

SOME CHARACTERIZATIONS OF REAL HYPERSURFACES OF TYPE (A) IN A NONFLAT COMPLEX SPACE FORM

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ABSTRACT. In this paper, we prove that if the structure Jacobi operator R_ξ is ξ -parallel and R_ξ commutes with the Ricci tensor S , then a real hypersurface with non-negative scalar curvature of a nonflat complex space form $\mathbf{M}_n(c)$ is a Hopf hypersurface. Further, we characterize such Hopf hypersurface in $\mathbf{M}_n(c)$.

1. Introduction

A complex n -dimensional Kähler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $\mathbf{M}_n(c)$. As is well-known, a connected complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbf{P}_n\mathbb{C}$, a complex Euclidean space \mathbb{C}_n or a complex hyperbolic space $\mathbf{H}_n\mathbb{C}$ according as $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface \mathbf{M} in a complex space form $\mathbf{M}_n(c)$, $c \neq 0$. Then \mathbf{M} has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler metric and complex structure J on $\mathbf{M}_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of \mathbf{M} and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [6] and \mathbf{M} is called *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $\mathbf{P}_n\mathbb{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary group $\mathbf{PU}(n+1)$. Takagi [13] completely classified such hypersurfaces as six model spaces which are said to be A_1 , A_2 , B , C , D and E .

On the other hand, real hypersurfaces in $\mathbf{H}_n\mathbb{C}$ have been investigated by Berndt [1], Montiel and Romero [8] and so on. Berndt [1] classified Hopf

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hypersurfaces in a complex hyperbolic space whose all principal curvatures are constant as four model spaces which are said to be A_0 , A_1 , A_2 and B .

We introduce the following theorem without proof which due to Okumura [10] for $c > 0$, Montiel and Romero [8] for $c < 0$ respectively.

Theorem O-MR. ([10], [8]). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies $A\phi - \phi A = 0$, then M is locally congruent to one of the following hypersurfaces:*

- (1) In case $P_n\mathbb{C}$
 - (A₁) a geodesic hypersphere of radius r , where $0 < r < \pi/2$;
 - (A₂) a tube of radius r over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n - 2$), where $0 < r < \pi/2$.
- (2) In case $H_n\mathbb{C}$
 - (A₀) a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube;
 - (A₁) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;
 - (A₂) a tube of a totally geodesic $H_k\mathbb{C}$ ($1 \leq k \leq n - 2$).

Let M be a real hypersurface of type (A₁) or (A₂) in a $P_n\mathbb{C}$, or type (A₀), (A₁) or (A₂) in a $H_n\mathbb{C}$. Then M is said to be *type (A)* for simplicity.

The curvature tensor field R on a Riemannian manifold (M, g) is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

for any vector fields X and Y on (M, g) . We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X . Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related with (the Jacobi vector equation) $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ , where $\dot{\gamma}$ denotes the velocity vector field along γ .

When we study a real hypersurface M in a complex space form, we will call the Jacobi operator on M with respect to the structure vector field ξ a *structure Jacobi operator* on M and will denote it by R_ξ , where R_ξ is defined by $R_\xi(X) = R(X, \xi)\xi$. However, recently it is known that there are no real hypersurfaces in $M_n(c)$ with parallel Jacobi operator R_ξ (see [11] and [12]). Some works have recently studied several conditions on the structure Jacobi operator R_ξ and given some results on the classification of real hypersurfaces of type (A) in a complex space form ([3], [4], [5], [8] and [10] etc.). One of them, Cho and Ki, one of the present authors, proved the following:

Theorem O-MR. ([4]). *Let M be a connected real hypersurface of $M_n(c)$ $c \neq 0$ whose shape operator A commutes R_ξ , that is, $R_\xi A = AR_\xi$. Then M satisfies $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

- (1) In case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,
 - (A₁) a geodesic hypersphere of radius r , where $0 < r < \pi/2$ and $r \neq \pi/4$;
 - (A₂) a tube of radius r over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n - 2$), where $0 < r < \pi/2$ and $r \neq \pi/4$.

- (2) In case that $\mathbf{M}_n(c) = \mathbf{H}_n\mathbb{C}$,
 - (A₀) a horosphere;
 - (A₁) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbf{H}_{n-1}\mathbb{C}$;
 - (A₂) a tube over a totally geodesic $\mathbf{H}_k\mathbb{C}$ ($1 \leq k \leq n - 2$).

In this paper, we study a real hypersurface in a nonflat complex space form $\mathbf{M}_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$, where S denotes the Ricci tensor of the hypersurface. We give another characterization of real hypersurfaces of type (A) in a nonflat complex space form by above two conditions. The main purpose of the present paper is to establish Theorem 6.1.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be orientable.

2. Preliminaries

Let \mathbf{M} be a real hypersurface immersed in a complex space form $\mathbf{M}_n(c)$, and N be a unit normal vector field of \mathbf{M} . By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor \tilde{g} of $\mathbf{M}_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y , where g denoted the Riemannian metric tensor of \mathbf{M} induced from \tilde{g} , and A is the shape operator of \mathbf{M} in $\mathbf{M}_n(c)$. For any vector field X tangent to \mathbf{M} we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $\mathbf{M}_n(c)$. Then we may see that \mathbf{M} induces an almost contact metric structure (ϕ, ξ, η, g) , that is

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, & g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(\xi) &= 1, & \phi\xi &= 0, & \eta(X) &= g(X, \xi) \end{aligned}$$

for any vector fields X and Y on \mathbf{M} .

