the structure vector ξ is tangent to M, (2.20) gives

$$\nabla_X\xi = FX, \tag{2.25}$$

$$g(A\xi,X) = u(X), \quad \text{that is,} \quad A\xi = U, \tag{2.26}$$

$$A_a\xi = 0, \quad A_{a^*}\xi = 0, \quad a = 2,\dots,q. \tag{2.27}$$

If the ambient manifold $\overline{M}$ is of constant curvature 1, then the equation of Codazzi implies that

$$\begin{aligned}&(\nabla_XA)Y - (\nabla_YA)X\\ &= \sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX + s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\},\end{aligned} \tag*{$(2.28)_1$}$$

$$(2.28)_2\quad \begin{aligned}&(\nabla_X A_a)Y - (\nabla_Y A_a)X = s_a(Y)AX - s_a(X)AY\\ &+\sum_{b=1}^{q}\{s_{ab}(X)A_bY - s_{ab}(Y)A_bX + s_{ab^*}(X)A_{b^*}Y - s_{ab^*}(Y)A_{b^*}X\},\end{aligned}$$

$(2.28)_3$

$$\begin{aligned}&(\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X = s_{a^*}(Y)AX - s_{a^*}(X)AY\\ &+\sum_{b=1}^{q}\{s_{a^*b}(X)A_bY - s_{a^*b}(Y)A_bX + s_{a^*b^*}(X)A_{b^*}Y - s_{a^*b^*}(Y)A_{b^*}X\},\end{aligned}$$

for any vector fields X, Y tangent to M(cf. [1, 2, 8]).

3. Main results

In this section we let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a $(2m+1)$-unit sphere S^{2m+1} and assume that the equality

$$h(FX, Y) = -h(X, FY) \tag{3.1}$$

holds on M. Then it follows from (2.16) and (3.1) that

$$FA = AF, \quad FA_a = A_aF, \quad FA_{a^*} = A_{a^*}F, \tag{3.2}$$

which together with $(2.19)_1$ and $(2.19)_2$ implies

$$(3.3)_1\qquad 2g((FA_a)X, Y) = s_a(X)u(Y) - s_a(Y)u(X),$$

$$(3.3)_2\qquad 2g((FA_{a^*})X, Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X),$$

from which and (2.9),

$$s_a(X) = s_a(U)u(X), \quad s_{a^*}(X) = s_{a^*}(U)u(X), \quad a = 1, \ldots, q. \tag{3.4}$$

Further, (3.2) and $(3.3)_1 - (3.3)_2$ yield

$$FA_a = A_aF = 0, \quad FA_{a^*} = A_{a^*}F = 0. \tag{3.5}$$

As a direct consequence of the first equation of (3.2) and (3.5), it follows from (2.11), (2.18), (2.26) and (2.27) that

$$AU = \lambda U + \xi, \quad \lambda := u(AU), \tag{3.6}$$

$$A_aX = s_{a^*}(X)U, \quad A_{a^*}X = -s_a(X)U. \tag{3.7}$$

Now we prepare a lemma for later use.

Lemma 3.1. *Let M be an $(n+1)(n \geq 3)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a $(2m+1)$-unit sphere S^{2m+1}. If, for any vector fields X, Y tangent to M, the equality* (3.1) *holds on M, then*

$$s_a = 0, \quad s_{a^*} = 0, \quad a = 1, \ldots q,$$

namely, the distinguished normal vector field N is parallel with respect to the normal connection. Moreover,

$$A_a = 0, \quad A_{a^*} = 0, \quad a = 1, \dots q.$$

Proof. Since S^{2m+1} is of constant curvature 1, applying F to the both sides of $(2.28)_2$ and using (3.4) – (3.5), we have

$$F((\nabla_X A_a)Y - (\nabla_Y A_a)X) = s_a(U)u(Y)FAX - s_a(U)u(X)FAY. \tag{3.8}$$

