

ASYMPTOTIC BEHAVIOR OF NONLINEAR VOLTERRA DIFFERENCE SYSTEMS

SUNG KYU CHOI, YOON HOE GOO, AND NAM JIP KOO

ABSTRACT. We study the asymptotic behavior of nonlinear Volterra difference system

$$x(n+1) = f(n, x(n)) + \sum_{s=n_0}^n g(n, s, x(s)), \quad x(n_0) = x_0$$

by using the resolvent matrix $R(n, m)$ of the corresponding linear Volterra system and the comparison principle.

1. Introduction

To obtain an asymptotic relationship between difference systems Poincaré (1885) and Perron (1921) began to study the long-time behavior of difference systems. The notions of asymptotic equilibrium and asymptotic equivalence become the main objectives in the theory of asymptotic behavior for difference systems. Two difference systems are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions tends to zero. If we know that two systems are asymptotically equivalent, and if we also know the asymptotic behavior of the solutions of one of the systems, then it is clear that we can obtain information about the asymptotic behavior of the solutions of the other systems [2].

For the study of the asymptotic behavior of difference systems, we refer to Choi et al. [5], Medina [9], Medina and Pinto [10], and Trench [11]. Also, Choi et al. [3, 4], Cuevas and Pinto [6], and Zouyousefain and Leela [12] studied the asymptotic behavior of Volterra difference systems.

In this paper we investigate asymptotic equilibrium of nonlinear Volterra difference system. To do this we develop the variation of constants formula for the quasi-linearized Volterra system by Zouyousefain and Leela [12] and

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then use the comparison principle which is one of the most efficient methods of obtaining information on the behavior of solutions of difference systems, even when they cannot be solved explicitly.

2. Main result

We consider the nonlinear Volterra difference system

$$(2.1) \quad x(n+1) = f(n, x(n)) + \sum_{s=n_0}^n g(n, s, x(s)), x(n_0) = x_0,$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $g : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$, n_0 is a nonnegative integer, and \mathbb{R}^m is the m -dimensional real euclidean space with any convenient norm $|\cdot|$. We assume that $f_x = \frac{\partial f}{\partial x}$ and $g_x = \frac{\partial g}{\partial x}$ exist and they are continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R}^m$, respectively. Also, we suppose that $f(n, 0) = 0$ and $g(n, s, 0) = 0$ for every $n \geq s \geq n_0$. Then one associated variational Volterra system of (2.1) is given by

$$(2.2) \quad v(n+1) = f_x(n, 0)v(n) + \sum_{s=n_0}^n g_x(n, s, 0)v(s)$$

and a fundamental matrix solution $\Phi(n, n_0, 0)$ of (2.2) is defined by

$$\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0}$$

[1, 7]. Denote $x(n) = x(n, n_0, x_0)$ by the solution of (2.1) with $x(n_0, n_0, x_0) = x_0$.

System (2.1) is said to have *asymptotic equilibrium* if there exist a single $\xi \in \mathbb{R}^m$ and $r > 0$ such that any solution $x(n)$ of (2.1) with $x_0 < r$ satisfies

$$(2.3) \quad x(n) = \xi + o(1) \text{ as } n \rightarrow \infty$$

and for every vector in \mathbb{R}^m , there exists a solution of (2.1) such that the asymptotic relationship (2.3) holds.

Consider the linear Volterra difference system

$$(2.4) \quad x(n+1) = A(n)x(n) + \sum_{s=n_0}^n B(n, s)x(s), x(n_0) = x_0,$$

and its perturbation

$$(2.5) \quad y(n+1) = A(n)y(n) + \sum_{s=n_0}^n B(n, s)y(s) + g(n), y(n_0) = x_0,$$

where $A(n)$ and $B(n, s)$ are $m \times m$ matrix functions on $\mathbb{N}(n_0)$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$, respectively, and $g(n)$ is a vector function on $\mathbb{N}(n_0)$.

