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ON WEAK Π-REGULARITY AND THE SIMPLICITY OF PRIME FACTOR RINGS

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ABSTRACT. A connection between weak π-regularity and the condition every prime ideal is maximal will be investigated. We prove that a certain 2-primal ring R is weakly π-regular if and only if every prime ideal is maximal. This result extends several known results nontrivially. Moreover a characterization of minimal prime ideals is also considered.

Throughout this paper R denotes an associative ring with identity. We use $P(R)$ and $N(R)$ to represent the prime radical and the set of nilpotent elements of R, respectively. Recall that an ideal P of a ring R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. A ring R is called *2-primal* if $P(R) = N(R)$ [1]. Hirano [10] considered the 2-primal condition in the context of strongly π-regular rings. Also the 2-primal condition was taken up independently by Sun [15], where he introduced a condition called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. We investigate in this paper a connection between weak π-regularity and the simplicity of prime factor rings. The connections between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal have been studied by many authors [2, 3, 6, 7, 8, 10, 13, and 16].The earliest results of this type seems to be by Cohen [7, Theorem 1]. Storrer [13] was able to provide the following results: If R is commutative ring then the following are equivalent: (1) R is π-regular, (2) $R/P(R)$ is regular, and (3) all prime ideals of R are maximal ideal. This result was generalized to cases of PI-rings and duo rings by many authors [6,8,10, and 16]. More recently Birkenmeier, Kim and Park [2, Theorem 8] showed that a reduced ring R is weakly regular if and only if every prime ideal is maximal. As a corollary they got Hirono's result [10, Corollary 1]. Also they improved their results in [3]. On the same direction, we shall prove that a certain 2-primal ring R is weakly π-regular if and only if every prime ideal is maximal. This result extends several known results including the main theorem [2, Theorem 8] nontrivially. And a characterization of minimal prime ideals in a certain 2-primal ring is also discussed. Moreover we conclude

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our paper with some examples which illustrate and delimit our results. To do this we consider a condition $(*)$: if $aRb \subseteq P(R)$, then there exists a positive integer n such that $a^nRb^n = 0$ for $a, b \in R$. All prime ideals are taken to be proper ideals. Let X be a nonempty subset of R, then $\ell(X)$ and $r(X)$ denote the left annihilator of X in R, and the right annihilator of X in R, respectively.

Definition 1. (1) A ring R is *right* (*left*) *weakly regular* if $a \in aRaR$ ($a \in RaRa$) for every $a \in R$. R is *weakly regular* if it is both left and right weakly regular [12].

(2) A ring R is called *π-regular* if for every $a \in R$ there exists a natural number $n = n(a)$, depending a, such that $a^n \in a^nRa^n$.

(3) A ring R is *right* (resp., *left*) *weakly π-regular* if for every $a \in R$ there exists a natural number $n = n(a)$, depending on a, such that $a^n \in a^nRa^nR$ (resp., $a^n \in Ra^nRa^n$). A ring R is *weakly π-regular* if it is both right and left weakly π-regular [9].

Every π-regular ring, biregular ring (including simple rings), and right duo ring satisfying d.c.c. on principal ideals is right weakly π-regular.

Lemma 2. *A ring R is 2-primal if and only if every minimal prime ideal is completely prime.*

Proof. See [14, Proposition 1.11]. □

Lemma 3. *If R is a 2-primal ring and $R/P(R)$ is right weakly π-regular, then every prime ideal of R is maximal.*

Proof. See [2, Lemma 5]. □

Definition 4. A ring R is called *pseudo symmetric* [14] if it satisfies the following two conditions:

(PS I) R/I is 2-primal whenever $I = 0$ or $I = r(aR)$ for some $a \in R$;

(PS II) For any $a, b, c \in R$, if $aR(bc)^n = 0$ for a positive integer n, then $a(RbR)^mc^m = 0$ for some positive integer m.

Lambek [11] calls a ring R symmetric provided $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$. Symmetric rings are pseudo symmetric. But there is a pseudo symmetric ring in Example 5.1(c) of [14] which is not symmetric. Recall that a ring R satisfies the condition $(*)$ if $aRb \subseteq P(R)$, then there exists a positive integer n such that $a^nRb^n = 0$.

Proposition 5. *If R satisfies condition* (PS II) (*e.g., a pseudo symmetric ring*), *then R is a 2-primal ring satisfying condition* $(*)$.