Since J is parallel, we verify from the Gauss and Weingarten formulas the following

$$(2.1) \quad \nabla_X \xi = \phi AX,$$

$$(2.2) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

Since the ambient manifold is of constant holomorphic sectional curvature c , we have the following Gauss and Codazzi equations respectively:

$$\begin{aligned} (2.3) \quad & R(X, Y)Z \\ &= \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(2.4) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any vector fields X , Y and Z on \mathbf{M} , where R denotes the Riemannian curvature tensor of \mathbf{M} .

Now let us denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$, $\mu^2 = \beta - \alpha^2$, $h = \text{trace}A$, and ∇f the gradient vector field of a function f defined on \mathbf{M} . In the following, we use the same terminology and notation as above unless otherwise stated. We shall denote the Ricci tensor of type (1,1) by S . Then it follows from (2.3) that

$$(2.5) \quad SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X,$$

which implies

$$(2.6) \quad S\xi = \frac{c}{2}(n-1)\xi + hA\xi - A^2\xi.$$

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector field ξ . Using (2.1) we see that

$$(2.7) \quad \phi U = -A\xi + \alpha\xi,$$

which shows that $g(U, U) = \beta - \alpha^2$. By definition of U and (2.1) we verify that

$$(2.8) \quad g(\nabla_X \xi, U) = g(A^2\xi, X) - \alpha g(A\xi, X).$$

Now, differentiating (2.7) covariantly along \mathbf{M} and using (2.1) and (2.2), we find that

$$(2.9) \quad \begin{aligned} & \eta(X)g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ &= g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y). \end{aligned}$$

Together with (2.4) we obtain

$$(2.10) \quad (\nabla_\xi A)\xi = 2AU + \nabla\alpha.$$

From (2.9) we also have

$$(2.11) \quad \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha,$$

where we have used (2.1) and (2.8).

If $A\xi - \eta(A\xi)\xi \neq 0$, we can put

$$(2.12) \quad A\xi = \alpha\xi + \mu W,$$

where W is a unit vector field orthogonal to ξ . Then from (2.7) it is clear that $U = \mu\phi W$ and hence $g(U, U) = \mu^2$, and W is also orthogonal to U . Using (2.1) we see that

$$(2.13) \quad \mu g(\nabla_X W, \xi) = g(AU, X).$$

From Gauss equation (2.3) we know that the structure Jacobi operator R_ξ is given by

$$(2.14) \quad R_\xi X = R(X, \xi)\xi = \frac{c}{4}\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi$$

for any vector field X on \mathbf{M} .

In what follows we assume that $\mu \neq 0$ on \mathbf{M} , that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in \mathbf{M} | \mu(p) \neq 0\}$. Suppose that Ω is not empty. Then Ω is an open subset of \mathbf{M} , and from now on we discuss our arguments on Ω .

3. ξ -parallel structure Jacobi operator

Let \mathbf{M} be a real hypersurface in a complex space form $\mathbf{M}_n(c)$, $c \neq 0$ satisfying $\nabla_\xi R_\xi = 0$, which means that the structure Jacobi operator is ξ -parallel.

Differentiating (2.14) covariantly, we find

$$\begin{aligned} &g((\nabla_X R_\xi)Y, Z) \\ &= -\frac{c}{4}\{\eta(Z)g(\nabla_X \xi, Y) + \eta(Y)g(\nabla_X \xi, Z) + (X\alpha)g(AY, Z) \\ &\quad + \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &\quad - g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\}. \end{aligned}$$

Together with (2.1) and (2.10) we have

$$\begin{aligned} &g((\nabla_\xi R_\xi)Y, Z) \\ &= -\frac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &\quad + \alpha g((\nabla_\xi A)Y, Z) - g(A\xi, Z)\{3g(AU, Y) + Y\alpha\} \\ &\quad - g(A\xi, Y)\{3g(AU, Z) + Z\alpha\}, \end{aligned}$$

where u is a 1-form defined by $u(X) = g(U, X)$ for any vector field X .

From the last equation and $\nabla_\xi R_\xi = 0$ we have

$$\begin{aligned} (3.1) \quad &\alpha(\nabla_\xi A)X + (\xi\alpha)AX \\ &= \frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)\{3AU + \nabla\alpha\} \\ &\quad + \{3g(AU, X) + X\alpha\}A\xi. \end{aligned}$$

Putting $X = \xi$ in this and using (2.10) we get

$$(3.2) \quad \alpha AU + \frac{c}{4}U = 0,$$

which shows that $\alpha \neq 0$ on Ω .

Putting $X = \alpha U$ in (3.1) and making use of (3.2) we find

$$(3.3) \quad \alpha^2(\nabla_\xi A)U - \frac{c}{4}(\xi\alpha)U = \frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}A\xi.$$

Because of (3.2), the equation (2.11) turns out to be

$$(3.4) \quad \alpha\nabla_\xi U = \frac{3}{4}c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha.$$

Differentiating (3.2) covariantly along Ω , we find

$$(3.5) \quad (X\alpha)AU + \alpha(\nabla_X A)U + \alpha A(\nabla_X U) + \frac{c}{4}\nabla_X U = 0.$$

If we replace $X = \alpha\xi$ in this equation and take account of (3.2) and (3.3), we can obtain

$$\frac{c}{4}\alpha\mu^2\xi + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}A\xi + \alpha^2A(\nabla_\xi U) + \frac{c}{4}\alpha\nabla_\xi U = 0.$$

Together with (3.4) we get

$$(3.6) \quad \alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \mu(\alpha^2 + \frac{3}{4}c)\{AW - \mu\xi - \frac{1}{\alpha}(\mu^2 - \frac{c}{4})W\} = 0,$$

where we have used (2.12).

