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HYPERCYCLICITY FOR TRANSLATIONS THROUGH RUNGE'S THEOREM

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Abstract. In this paper, we first adapt Runge's Theorem to work on certain domains in any complex Banach space. Then, using this result, we extend Birkhoff's Theorem on the hypercyclicity of translations on $H(\mathbb{C})$ and Costakis' and Sambarino's result on the existence of common hypercyclic functions for uncountable families of translations on $H(\mathbb{C})$ to subspaces of $H_b(E)$ (in some cases all of $H_b(E)$), E being in a large class of Banach spaces.

A continuous linear operator $T : E \to E$ on a Fréchet space E is called **hypercyclic** if there is a vector $x \in E$ such that its orbit under T, given by $\mathcal{O}(x;T) = \{x, Tx, T^2x, \dots\}$, is dense in E (in this case the vector x is called a hypercyclic vector for the operator T).

The first known example of hypercyclic operator comes through Birkhoff's Theorem [3], in 1929. Birkhoff showed that there is a function f in the space $H(\mathbb{C})$ of entire functions on $\mathbb{C}$ and a sequence (a_n) of positive numbers such that the translates $\{f(z), f(z+a_1), f(z+a_2), \dots\}$ are dense in $H(\mathbb{C})$ (considering the compact-open topology). Actually this doesn't match with the above definition, but in Birkhoff's proof the $a_1, a_2, \dots$ can be chosen as multiples of any real positive number a. So we have that, for any $a > 0$, the translation by a, $T_a : H(\mathbb{C}) \to H(\mathbb{C})$, given by $T_a(f)(z) = f(z+a)$, is a hypercyclic operator. It's also easy to see that if $f \in H(\mathbb{C})$ is hypercyclic for the translation $T_a, a > 0$, then $g(z) = f(e^{-i\theta}z)$ is hypercyclic for $T_{ae^{i\theta}}$, for each $\theta \in [0, 2\pi]$. So, for every $b \neq 0$ in $\mathbb{C}$, the translation $T_b : H(\mathbb{C}) \to H(\mathbb{C})$ is a hypercyclic operator. In the study of hypercyclicity for translations on $H(\mathbb{C})$ one particular tool has been shown to be very useful, namely:

Theorem 1 (Runge). *If f is holomorphic in a neighborhood of a compact set $K \subset \mathbb{C}$ and $\mathbb{C} \backslash K$ is connected, then f can be uniformly approximated on K by polynomials. (see* [4]*, p.85)*

Runge's Theorem becomes a natural tool when we deal with hypercyclicity for translations not only because we are considering the compact-open topology

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on $H(\mathbb{C})$, but mainly because we can produce compact sets (disjoint unions of balls) as in Runge's Theorem with translations of balls. Here are some examples to show this: In [2], Aron and Markose give a simpler (and shorter) proof of Birkhoff's result using Runge's Theorem. In [5], Costakis and Sambarino, using Runge's Theorem strongly again, prove that there's a dense G_δ set of common hypercyclic entire functions for all translations T_b, $b \neq 0$, in $H(\mathbb{C})$. In both cases the compact set K used in Runge's Theorem can be consider as a finite union of disjoint closed balls. So it's natural to ask if we have similar results (to Runge's) for simpler sets in other spaces and what we can do with such results regarding hypercyclicity for translations. To simplify towards the result we're looking for, let's state:

Theorem 2. *Let B_1 and B_2 be two disjoint closed balls in a complex Banach space E. If f is a holomorphic and bounded function in a uniform neighborhood of $B_1 \cup B_2$ (by a uniform neighborhood of a set A we mean $A + B(0,\delta)$ for some $\delta > 0$), then f can be uniformly approximated by polynomials on $B_1 \cup B_2$.*

Proof. Let $\epsilon > 0$. First, since f is holomorphic and bounded in a uniform neighborhood of $B_1 \cup B_2$, there are polynomials p_1 and p_2 in $\mathcal{P}(E)$-polynomials from E to $\mathbb{C}$-such that $\sup_{x \in B_j} |p_j(x) - f(x)| < \epsilon/2$, $j = 1, 2$. In fact, there's a $\delta > 0$ such that f is holomorphic and bounded in $(B_1 \cup B_2) + B(0, \delta)$, so the radius of boundedness of f ([7], p.52) in the center of each ball is strictly greater than the radius of the ball. But since the radius of convergence (in each center) is bigger than or equal to the radius of boundedness, the Taylor Series of f around each center will converge uniformly on the corresponding ball. So we can get p_1 and p_2 as Taylor polynomials.