Proof. Let $a \in R$ with $a^n = 0$ for some n. Then there is m such that $(RaR)^m = 0$ and so $a \in P(R)$. Thus R is 2-primal. Next let $xRy \subseteq P(R)$. Then $(xy)^n = 0$ for some positive integer n. Thus $xR(xy)^n = 0$, so by condition (PS II) $x(RxR)^my^m = 0$ for some positive integer m. Now $x^{m+1}Ry^m \subseteq x(RxR)^my^m = 0$, hence $x^{m+1}Ry^{m+1} = 0$. Therefore condition (PS II) satisfies condition $(*)$. □

Obviously semiprime rings and commutative rings satisfy the condition $(*)$. Also Proposition 5 provides a class of rings satisfying condition $(*)$ which is neither semiprime nor commutative. But it is not clear whether the converse of Proposition 5 is true or not. The following result generalizes the main theorem in [2]. Without using a result of completely semiprimeness [11] and WCI condition[3] we shall give a direct and simple proof. Actually our condition $(*)$ is less technical, hence more natural than the WCI condition.

Theorem 6. *Let R be a 2-primal ring satisfying condition $(*)$. Then the following conditions are equivalent:*
(1) *R is right weakly π-regular.*
(2) *$R/P(R)$ is right weakly π-regular.*
(3) *Every prime ideal of R is maximal.*

Proof. (1)$\Rightarrow$(2) is clear. Lemma 3 yields (2)$\Rightarrow$(3). So we will assume that every prime ideal of R is maximal and show that R is right weakly π-regular. Suppose that R is not a right weakly π-regular ring. Then there exists an element $a \in R$ which is not a right weakly π-regular element. So we have $a^m \notin a^mRa^mR$ for every positive integer m. Hence $a \neq 0$ and $a \notin aRaR$. Then RaR is contained in a maximal ideal which is also a prime ideal. Now since every prime ideal is maximal, then every prime ideal is completely prime. Let T be the union of all prime ideals which contain a. Let $S = R \setminus T$. Since every prime ideal is completely prime, S is a multiplicatively closed set. Let F be the multiplicatively closed system generated by the set $\{a\} \cup S$. Now we assert that $0 \in F$. Suppose this was not true, then partial order the collection of ideals disjoint with F by set inclusion. By Zorn's lemma, we get an ideal M which is maximal disjoint with F. Then M is a prime ideal and so a maximal ideal by hypothesis. Since $a \notin M$, $M + RaR = R$. Thus there exists $b \in M$ and $c \in RaR$ such that $b + c = 1$. It follows that $b \notin T$. Thus $b \in S \subseteq F$, which implies $b \in F \cap M = \emptyset$, a contradiction. Thus $0 \in F$, so

$$0 = a^{n_1}s_1a^{n_2}s_2 \cdots a^{n_t}s_t,$$

where $s_i \in S$, and we may assume that the integers $n_1, n_2, \ldots, n_t$ are positive. For any prime ideal P, we have that $0 = a^{n_1}s_1a^{n_2}s_2 \cdots a^{n_t}s_t \in P$. Since P is completely prime, either $a \in P$ or $s_i \in P$ for some $i \in \{1, 2, \ldots, t\}$. Let $s = s_1s_2 \cdots s_t$. Then for any prime ideal P, either $a \in P$ or $s \in P$. Therefore $aRs \subseteq P(R)$ and so there exists a positive integer k such that $a^kRs^k = 0$ by hypothesis. Observe that a prime ideal can not contain both a^k and s^k. Otherwise a prime ideal would contain both of them which would contradict the definition of S and T. Hence $Ra^kR + Rs^kR = R$. So $a^kR = a^kRa^kR + a^kRs^kR = a^kRa^kR$. This means that a is a right weakly π-regular element, a contradiction. Consequently R is a right weakly π-regular ring. Moreover R is a weakly π-regular ring since the conditions "right weakly π-regular" in Theorem 6 can be replaced by the condition "left weakly π-regular" and we have $sRa \subseteq P(R)$ also. □

Immediately we have the following from Theorem 6.

Corollary 7. ([2, Theorem 8]) *Let R be a reduced ring. Then the following conditions are equivalent:*
(1) *R is weakly regular.*
(2) *R is right weakly π-regular.*
(3) *Every prime ideal of R is maximal.*

Corollary 8. ([2, Corollary 9]) *Let R be a 2-primal ring. Then the following conditions are equivalent:*
(1) *$R/P(R)$ is weakly regular.*
(2) *$R/P(R)$ is right weakly π-regular.*
(3) *Every prime ideal of R is maximal.*

Note that a 2-primal ring satisfying the condition $(*)$ has the WCI condition. Through the WCI condition seems more technical but Theorem 6 can be also followed as a corollary of [3, Theorem 2.8].