Using (2.4) and (2.7), we verify from (3.5) that

$$(3.7) \quad \frac{c}{4}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} \\ + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X, Y) = 0,$$

where w is a 1-form defined by $w(X) = g(W, X)$, and the exterior derivative du of 1-form u is given by

$$du(X, Y) = \frac{1}{2}\{Yu(X) - Xu(Y) - u([X, Y])\}.$$

If we replace X by U in (3.7), then we obtain

$$(3.8) \quad \frac{c}{4}(\mu^2\nabla\alpha - (U\alpha)U) + \alpha^2A\nabla_U U + \frac{c}{4}\alpha\nabla_U U = 0,$$

because U and W are mutually orthogonal.

Combining (2.9) to (3.1) and using the Codazzi equation (2.4) and (3.2), we obtain

$$\alpha^2\phi\nabla_X U = -\alpha^2(X\alpha)\xi + \frac{c}{4}\alpha u(X)\xi - \alpha(\xi\alpha)AX - \frac{c}{4}\alpha^2\phi X \\ + g(A\xi, X)(\alpha\nabla\alpha - \frac{3}{4}cU) + (\alpha(X\alpha) - \frac{3}{4}cu(X))A\xi \\ + \frac{c}{4}(u(X)\xi + \eta(X)U) - \alpha^2A\phi AX + \alpha^3\phi AX.$$

Applying this by ϕ and using (2.8) and (2.12), we have

$$(3.9) \quad \alpha^2\nabla_X U + \alpha^2g(AW, X)\xi - \alpha g(A\xi, X)\phi\nabla\alpha \\ = \alpha(\xi\alpha)\phi AX + \frac{c}{4}\alpha^2(X - \eta(X)\xi) + \frac{3}{4}c\mu g(A\xi, X)W + \alpha(X\alpha)U \\ - \frac{3}{4}cu(X)U + \alpha^3AX - \frac{c}{4}\alpha\mu\eta(X)W - \alpha^3\eta(X)A\xi - \alpha^2\phi A\phi AX.$$

Putting $X = U$ in (3.9) and taking account of (2.7), (2.12) and (3.2), we obtain

$$(3.10) \quad \alpha^2\nabla_U U = -\frac{c}{4}\mu(\xi\alpha)W + \{\alpha(U\alpha) - \frac{3}{4}c\mu^2\}U + \frac{c}{4}\mu\alpha\phi AW.$$

4. Real hypersurfaces satisfying $R_\xi S = SR_\xi$

Let M be a real hypersurface in $M_n(c)$, $c \neq 0$ satisfying $R_\xi S = SR_\xi$, which means that the Ricci tensor S of type (1,1) and the Jacobi operator R_ξ with respect to the structure vector field ξ commute to each other. Then we have

$$\begin{aligned} & g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ = & g(A^2\xi, Y)g(hA\xi - \frac{c}{4}\xi, X) - g(A^2\xi, X)g(hA\xi - \frac{c}{4}\xi, Y) \\ & + \frac{c}{4}h\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}, \end{aligned}$$

where we have used (2.6) and (2.14), which shows that

$$(4.1) \quad \alpha A^3\xi = (\alpha h - \frac{c}{4})A^2\xi + (\gamma - \beta h + \frac{c}{4}h)A\xi + \frac{c}{4}h(\beta - h\alpha)\xi.$$

Combining above two equations, using (2.12) we see that

$$\mu\{g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)\} = \beta\{\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)\}.$$

Putting $Y = A\xi$ in this equation, we find that

$$\mu^2 g(A^2\xi, X) = \mu\gamma w(X) - \beta\alpha g(A\xi, X) + \beta^2\eta(X).$$

It follows that

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi$$

and consequently

$$(4.2) \quad A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi,$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$ on Ω .

From (2.12) and (4.2), we see that

$$(4.3) \quad AW = \mu\xi + (\rho - \alpha)W$$

and

$$(4.4) \quad A^2W = \rho AW + (\beta - \rho\alpha)W$$

because of $\mu \neq 0$.

Now, differentiating (4.3) covariantly, we find

$$(4.5) \quad (\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X \xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W,$$

which implies that

$$(4.6) \quad g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha,$$

$$(4.7) \quad \mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu,$$

where we have used (2.4) and (2.13).

Replacing X by ξ in (4.5) and making use of (4.7), we find

$$\begin{aligned} (4.8) \quad & (\rho - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{A\nabla_\xi W - (\rho - \alpha)\nabla_\xi W\} \\ & = \mu(\xi\mu)\xi + \mu^2U + \mu(\xi\rho - \xi\alpha)W. \end{aligned}$$

Since we have $\phi U = -\mu W$, it is verified that

$$g(AU, X)\xi - \phi \nabla_X U = (X\mu)W + \mu \nabla_X W.$$

Putting $X = \xi$ in this equation, and using (2.7) and (2.11), we obtain

$$(4.9) \quad \mu \nabla_\xi W = 3AU - \alpha U + \nabla \alpha - (\xi \alpha)\xi - (\xi \mu)W,$$

which shows that

$$(4.10) \quad \xi \mu = W\alpha.$$

Using the last two equations, (4.8) turns out to be

$$(4.11) \quad \begin{aligned} & 3A^2U - 2\rho AU + A\nabla \alpha + \frac{1}{2}\nabla \beta - \rho \nabla \alpha + (\alpha\rho - \beta - \frac{c}{4})U \\ &= 2\mu(W\alpha)\xi + \mu(\xi\rho)W - (\rho - 2\alpha)(\xi\alpha)\xi. \end{aligned}$$