Let $M = \sup_{x \in B_1 \cup B_2} \{|p_1(x)| + |p_2(x)|\}$.

By the Hahn-Banach Separation Theorem, there exists $\varphi \in E'$ such that $K_1 = \overline{\varphi(B_1)}$ and $K_2 = \overline{\varphi(B_2)}$ are disjoint convex compact sets in $\mathbb{C}$. Now we can apply Runge's Theorem to the compact set $K = K_1 \cup K_2 \subset \mathbb{C}$ (since $\mathbb{C} \backslash K$ is path-connected) and find $q \in \mathcal{P}(\mathbb{C})$ such that $q \sim 1$ on K_1 and $q \sim 0$ on K_2.

So the polynomial $h = q \circ \varphi : E \to \mathbb{C} \in \mathcal{P}(E)$ satisfies: $h \sim 1$ on B_1 and $h \sim 0$ on B_2; that is $\sup_{x \in B_1} |h(x) - 1| < \epsilon/2M$ and $\sup_{x \in B_2} |h(x)| < \epsilon/2M$.

Finally, $p = p_1 \cdot h + p_2 \cdot (1 - h)$ is the polynomial we're looking for. □

Corollary 3. *Let $B_1, B_2, \ldots, B_n$ be disjoint closed balls in a complex Banach space E such that for each $j = 1, 2, \ldots, n-1$ there's a closed ball A_j with $B_1 \cup \cdots \cup B_j \subset A_j$ and $A_j \cap B_{j+1} = \phi$. If f is a holomorphic and bounded function in a uniform neighborhood of $B_1 \cup B_2 \cup \cdots \cup B_n$, then f can be uniformly approximated by polynomials on $B_1 \cup B_2 \cup \cdots \cup B_n$; that is, given $\epsilon > 0$, there exists $p \in \mathcal{P}(E)$ such that $\sup_{x \in B_1 \cup \cdots \cup B_n} |p(x) - f(x)| < \epsilon$.*

Now, using Theorem 2 and its Corollary, we are able to extend some results about hypercyclicity of translations on $H(\mathbb{C})$ to more general spaces of holomorphic functions. Before doing this, let's take a look at the function spaces for which we'll obtain our hypercyclicity results. At first, we take a complex Banach space E and try the translations on $H(E)$. But a quick look at Theorem 2 (we need our functions to be bounded on uniform neighborhoods of balls) takes us to the Fréchet algebra $H_b(E)$ of the entire functions of bounded type (bounded on bounded sets of E), with the topology of uniform convergence on the balls of E. It's clear that this adjustment is not necessary if E is finite dimensional. But another detail arises when E is infinite dimensional, since separability is a necessary condition in any discussion of hypercyclicity. If E is infinite dimensional we can't be certain that $H_b(E)$ is separable, although there are some known cases where this can occur. So we're dealing with closed (after all we need completeness) separable subspaces of $H_b(E)$ and these subspaces must be translation-invariant, for the translations to be well defined as linear continuous operators.

Therefore, our results on hypercyclicity for translations shall be valid on closed, separable, translation-invariant subspaces X of $H_b(E)$, E being any Banach space. Let's enumerate some concrete examples:

1) E finite dimensional: $X = H(\mathbb{C}^n)$;
2) E infinite dimensional with $H_b(E)$ separable: $X = H_b(E)$ (examples: $E = c_0$, $E =$Tsirelson) ;
3) If E is a Banach space such that E' is separable, then the space $X = H_{bc}(E)$ of functions in $H_b(E)$ whose Taylor polynomials are in the closure of polynomials of finite type is a closed, separable subspace of $H_b(E)$. But we also need it to be translation-invariant, so if we require E' to have the approximation property, we'll have $H_{bc}(E) = H_{wu}(E)$ ([1], p.56), $H_{wu}(E)$ being the space of functions that are weakly-uniformly continuous on the balls of E, and it's not difficult to see that $H_{wu}(E)$ is translation-invariant.