On the other hand, differentiating $FA_a = 0$ covariantly along M and using (2.22), (2.27), (3.4), and (3.8) we can easily obtain

$$\begin{aligned} &F(\nabla_X A_a)Y \\ &= s_{a^*}(U)u(X)u(Y)\xi + s_{a^*}(U)u(AX)u(Y)U - s_{a^*}(U)u(Y)AX, \end{aligned}$$

from which and (3.6),

$$\begin{aligned} &F((\nabla_X A_a)Y - (\nabla_Y A_a)X) \\ &= s_{a^*}(U)\{\eta(X)u(Y) - u(X)\eta(Y)\}U - s_{a^*}(U)\{u(Y)AX - u(X)AY\}. \end{aligned} \tag{3.9}$$

Comparing (3.9) with (3.8), it is clear that

$$\begin{aligned} &s_a(U)\{u(Y)FAX - u(X)FAY\} \\ &= s_{a^*}(U)\{\eta(X)u(Y) - u(X)\eta(Y)\}U - s_{a^*}(U)\{u(Y)AX - u(X)AY\}, \end{aligned}$$

from which, putting $Y = U$ and using (3.6), we have

$$\begin{aligned} &s_a(U)g(FAX, Y) \\ &= s_{a^*}(U)\{\eta(X)u(Y) + u(X)\eta(Y) + \lambda u(X)u(Y) - g(AX, Y)\}, \end{aligned}$$

and consequently

$$s_a(U)\{g(FAX, Y) - g(FAY, X)\} = 2s_a(U)g(FAX, Y) = 0 \tag{3.10}$$

with the aid of the fact that F is skew-symmetric and (3.2).

Now we assume that $s_a(U) \neq 0$. Then it follows from (2.23), (2.24) and (3.10) that

$$FAX = 0, \quad \nabla_X U = 0, \quad \nabla_X u = 0. \tag{3.11}$$

Furthermore, (2.11), (3.6) and the first equation of (3.11) imply

$$AX = \{\lambda u(X) + \eta(X)\}U + u(X)\xi. \tag{3.12}$$

Differentiating (3.12) covariantly along M and using (2.25) and (3.11), we have

$$(\nabla_Y A)X = \{(Y\lambda)u(X) + g(X, FY)\}U + u(X)FY,$$

from which together with (2.9), $(2.28)_1$ and (3.5),

$$\begin{aligned} F((\nabla_Y A)X - (\nabla_X A)Y) =& u(X)\{-Y + u(Y)U + \eta(Y)\xi\} \\ &- u(Y)\{-X + u(X)U + \eta(X)\xi\} = 0. \end{aligned}$$

and consequently $X = u(X)U + \eta(X)\xi$, which is a contradiction because of $n \geq 3$. Hence $s_a(U) = 0$, which and (3.4) imply

$$s_a(X) = 0, \quad a = 1, \dots, q \tag{3.13}$$

everywhere on M.

Next, combining (3.7) and (3.12), we have

$$A_{a^*} = 0, \quad a = 1, \dots, q,$$

from which, using $(2.28)_3$ and (3.5),

$$s_{a^*}(U)\{u(Y)FAX - u(X)FAY\} = 0.$$

Putting $Y = U$ in the above equation and using (2.9), we have $s_{a^*}(U)\ FAX = 0$ and hence, by the same method as in the case of (3.13),

$$s_{a^*}(X) = 0, \quad a = 1, \dots, q \tag{3.14}$$

everywhere on M. Further (3.7) and (3.14) give

$$A_a = 0, \quad a = 1, \dots, q.$$

□

For the submanifold M given in Lemma 3.1, we can easily see that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection $\nabla^\perp$ from our assumption. Thus we may apply Erbacher's reduction theorem ([3, p.339]) and this yields

Theorem 3.2. *Let M be as in Lemma 3.1. If the equality appeared in* (3.1) *holds on M, then there exists an $(n+2)$-dimensional totally geodesic unit sphere S^{n+2} such that $M \subset S^{n+2}$.*

Combining Theorem P-K stated in section 1 and Lemma 3.1, we have

Theorem 3.3. *Let M be as in Lemma 3.1. If the equality appeared in* (3.1) *holds on M, then M is locally isometric to*

$$S^{2n_1+1}(r_1) \times S^{2n_2+1}(r_2) \quad (r_1^2 + r_2^2 = 1)$$

for some integers n_1, n_2 with $n_1 + n_2 = (n-1)/2$.

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