Differentiating (4.2) covariantly and making use of (2.1), we find

$$\begin{aligned} & (\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - \rho A\phi AX \\ &= (X\rho)A\xi + \rho(\nabla_X A)\xi + X(\beta - \rho\alpha)\xi + (\beta - \rho\alpha)\phi AX, \end{aligned}$$

which together with (2.4) implies that

$$\begin{aligned} & \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \frac{c}{2}(\rho - \alpha)g(\phi Y, X) - g(A^2\phi AX, Y) \\ &+ g(A^2\phi AY, X) + 2\rho g(\phi AX, AY) - (\beta - \rho\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ &= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Y\rho)g(A\xi, X) - (X\rho)g(A\xi, Y) \\ &+ Y(\beta - \rho\alpha)\eta(X) - X(\beta - \rho\alpha)\eta(Y). \end{aligned}$$

If we replace X by μW to both sides of the last equation and take account of (2.4), (2.10), (4.3), (4.4), and (4.7), then we obtain

$$(4.12) \quad \begin{aligned} & (3\alpha - 2\rho)A^2U + 2(\rho^2 + \beta - 2\rho\alpha + \frac{c}{4})AU + (\rho - \alpha)(\beta - \rho\alpha - \frac{c}{2})U \\ &= \mu A\nabla \mu + (\alpha\rho - \beta)\nabla \alpha - \frac{1}{2}(\rho - \alpha)\nabla \beta + \mu^2\nabla \rho \\ &\quad - \mu(W\rho)A\xi - \mu W(\beta - \rho\alpha)\xi. \end{aligned}$$

From (4.6) and (4.7) we see that

$$(4.13) \quad W\mu = \xi\rho - \xi\alpha.$$

5. Real hypersurfaces with $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$

We will continue our arguments under the hypotheses $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$. Then (3.6) turns out to be

$$\alpha A\phi \nabla \alpha + \frac{c}{4}\phi \nabla \alpha + (U\alpha)A\xi + \frac{1}{\alpha}\mu(\alpha^2 + \frac{3}{4}c)(\rho\alpha + \frac{c}{4} - \beta)W = 0$$

by virtue of (4.3). Taking an inner product this with W and using (4.3), we find

$$(\beta - \rho\alpha - \frac{c}{4})\{\alpha(U\alpha) - \mu^2(\alpha^2 + \frac{3}{4}c)\} = 0,$$

which implies

$$(5.1) \quad \alpha(U\alpha) = \mu^2(\alpha^2 + \frac{3}{4}c).$$

In fact, if not, then we have $\beta = \rho\alpha + \frac{c}{4}$ on this subset. So (4.2) becomes $A^2\xi = \rho A\xi + \frac{c}{4}\xi$. From this and (2.14) we see that $R_\xi A = AR_\xi$ on the set Ω . According to Theorem CK([4]), we verify that $\Omega = \emptyset$ because $\nabla_\xi R_\xi = 0$ was assumed. Accordingly (5.1) is proved.

Because of (4.3) and (5.1), the equation (3.10) can be written as

$$(5.2) \quad \alpha^2 \nabla_U U = -\frac{c}{4}\mu(\xi\alpha)W + \alpha(\alpha\mu^2 + \frac{c}{4}(\rho - \alpha))U,$$

which together with (3.2) yields that

$$\alpha^2 A \nabla_U U = -\frac{c}{4}\mu(\xi\alpha)AW - \frac{c}{4}(\alpha\mu^2 + \frac{c}{4}(\rho - \alpha))U.$$

Substituting (5.1) and the last two equations into (3.8) and making use of (4.3), we get

$$\alpha\mu\nabla\alpha = \alpha\mu(\xi\alpha)\xi + (\rho\alpha - \alpha^2 + \frac{c}{4})(\xi\alpha)W + (\alpha^2 + \frac{3}{4}c)\mu U,$$

which enables us to obtain

$$(5.3) \quad \mu\alpha(W\alpha) = (\rho\alpha - \alpha^2 + \frac{c}{4})\xi\alpha.$$

From above two equations it follows that

$$(5.4) \quad \alpha\nabla\alpha = \alpha(\xi\alpha)\xi + \alpha(W\alpha)W + (\alpha^2 + \frac{3}{4}c)U.$$

Using (3.2), the equation (3.1) can be written as

$$\begin{aligned} \alpha^2(\nabla_\xi A)X &= -\alpha(\xi\alpha)AX + \frac{c}{4}\alpha\{u(X)\xi + \eta(X)U\} \\ &\quad + (\alpha(X\alpha) - \frac{3}{4}cu(X))A\xi + \eta(AX)(\alpha\nabla\alpha - \frac{3}{4}cU). \end{aligned}$$

Putting $X = \mu W$ in this and taking account of (3.2) and (4.7), we find

$$\alpha(\frac{1}{2}\alpha\nabla\beta - \beta\nabla\alpha) + \frac{c}{4}(3\beta - 2\alpha^2 - \rho\alpha)U = -\mu\alpha(\xi\alpha)AW + \mu\alpha(W\alpha)A\xi,$$

where we have used $\mu^2 = \beta - \alpha^2$, or using (2.12), (4.3) and (5.3),

$$(5.5) \quad \alpha^2\nabla\beta - \beta\nabla\alpha^2 + \frac{c}{2}(3\beta - 2\alpha^2 - \rho\alpha)U = (\xi\alpha)\{2\alpha(\rho\alpha - \beta)\xi + \frac{c}{2}A\xi\}.$$