Finally, one additional condition should be pointed out:

Remark. (on Theorem 2): Applying Theorem 2 (and its Corollary), we will be able to obtain a polynomial (so, an entire function) which approximates f on disjoint closed balls. However, we may be working with a space that may not contain all polynomials (e.g. 3 above) and since it's crucial to produce a function in this space, we must ask for one condition on the space to guarantee this. Let's take a closer look at the proof of Theorem 2 to find a reasonable condition: The polynomial (which we want to be in our space) produced is given by: $p = p_1 \cdot h + p_2 \cdot (1-h)$. Each part of this sum should be in our space. Since h and $(1-h)$ are finite type polynomials, p_1 and p_2 should produce polynomials in the space when multiplied by finite type polynomials. But how did we obtain p_1 and p_2 in the proof? p_1 and p_2 were Taylor approximations to f on the respective balls. This motivates our **Condition(*)**:

Definition 4. A space $X \subset H(E)$, E complex Banach, satisfies Condition(*) if the Taylor polynomials of the functions in X can be approximated by polynomials whose products by finite type polynomials are still in the space X.

Condition(*) is quite natural, in light of our previous known examples: The cases 1) $X = H(\mathbb{C}^n)$ and 2) $X = H_b(E)$ separable trivially satisfy Condition(*), since both types of spaces contain every polynomial. The third group of examples, $X = H_{bc}(E)$ (E' separable and with the approximation property), also satisfies Condition(*), by its very definition: Taylor polynomials of functions in $H_{bc}(E)$ are in the closure (so can be approximated by) of polynomials of finite type!

The first result we'll extend is Birkhoff's, about hypercyclicity of translation operators. To do this, we'll simply adapt Aron's and Markose's proof [2] to a more general setting, replacing Runge's Theorem (Theorem 1) by our Theorem 2.

Theorem 5. *Let E be a Banach space and X be a closed, separable, translation-invariant subspace of $H_b(E)$. Suppose also that X satisfies Condition(*). If $v \neq 0$ is any non-zero vector in E, then the translation operator $T_v : X \to X$, given by $T_v(f)(x) = f(x+v)$, is hypercyclic.*

Proof. Let $(g_j)_j$ be a dense sequence in X such that each g_j occurs infinitely often in the sequence. Given $v \neq 0$ in E, we want to show that $T_v : X \to X$ is hypercyclic. Let $(D_j)_j$ be a sequence of disjoint closed balls in E, each D_j of radius j and center $c_j = n_j.v$, with $n_j \in \mathbb{N}$, $n_1 < n_2 < n_3 < \cdots$. It's not difficult to see that there's a sequence $(E_j)_j$ of closed balls (in E) centered at the origin such that $D_1 \cup D_2 \cup \cdots \cup D_j \subset E_j$ and $E_j \cap D_{j+1} = \phi$ for every $j \in \mathbb{N}$. Since $E_1 \cap D_2 = \phi$ (disjoint closed balls), it follows from Theorem 2 and Condition(*) that there's a polynomial $Q_1 \in X$ such that

$$\sup_{x \in E_1} |Q_1(x)| < \frac{1}{2} \quad \text{and} \quad \sup_{x \in D_2} |Q_1(x) - [g_1(x - c_2)]| < \frac{1}{2}\,.$$

In fact, use (within Theorem 2)

$$f(x) = \begin{cases} 0 & \text{in a uniform neighborhood of } E_1 \\ g_1(x - c_2) & \text{in a uniform neighborhood of } D_2\,. \end{cases}$$

Next, using again Theorem 2 and Condition(*), choose a polynomial $Q_2 \in X$ such that

$$\sup_{x \in E_2} |Q_2(x)| < \frac{1}{2^2} \quad \text{and} \quad \sup_{x \in D_3} |Q_2(x) - [g_2(x - c_3) - Q_1(x)]| < \frac{1}{2^2}\,.$$

Proceeding in this way, we construct a sequence of polynomials $(Q_n)_n \subset X$ such that for every $n \geq 1$:

$$\text{(I)} \quad \sup_{x \in E_n} |Q_n(x)| < \frac{1}{2^n}$$

$$\text{(II)} \quad \sup_{x \in D_{n+1}} \left| Q_n(x) - \left[g_n(x - c_{n+1}) - \sum_{j=1}^{n-1} Q_j(x) \right] \right| < \frac{1}{2^n}.$$