For the nonlinear Volterra difference system (2.1), we set

$$f_x(n, 0) = A(n) \text{ and } g_x(n, s, 0) = B(n, s) \text{ for } n \geq s \geq n_0.$$

Then, by using the mean value theorem, (2.1) can be written as

$$(2.6) \quad \begin{aligned} x(n+1) &= A(n)x(n) + \sum_{s=n_0}^n B(n,s)x(s) + F(n,x(n)) \\ &\quad + \sum_{s=n_0}^n G(n,s,x(s)), \quad x(n_0) = x_0, \end{aligned}$$

where

$$\begin{aligned} F(n,x) &= \int_0^1 [f_x(n,x\theta) - f_x(n,0)]d\theta \cdot x, \\ G(n,s,x) &= \int_0^1 [g_x(n,s,x\theta) - g_x(n,s,0)]d\theta \cdot x. \end{aligned}$$

Let $S(\mathbb{N}(n_0), \mathbb{R}^m)$ be the set of all sequences $y : \mathbb{N}(n_0) \rightarrow \mathbb{R}^m$. We assume that the operator T defined on $S(\mathbb{N}(n_0), \mathbb{R}^m)$ satisfies the following :

- (i) $|T| : S(\mathbb{N}(n_0), \mathbb{R}^+) \rightarrow S(\mathbb{N}(n_0), \mathbb{R}^+)$ is nondecreasing.
- (ii) For any $y(n)$ and $z(n)$ in $S(\mathbb{N}(n_0), \mathbb{R}^m)$, $|y(n)| \leq |z(n)|$ for each $n_0 \leq n \leq n_1$ implies $|Ty(n)| \leq |Tz(n)|$,
- (iii) $|Ty(n)| \leq |T||y(n)|$ for any $y(n) \in S(\mathbb{N}(n_0), \mathbb{R}^m)$. If we impose on T various meanings, there will be appear various types of equations [9].

The following is the most basic comparison principle.

Lemma 2.1. [8, Theorem 1.6.1] *Let $g(n,r)$ be a function nondecreasing with respect $r \geq 0$ for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for $n \geq n_0$, the inequalities*

$$\begin{aligned} v(n+1) &\leq g(n,v(n)), \\ u(n+1) &\geq g(n,u(n)) \end{aligned}$$

hold. Then $v(n_0) \leq u(n_0)$ implies $v(n) \leq u(n)$ for all $n \geq n_0$.

We need the following comparison principle which is a modification of Lemma 2.1.

Lemma 2.2. *Let $\iota(n,r,u)$ be a nondecreasing function in r and u for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for $n \geq n_0$,*

$$v(n) - \sum_{l=n_0}^{n-1} \iota(l,v(l),|T|v(l)) < u(n) - \sum_{l=n_0}^{n-1} \iota(l,u(l),|T|u(l)).$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \geq n_0$.

Proof. If we assume that this is not the case, then we have $v(k) = u(k)$ and $v(l) < u(l)$ for some $k \in \mathbb{N}(n_0)$ with $n_0 \leq l < k$. Since $|T|v(l) \leq |T|u(l)$ for $n_0 \leq l < k$, we obtain

$$v(k) < u(k) + \sum_{l=n_0}^{k-1} \iota(l,v(l),|T|v(l)) - \sum_{l=n_0}^{k-1} \iota(l,u(l),|T|u(l)) \leq u(k),$$

which is a contradiction. □

The following lemma states a result corresponding to Fubini's theorem, which can be proved by the induction.

Lemma 2.3. [12] *Let $L(n, s)$ and $K(n, s)$ be $m \times m$ matrices defined for $s, n \geq n_0$ such that L and K are zero matrices for $s, n \leq n_0$. Then the relation*

$$\sum_{s=n_0}^{n-1} L(n, s+1) \sum_{\sigma=n_0}^{s-1} K(s, \sigma)x(\sigma) = \sum_{s=n_0}^{n-1} \sum_{\sigma=s+1}^{n-1} L(n, \sigma+1)K(\sigma, s)x(s)$$

holds, where $x : \mathbb{N}(n_0) \rightarrow \mathbb{R}^m$ is a vector function.