If we use (2.12), (4.3) and (5.4), then (3.9) is reduced to

$$\begin{aligned} & \alpha \nabla_X U + \alpha \mu^2 \eta(X) \xi + g(A\xi, X) \left\{ \mu \alpha W - \frac{1}{\mu} (W\alpha) U \right\} \\ = & (\xi \alpha) \phi AX + \frac{c}{4} \alpha (X - \eta(X) \xi) + \{(W\alpha) w(X) + \alpha u(X)\} U \\ & + \alpha^2 AX - \alpha^3 \eta(X) \xi - \frac{c}{4} \mu \eta(X) W - \alpha \phi A \phi AX, \end{aligned}$$

which implies that

$$\begin{aligned} (5.6) \quad \alpha du(X, Y) = & \mu(\rho \alpha - \alpha^2 - \frac{c}{4})(\eta(X)w(Y) - \eta(Y)w(X)) \\ & + \{\xi \alpha + \frac{\alpha}{\mu}(W\alpha)\}(\eta(X)u(Y) - \eta(Y)u(X)) \\ & + 2(W\alpha)(w(X)u(Y) - w(Y)u(X)) \\ & + (\xi \alpha)\{g(\phi AX, Y) - g(\phi AY, X)\} \\ & - \alpha\{g(\phi A \phi AX, Y) - g(\phi A \phi AY, X)\}. \end{aligned}$$

Putting $X = \mu \xi$ in (5.6) and using (2.7), (3.2) and (4.3), we find

$$(5.7) \quad \mu du(\xi, Y) = (W\alpha)u(Y) + \mu^2(\rho - \alpha)w(Y).$$

Since we have $\mu^2 = \beta - \alpha^2$, it is, using (4.10) and (5.3), seen that

$$(5.8) \quad \alpha(\xi \beta) = (2\rho \alpha + \frac{c}{2})\xi \alpha.$$

On the other hand, we obtain from (3.1)

$$\alpha^2(\xi h) = \alpha(2\alpha - h)\xi \alpha + 2\mu \alpha(W\alpha)$$

because of (2.12) and (3.2), which together with (5.3) gives

$$(5.9) \quad \alpha^2(\xi h) = \{\alpha(2\rho - h) + \frac{c}{2}\}\xi \alpha.$$

By the way, combining (4.1) to (4.2), we verify that $(h - \rho)(\beta - \rho \alpha - \frac{c}{4}) = 0$ on Ω . Hence $h - \rho = 0$ by virtue of $\beta - \rho \alpha - \frac{c}{4} \neq 0$ on Ω . Accordingly (5.9) reformed as

$$(5.10) \quad \alpha^2(\xi \rho) = (\rho \alpha + \frac{c}{2})\xi \alpha.$$

Using (5.9) and (5.10), we see that

$$(5.11) \quad \xi(\rho \alpha - \alpha^2) = \xi \mu^2 = \{2(\rho - \alpha) + \frac{c}{4\alpha}\}\xi \alpha.$$

Now, we put $\sigma = \beta - \rho \alpha$. Then we obtain

$$(5.12) \quad \xi \sigma = 0$$

by virtue of (5.10) and (5.11).

Owing to (3.2) and (5.4), we verify from (4.9) that

$$(5.13) \quad \nabla_\xi W = 0.$$

If we take a inner product (5.5) with \bar{U} and use (5.1), then we get

$$(5.14) \quad \frac{1}{2}\alpha(U\beta) = \{\beta\alpha + \frac{c}{4}(2\alpha + \rho)\}\mu^2.$$

This, together with (5.1) implies that

$$(5.15) \quad \alpha\mu(U\mu) = \{\alpha\mu^2 + \frac{c}{4}(\rho - \alpha)\}\mu^2.$$

Applying (4.12) by α^2U and making use of (3.2), (5.1) and (5.14), we find

$$\alpha^2(U\rho) = \alpha^2(2\beta\rho - \rho^2\alpha - \beta\alpha) + \frac{c}{4}\alpha(2\beta - \rho^2 - \alpha^2) - \frac{c^2}{16}\rho,$$

which connected to (5.1) and (5.14) yields

$$(5.16) \quad \begin{aligned} \alpha(U\sigma) &= \{2\beta\alpha - \rho\alpha^2 + \frac{c}{4}(4\alpha - \rho)\}\mu^2 + \alpha^2(\alpha\rho^2 + \beta\alpha - 2\beta\rho) \\ &\quad + \frac{c}{4}\alpha(\rho^2 + \alpha^2 - 2\beta) + \frac{c^2}{16}\rho. \end{aligned}$$

Now, we prove

Lemma 1. $\xi\alpha = 0$ and $W\alpha = 0$ on Ω .