By **(I)**, the sequence $\left(\sum_{j=1}^{n} Q_j\right)_n$ is a Cauchy sequence in X. So we have $h = \sum_{j=1}^{\infty} Q_j \in X$ and we'll now show that h is hypercyclic for T_v. To do this, it suffices to show that given $R > 0$, $\epsilon > 0$ and $g \in (g_j)_j$, there's an $l \in \mathbb{N}$ such that

$$\sup_{\|x\| \leq R} |h(x + c_l) - g(x)| < \epsilon.$$

Indeed, we can choose $l \in \mathbb{N}$ such that $l > R$, $\frac{1}{2^{l-1}} < \frac{\epsilon}{2}$ and $g_{l-1} = g$. If $\|x\| \leq R$ then $w = x + c_l \in B[0; R] + c_l \subset B[0; l] + c_l = D_l \subset E_l$. Then

$$\sup_{\|x\| \leq R} |h(x + c_l) - g(x)| \leq \sup_{w \in D_l} |h(w) - g_{l-1}(w - c_l)|$$
$$\leq \sup_{w \in D_l} \left| h(w) - \sum_{j=1}^{l-1} Q_j(w) \right| + \sup_{w \in D_l} \left| \left[\sum_{j=1}^{l-1} Q_j(w) \right] - g_{l-1}(w - c_l) \right|.$$

In the first expression above, we have

$$\sup_{w \in D_l} \left| h(w) - \sum_{j=1}^{l-1} Q_j(w) \right| \leq \sup_{w \in E_l} \left| \sum_{j=l}^{\infty} Q_j(w) \right|$$
$$\leq \sum_{j=l}^{\infty} \sup_{w \in E_j} |Q_j(w)| < \frac{1}{2^l} + \frac{1}{2^{l+1}} + \cdots = \frac{1}{2^{l-1}} < \frac{\epsilon}{2}.$$

For the second part, from **(II)**:

$$\sup_{w \in D_l} \left| \left[\sum_{j=1}^{l-1} Q_j(w) \right] - g_{l-1}(w - c_l) \right| < \frac{1}{2^{l-1}} < \frac{\epsilon}{2}.$$

Thus, $\sup_{\|x\| \leq R} |h(x + c_l) - g(x)| < \epsilon$, as required. □

Our next result refers to the existence of common hypercyclic functions for translations. It's known that every hypercyclic operator on a Fréchet space has a residual set of hypercyclic vectors [6]. Since a countable intersection of residual sets is still a residual set, it's easy to see that countable families of hypercyclic operators share common hypercyclic vectors (vectors which are hypercyclic for all operators in the family). The subject becomes more interesting when we consider uncountable families of hypercyclic operators and ask whether or not they share common hypercyclic vectors. In [5], Costakis and Sambarino proved that there's a residual set of common hypercyclic functions for all translations on $H(\mathbb{C})$. In their proof, they used strongly Runge's Theorem, with the compact set considered each time being a finite union of disjoint closed balls in $\mathbb{C}$. So it's natural to ask how we can use Corollary 3 of Theorem 2 to extend their result. The result we get is the following:

Theorem 6. *Let E be a Banach space and X be a closed, separable, translation-invariant subspace of $H_b(E)$. Suppose also that X satisfies Condition(*). For any $v \neq 0$ in E, the (uncountable) family of translations $\{ T_{c \cdot v} \; ; \; c \neq 0 \text{ in } \mathbb{C} \}$ share a common residual set of hypercyclic functions.*

Just as the analogous result of Costakis and Sambarino in [5], Theorem 6 is an immediate consequence of two theorems, which we will state below. It should be pointed out that the proofs are essentially Costakis' and Sambarino's proofs (with the obvious adaptations) for the existence of a residual set of common hypercyclic functions for all translations on $H(\mathbb{C})$. Now, our Corollary 3 of Theorem 2 fits perfectly in these proofs, replacing Runge's Theorem and allowing us to obtain a similar result for more general spaces. So, Theorem 6 is obtained by combining the following results:

Theorem 7. *Let E be a Banach space and X be a closed, separable, translation-invariant subspace of $H_b(E)$. Suppose also that X satisfies Condition(*). If v is a unit vector in E then the set of common hypercyclic functions for all translations $T_{e^{2\pi i\theta} \cdot v} : X \to X$, $\theta \in [0,1]$, is residual in X.*

Theorem 8. *Let E be a Banach space and X be a closed, separable, translation-invariant subspace of $H_b(E)$. Suppose also that X satisfies Condition(*). If v is a unit vector in E and $f \in X$ is hypercyclic for the translation $T_v : X \to X$, then f is hypercyclic for every translation of the family $\{T_{r \cdot v}\}_{r>0}$.*

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