We define the resolvent matrix $R(n, m)$ of the linear system (2.4) as the unique solution of the matrix difference equation

$$(2.7) \quad R(n, m) = R(n, m+1)A(m) + \sum_{r=m}^{n-1} R(n, r+1)B(r, m), \quad n-1 \geq m \geq n_0.$$

with $R(m, m) = I$ [7]. Then we can obtain the expression of a solution of the perturbation (2.5) via the following variation of constants formula.

Theorem 2.4. *The unique solution $y(n, n_0, y_0)$ of (2.5) satisfying $y(n_0) = y_0$ is given by*

$$(2.8) \quad y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s),$$

where $R(n, m)$ is the unique solution of the matrix difference equation (2.7).

Proof. Letting $p(s) = R(n, s)y(s)$, $n-1 \geq s \geq n_0$, we have

$$(2.9) \quad \begin{aligned} \Delta p(s) &= R(n, s+1)y(s+1) - R(n, s)y(s) \\ &= [R(n, s+1)A(s) - R(n, s)]y(s) \\ &\quad + R(n, s+1) \sum_{r=n_0}^s B(s, r)y(r) + R(n, s+1)g(s). \end{aligned}$$

Summing both sides of (2.9) from $n-1$ to n_0 , we obtain, by Lemma 2.3,

$$\begin{aligned} p(n) &= y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s) \\ &\quad + \sum_{s=n_0}^{n-1} [R(n, s+1)A(s) - R(n, s)]y(s) \\ &\quad + \sum_{s=n_0}^{n-1} R(n, s+1) \sum_{r=n_0}^{s-1} B(s, r)y(r) + \sum_{s=n_0}^{n-1} R(n, s+1)B(s, s)y(s) \end{aligned}$$

$$\begin{aligned}
&= R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s) \\
&\quad + \sum_{s=n_0}^{n-1} \left[R(n, s+1)A(s) - R(n, s) + \sum_{r=s}^{n-1} R(n, r+1)B(r, s) \right] y(s).
\end{aligned}$$

From (2.7) we have the result

$$y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s).$$

□

Example 2.5. [4] We consider the linear Volterra difference equation

$$\begin{aligned}
(2.10) \quad x(n+1) &= A(n)x(n) + \sum_{s=n_0}^n B(n, s)x(s) \\
&= 2x(n) + \sum_{s=n_0}^n 2^{n-s}x(s),
\end{aligned}$$

where $A(n) = 2$ and $B(n, s) = 2^{n-s}$. Then any solution $x(n, n_0, x_0)$ of (2.10) through the initial point $x(n_0, n_0, x_0) = x_0$ is given by

$$x(n, n_0, x_0) = \frac{x_0}{3}[1 + 2 \cdot 4^{n-n_0}], \quad n \in \mathbb{N}(n_0).$$

Thus the resolvent matrix solution $R(n, m)$ with $R(m, m) = 1$ of the difference equation

$$\begin{aligned}
R(n, m) &= R(n, m+1)A(n) + \sum_{r=m}^{n-1} R(n, r+1)B(r, m) \\
&= R(n, m+1)2 + \sum_{r=m}^{n-1} R(n, r+1)2^{r-m}, \quad n-1 \geq m \geq n_0,
\end{aligned}$$

is given by

$$R(n, m) = \frac{1}{3}[1 + 2 \cdot 4^{n-m}].$$

It follows from the simple calculation that

$$\begin{aligned}
&R(n, m+1)A(n) + \sum_{r=m}^{n-1} R(n, r+1)B(r, m) \\
&= \frac{2}{3}[1 + 2 \cdot 4^{n-m-1}] + \sum_{r=m}^{n-1} \frac{1}{3}[1 + 2 \cdot 4^{n-r-1}]2^{r-m} \\
&= \frac{1}{3}[2 + 4^{n-m}] + \frac{1}{3}[2^{n-m} - 1] + \frac{2^{2n-m-1}}{3}[2^{1-m} - 2^{1-n}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}[1 + 2 \cdot 4^{n-m}] \\
&= R(n, m).
\end{aligned}$$