Proof. By the definition we have

$$\alpha\mu\nabla\mu = \alpha(\frac{1}{2}\nabla\beta - \alpha\nabla\alpha).$$

Substituting (5.4) and (5.5) into this, and taking account of (2.12), (4.3) and (5.3), we obtain

$$\frac{1}{2}\alpha^2\nabla\mu^2 = \{(\alpha^2 + \frac{c}{4})\mu^2 - \frac{c}{4}\sigma\}U + \varepsilon\{(\mu^2 - \sigma)A\xi - \frac{c}{4}\xi\},$$

where we have put $\varepsilon = \xi\alpha$. This is rewritten as

$$(5.17) \quad \frac{1}{2}a^2(Y\mu^2) = \{(\alpha^2 + \frac{c}{4})\mu^2 - \frac{c}{4}\sigma\}u(Y) + \varepsilon\{(\mu^2 - \sigma)\eta(AY) - \frac{c}{4}\alpha\eta(Y)\}.$$

Differentiating this with respect to a vector field X again, and taking the skew-symmetric parts for X and Y , we eventually have

$$\begin{aligned} &\{\alpha(X\alpha) + \varepsilon\eta(AX) + (\alpha^2 + \frac{c}{4})u(X)\}(Y\mu^2) \\ &\quad - \{\alpha(Y\alpha) + \varepsilon\eta(AY) + (\alpha^2 + \frac{c}{4})u(Y)\}(X\mu^2) \\ &\quad + (X\alpha)(\frac{c}{4}\varepsilon\eta(Y) - 2\alpha\mu^2u(Y)) - (Y\alpha)(\frac{c}{4}\varepsilon\eta(X) - 2\alpha\mu^2u(X)) \end{aligned}$$

$$\begin{aligned}
& +(X\sigma)(\varepsilon\eta(AY) + \frac{c}{4}u(Y)) - (Y\sigma)(\varepsilon\eta(AX) + \frac{c}{4}u(X)) \\
= & \{(\alpha^2 + \frac{c}{4})\mu^2 - \frac{c}{4}\sigma\}du(X, Y) + \alpha(\rho - \alpha)\varepsilon(2g(A\phi AX, Y) - \frac{c}{2}g(\phi X, Y)) \\
& - \frac{c}{4}\alpha\varepsilon(g(\phi AX, Y) - g(\phi AY, X)) + \alpha(X\varepsilon)\{(\rho - \alpha)\eta(AY) - \frac{c}{4}\eta(Y)\} \\
& - \alpha(Y\varepsilon)\{(\rho - \alpha)\eta(AX) - \frac{c}{4}\eta(X)\},
\end{aligned}$$

where we have used the Codazzi equation (3.2). Putting $X = U$ and $Y = \alpha\xi$ in the last equation and making use of (3.2), (4.10) and (5.12), we find

$$\begin{aligned}
& 4\varepsilon\alpha^2\mu(U\mu) - 2\mu\alpha(W\alpha)\{\alpha(U\alpha) + (\alpha^2 + \frac{c}{4})\mu^2\} \\
& - 2\alpha^2\mu^4\varepsilon - \frac{c}{4}\varepsilon\alpha^2(U\sigma) - \varepsilon\alpha^2(U\sigma) \\
(5.18) \quad = & \{(\alpha^2 + \frac{c}{4})\mu^2 - \frac{c}{4}\sigma\}\alpha du(\xi, U) - \alpha^2(\rho\alpha - \alpha^2 - \frac{c}{4})U\varepsilon \\
& + \frac{c}{2}\alpha(\alpha - \rho)\varepsilon\mu^2 - \frac{c}{4}\varepsilon\alpha^2\mu^2.
\end{aligned}$$

Let Ω_0 be a set of points such that $(\xi\alpha)_p \neq 0$ at $p \in \Omega$ and suppose that $\Omega_0 \neq \emptyset$. We then have from the last equation

$$\begin{aligned}
\frac{1}{\varepsilon}\alpha^2(\rho\alpha - \alpha^2 - \frac{c}{4})U\varepsilon = & \alpha^2(U\sigma) - 2\alpha^2\mu^4 + \frac{3}{2}c\alpha(\alpha - \rho)\mu^2 \\
& + (\rho\alpha - \alpha^2 + \frac{c}{4})\{5\mu^2(\alpha^2 + \frac{9}{4}c) - \frac{c}{4}\sigma\}
\end{aligned}$$

on Ω_0 , where we have used (5.1), (5.3), (5.7) and (5.15). Since we have $\rho\alpha - \alpha^2 = \mu^2 - \sigma$, this is reformed as

$$\begin{aligned}
(5.19) \quad \frac{1}{\varepsilon}\alpha^2(\mu^2 - \sigma - \frac{c}{4})U\varepsilon - \alpha^2(U\sigma) \\
= & 3(\alpha^2 + \frac{c}{4})\mu^4 + \{\frac{c}{4}(5\sigma + \frac{3}{4}c) - (5\alpha^2 + \frac{9}{4}c)(\sigma - \frac{c}{4})\}\mu^2 + \frac{c}{4}\sigma(\sigma - \frac{c}{4})
\end{aligned}$$

on Ω_0 .

On the other hand, we have from (5.5)

$$\alpha^2(Y\beta) - \beta(X\alpha^2) + \frac{c}{2}(2\mu^2 + \sigma)u(Y) = \varepsilon\{\frac{c}{2}\eta(AY) - 2\alpha\sigma\eta(Y)\}.$$