Following [14, 3, 4], for a prime ideal P of a ring R, we define

$$O_P = \{a \in R \mid as = 0 \text{ for some } s \in R \setminus P\},$$

$$\overline{O}_P = \{x \in R \mid x^n \in O_P \text{ for some positive integer } n\},$$

$$O(P) = \{a \in R \mid |aRs = 0 \text{ for some } s \in R \setminus P\},$$

and

$$\overline{O}(P) = \{x \in R \mid |x^n \in O(P) \text{ for some positive integer } n\}.$$

Observe that $O(P)$ is a two-sided ideal and $O(P) \subseteq P \cap O_P$. These definitions have been used to characterize minimal prime ideals of rings by many authors. Note that if R is a 2-primal ring then the condition $(*)$ is equivalent to the condition (CZ2)[4]. But in general they do not equivalent to each other. Actually there exists a semiprime ring (hence satisfies condition$(*)$) which does not satisfy condition (CZ2). In the following we have included our proof which is direct somewhat simple than [4, Theorem 2.3 and Corollary 2.4].

Theorem 9. *Let R be a ring satisfying condition $(*)$. Then R is 2-primal if and only if $P = \overline{O}(P) = \overline{O}_P$ for every minimal prime ideal P.*

Proof. Assume that R is 2-primal. Let P be any minimal prime ideal of R. Then by Lemma 2, $S = R \setminus P$ is multiplicatively closed. Also note that $\overline{O}(P) \subseteq \overline{O}_P \subseteq P$ by [4, Proposition 1.2]. So it remains to show that $P \subseteq \overline{O}(P)$. Let a be a nonzero element of P and F be the multiplicative system generated by $S \cup \{a\}$. By the similar method of the proof in Theorem 6, we have $0 = a^{n_1}s_1a^{n_2}s_2 \cdots a^{n_k}s_k$, where $s_i \in S$ and n_i positive integers. Hence $aRs \subseteq P(R)$ where $s = s_1s_2 \cdots s_k \in S$. So there exists a positive integer m such that $a^mRs^m = 0$ by hypothesis. Consequently $a \in \overline{O}(P)$ and so $P = \overline{O}(P) = \overline{O}_P$. Conversely, assume that $P = \overline{O}(P) = \overline{O}_P$ for every minimal prime ideal of R. Since $N(R) \subseteq \overline{O}(P)$ for every prime ideal P of R, R is 2-primal. $\square$

Note that in Theorem 6, condition "R is right weakly π-regular" can not be replaced by the condition "R is π -regular". The next example shows that in a 2-primal ring with condition $(*)$ "weak π-regularity" is not equivalent to "π-regularity".

Example 10. Let W be a simple domain with identity which is not a division ring, and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in W \right\}.$$

Observe that R is isomorphic to the split-null extension $S(W, W)$.

Claim. R is a pseudo symmetric ring, and every prime ideal is maximal; but it is not π-regular and is neither right nor left weakly regular. However it is a weakly π-regular ring.

Proof. Combine Shin [14, Proposition 1.6] with Birkenmeier, Kim and Park [3, Example 3.1]. However, by Proposition 5 and Theorem 6, it is a weakly π-regular ring. □

The next Corollary 11 shows that a 2-primal ring with the condition$(*)$ needs not be a right(or left) duo ring.

Corollary 11. ([10, Corollary 1]) *Let R be a right (or left) duo ring. Then R is π-regular if and only if every prime ideal is maximal.*

Proof. See [2, Corollary 10]. □

Corollary 12. *Let R satisfy condition* (PS II) (*e.g., a pseudo symmetric ring*). *Then the following conditions are equivalent:*

(1) *R is weakly π-regular.*
(2) *$R/P(R)$ is right weakly π-regular.*
(3) *Every prime ideal of R is maximal.*

Example 13. There is a ring R with (PS I) (hence 2-primal) such that every prime ideal is maximal, but it is neither right nor left weakly π-regular. Assume that $W_1[F]$ is the first Weyl algebra over a field F of characteristic zero. Let

$$R = \begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix}.$$

Then since every prime ideal of R is completely prime, R satisfies (PS I). Furthermore, we see that every prime ideal of R is maximal. Also as in [2, Example 12], $R/P(R)$ is weakly regular, hence it is weakly π-regular. But R is neither right nor left weakly π-regular.

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