Using the same method as that used to derive (5.18) from (5.17), we can deduce from the last equation the following :

$$\begin{aligned}
& \alpha^2(\frac{c}{4} - \sigma)U\varepsilon - \alpha^2\varepsilon(U\sigma) \\
= & \frac{c}{4}\varepsilon\sigma(\rho\alpha - \alpha^2 + \frac{c}{4}) + \{(4\rho\alpha + c)(\alpha^2 + \frac{3}{4}c) - 4\beta\alpha^2 - c(\rho\alpha + 2\alpha^2) \\
& - \frac{c}{2}(\rho\alpha - \alpha^2 + \frac{c}{4}) + \frac{3}{4}c\sigma - \frac{c^2}{8}\}\mu^2,
\end{aligned}$$

which shows that

$$(5.20) \quad \begin{aligned} & \frac{1}{\varepsilon} \alpha^2 \left(\frac{c}{4} - \sigma \right) - \alpha^2 (U\sigma) \\ &= \frac{3}{2} c \mu^4 + \frac{c}{4} \sigma (\mu^2 - \sigma + \frac{c}{4}) + (c\alpha^2 - \frac{3}{4} c\sigma - 4\alpha^2 \sigma + \frac{c^2}{2}) \mu^2 \end{aligned}$$

on Ω_0 . From (5.19) and (5.20) it follows that

$$(5.21) \quad \begin{aligned} \frac{1}{\varepsilon} (\mu^2 - \frac{c}{2}) \alpha^2 (U\varepsilon) &= 3(\alpha^2 - \frac{c}{4}) \mu^4 + \frac{c}{2} \sigma (\sigma - \frac{c}{4}) \\ &+ (\frac{c}{4} \alpha^2 - \frac{c}{2} \sigma - \alpha^2 \sigma + \frac{c^2}{4}) \mu^2 \end{aligned}$$

on Ω_0 .

Now, we know from (5.4)

$$(5.22) \quad Y\alpha = \varepsilon\eta(Y) + (W\alpha)w(Y) + \frac{1}{\alpha} (\alpha^2 + \frac{3}{4}c)u(Y).$$

In the same way as above, it is, using (5.22), verified that

$$\begin{aligned} & \alpha^2 \{ (X\varepsilon)\eta(Y) - (Y\varepsilon)\eta(X) \} + \alpha^2 \varepsilon \{ g(\phi AX, Y) - g(\phi AY, X) \} \\ & + \alpha^2 \{ X(W\alpha)w(Y) - Y(W\alpha)w(X) \} \\ & + (\alpha^2 - \frac{3}{4}c) \{ (X\alpha)u(Y) - (Y\alpha)u(X) \} + \alpha(\alpha^2 + \frac{3}{4}c)du(X, Y) = 0. \end{aligned}$$

Putting $X = U$ and $Y = \xi$ in this and using (2.13), (3.2) and (5.13), we find

$$\alpha^2 (U\varepsilon) - (2\alpha^2 - \frac{3}{4}c)\mu^2 \varepsilon - \frac{c}{4} \alpha \mu (W\alpha) + \alpha(\alpha^2 + \frac{3}{4}c)du(\xi, U) = 0,$$

which together with (5.3) and (5.7) implies that

$$\frac{1}{\varepsilon} \alpha^2 (U\varepsilon) = (\alpha^2 - \frac{5}{4}c)\mu^2 + (\sigma - \frac{c}{4})(\alpha^2 + \frac{c}{2})$$

on Ω_0 . Substituting this into (5.21), we find on Ω_0

$$(2\alpha^2 + \frac{c}{2})\mu^4 + (c\alpha^2 - 2\sigma\alpha^2 + c\sigma - \frac{c^2}{4})\mu^2 + \frac{c}{2}(\sigma - \frac{c}{4})(\alpha^2 + \sigma + \frac{c}{2}) = 0.$$

Differentiating this with respect to a vector field X , and taking several choices of X , we eventually have $\varepsilon = 0$ and hence $\Omega_0 = \emptyset$, where we have used (5.1), (5.11), (5.12), (5.15) and (5.16). This means that $\xi\alpha = 0$ is accomplished everywhere on Ω . From this and (5.3) it follows that $W\alpha = 0$. This completes the proof of Lemma 1. \square

According to Lemma 1, equations (5.4), (5.5) and (5.6) are reduced respectively to

$$(5.23) \quad \alpha \nabla \alpha = (\alpha^2 + \frac{3}{4}c)U,$$

$$(5.24) \quad \alpha^2 \nabla \beta - \beta \nabla \alpha^2 + \frac{c}{2}(3\beta - 2\alpha^2 + \rho\alpha)U = 0,$$

$$(5.25) \quad \alpha du(X, Y) = \mu(\rho\alpha - \alpha^2 - \frac{c}{4})(\eta(X)w(Y) - \eta(Y)w(X)) + \alpha\{g(\phi A\phi AY, X) - g(\phi A\phi AX, Y)\}.$$

6. Proof of main theorem

We will continue our discussions for real hypersurfaces \mathbf{M} in $\mathbf{M}_n(c)$, $c \neq 0$ which satisfies $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$ as in section 5.

Using the formulas (2.9), (3.2), (4.12), (5.2), (5.23), (5.24) and (5.25), we will prove

Theorem 6.1. *Let \mathbf{M} be a real hypersurface with non-negative scalar curvature in a nonflat complex space form $\mathbf{M}_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$, where S denotes the Ricci tensor of \mathbf{M} . Then \mathbf{M} is a Hopf hypersurface in $\mathbf{M}_n(c)$. Further, \mathbf{M} is locally congruent to one of the following hypersurfaces:*

- (1) *In case that $\mathbf{M}_n(c) = \mathbf{P}_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*
 - (A₁) *a geodesic hypersphere of radius r , where $0 < r < \pi/2$ and $r \neq \pi/4$;*
 - (A₂) *a tube of radius r over a totally geodesic $\mathbf{P}_k\mathbb{C}$ ($1 \leq k \leq n - 2$), where $0 < r < \pi/2$ and $r \neq \pi/4$.*
- (2) *In case that $\mathbf{M}_n(c) = \mathbf{H}_n\mathbb{C}$,*
 - (A₀) *a horosphere;*
 - (A₁) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbf{H}_{n-1}\mathbb{C}$;*
 - (A₂) *a tube over a totally geodesic $\mathbf{H}_k\mathbb{C}$ ($1 \leq k \leq n - 2$).*

Proof. Combining (5.23) to (5.24), we find

$$(6.1) \quad \alpha \nabla \beta = \{2\beta\alpha + \frac{c}{2}(2\alpha + \rho)\}U.$$

Because of (2.12) and (3.2), we can write (4.12) as

$$(6.2) \quad \frac{c}{4}(2\rho - 3\alpha)AU + \{\alpha(\rho - \alpha)(\beta - \rho\alpha - \frac{c}{4}) - \frac{c}{2}(\rho^2 + \beta - 2\rho\alpha + \frac{c}{4})\}U = \alpha\mu A\nabla\mu + \alpha(\alpha\rho - \beta)\nabla\alpha - \frac{1}{2}(\rho - \alpha)\alpha\nabla\beta + \alpha\mu^2\nabla\rho - \mu^2\alpha(W\rho)W,$$

where we have used (4.13), (5.10) and Lemma 1.

Differentiating (5.23) covariantly and taking the skew-symmetric parts obtained, we find $(\alpha^2 + \frac{3}{4}c)du(X, Y) = 0$, where we have used (5.23) itself.

Similarly, from (6.1) we have $\{2\beta\alpha + \frac{c}{4}(2\alpha + \rho)\}du(X, Y) = 0$. Then we have by above two equations

$$(6.3) \quad \alpha^2 + \frac{3}{4}c = 0, \quad \beta\alpha + \frac{c}{4}(2\alpha + \rho) = 0.$$

In fact, if not, then we have $du = 0$. By taking a inner product (5.25) with ξ , we verify that $\rho = \alpha$. Hence (6.1) is led to $\frac{1}{2}\nabla\beta = (\beta + \frac{3}{4}c)U$. From this and (5.23) we see that $\mu\nabla\mu = \mu^2U$. Using these facts, (6.2) implies that $(\mu^2 + \frac{c}{4})AU = 0$, which enables us to obtain $\mu^2 + \frac{c}{4} = 0$ and thus $\nabla\mu = 0$. So we have $\mu = 0$, a contradiction. Consequently we have $\nabla\alpha = 0$ and $\nabla\beta = 0$ by virtue of (6.1). Therefore we see that $\nabla\rho = 0$ and hence $\nabla\mu = 0$. Accordingly (6.2) is reformed as

$$\frac{c}{4}(2\rho - 3\alpha)AU + \{\alpha(\rho - \alpha)(\beta - \rho\alpha - \frac{c}{2}) - \frac{c}{2}(\rho^2 + \beta - 2\rho\alpha + \frac{c}{4})\}U = 0,$$

which together with (3.2) and (6.3) implies that

$$\frac{c}{4}(3\alpha - 2\rho) - 2\alpha(\rho^2 + \beta - 2\rho\alpha + \frac{c}{4}) - 3(\rho - \alpha)(\beta - \rho\alpha - \frac{c}{2}) = 0.$$

From this and (6.3) we see that $6\rho = 7\alpha$, which together with (4.1), (4.2) and (6.3) implies that $\rho = h$.

On the other hand, we see, using (2.5), that the scalar curvature r on M is given by

$$r = c(n^2 - 1) + h^2 - Tr({}^tAA),$$

which together with $\alpha^2 + \frac{3}{4}c = 0$ and $6h = 7\alpha$ will produce a contradiction because r is non-negative. Accordingly we conclude that $\Omega = \emptyset$. Therefore we verify that the subset Ω in \mathbf{M} on which $A\xi - \eta(A\xi)\xi \neq 0$ is an empty set. Namely, in $\mathbf{M}_n(c)$, $c \neq 0$, very real hypersurface satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$ is a Hopf hypersurface. Therefore we have $U = 0$ and moreover, the function α should be constant on \mathbf{M} ([6]). Thus, (3.1) is reformed as $\alpha\nabla_\xi A = 0$. This, together with (2.4) and (2.9) yields $\alpha(A\phi - \phi A) = 0$.

When the constant α identically vanishes, by Cecil and Ryan [2] we assert that \mathbf{M} is a tube of radius $\frac{\pi}{4}$ over certain Kähler submanifold in $\mathbf{P}_n\mathbb{C}$. But we here note that any Hopf hypersurfaces in $\mathbf{H}_n\mathbb{C}$ the function α never vanishing (see [1], [8], [9]). For the non-vanishing constant α , by virtue of Theorem O-MR due to Okumura [10] for $c > 0$ and Montiel and Romero [8] for $c < 0$ respectively we complete the proof of Theorem 6.1. \square

From above arguments and (6.3) we have

Theorem 6.2. *Let \mathbf{M} be a real hypersurface in a complex projective space $P_n(\mathbb{C})$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$, where S denotes the Ricci tensor of \mathbf{M} . Then \mathbf{M} is a Hopf hypersurface in $P_n(\mathbb{C})$. Further, \mathbf{M} is locally congruent to one of the following hypersurfaces:*

- (A₁) *a geodesic hypersphere of radius r , where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (A₂) *a tube of radius r over a totally geodesic $\mathbf{P}_k\mathbb{C}$ ($1 \leq k \leq n - 2$), where $0 < r < \pi/2$ and $r \neq \pi/4$,*

where $\eta(A\xi) \neq 0$.

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