In fact, we note that the fundamental matrix of (2.10) is given by

$$\frac{\partial x(n, n_0, x_0)}{\partial x_0} = \Phi(n, n_0) = \frac{1}{3}[1 + 2 \cdot 4^{n-n_0}]$$

and

$$\Phi(n, n_0) = R(n, n_0).$$

Remark 2.6. In the special case when $f(n, x) = A(n)x(n)$ and $g(n, s, x) = B(n, s)x(s)$ in the nonlinear system (2.1) we note that the resolvent matrix $R(n, m)$ for Equation (2.7) is closely related to the fundamental matrix $\Phi(n, n_0)$. By the uniqueness of solution, it is easy to see that $R(n, n_0) = \Phi(n, n_0)$, and $R(n, m) = \Phi(n - m)$ for the equation of convolution type such as $B(n, s) = B(n - s)$.

The following main result can be obtained by improving the scalar Volterra difference equation in Theorem 3.2 in [4].

Theorem 2.7. *Assume that*

- (i) (2.4) has asymptotic equilibrium,
- (ii) F and G perturbed terms in (2.6) satisfies

$$|F(n, x)| \leq \omega_1(n, |x|), |G(n, s, x)| \leq \omega_2(n, s, |x|), n \geq s \geq n_0, x \in \mathbb{R}^m,$$

where $\omega_1 : \mathbb{N}(n_0) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega_2 : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and $\omega_1(n, u)$ and $\omega_2(n, s, u)$ are continuous and nondecreasing in u for n and (n, s) , respectively.

Also, we consider the scalar Volterra difference equation

$$\begin{aligned}
(2.11) \quad u(n+1) &= u(n) + M[\omega_1(n, u(n)) \\
&\quad + \sum_{s=n_0}^n \omega_2(n, s, u(s))], \quad u(n_0) = u_0 > 0,
\end{aligned}$$

where M is an upper bounded number of $R(n, m)$.

- (iii) All solutions of (2.11) are bounded on $\mathbb{N}(n_0)$.

Then (2.1) has asymptotic equilibrium provided $M|x_0| < u_0$.

Proof. Let $x(n)$ be any solution of (2.1). From the variation of constants formula, and (i)-(ii), we obtain

$$\begin{aligned}
|x(n)| &= |R(n, n_0)x_0 + \sum_{s=n_0}^{n-1} R(n, s+1)[F(s, x(s)) + \sum_{\sigma=n_0}^s G(s, \sigma, x(\sigma))]| \\
&\leq M|x_0| + M \sum_{s=n_0}^{n-1} \left[\omega_1(s, |x(s)|) + \sum_{\sigma=n_0}^s \omega_2(s, \sigma, |x(\sigma)|) \right],
\end{aligned}$$

where M is a constant from the boundedness of $R(n, m)$ for each $n \geq m \geq t_0$ and $d = M|x_0|$, we have the following difference inequality :

$$\begin{aligned} & |x(n)| - M \sum_{s=n_0}^{n-1} \left[\omega_1(s, |x(s)|) + \sum_{\sigma=n_0}^s \omega_2(s, \sigma, |x(\sigma)|) \right] \\ & \leq d \\ & < u_0 \\ & = u(n) - M \sum_{s=n_0}^{n-1} \left[\omega_1(s, u(s)) + \sum_{\sigma=n_0}^s \omega_2(s, \sigma, u(\sigma)) \right]. \end{aligned}$$

Setting $\iota(n, s, u) = M[\omega_1(n, u) + |T|u]$ with $|T|u(n) = \sum_{s=n_0}^n \omega_2(n, s, u(s))$, it follows from Lemma 2.2 that

$$|x(n)| < u(n), \text{ for each } n \geq n_0$$

provided $d < u_0$.

Now, we prove that the solution $x(n)$ of (2.1) converges to a vector as $n \rightarrow \infty$. Consider the sequence

$$(2.12) \quad v(n, n_0, x_0) = \sum_{s=n_0}^{n-1} R(n, s + 1) \left[F(s, x(s)) + \sum_{\sigma=n_0}^s G(s, \sigma, x(\sigma)) \right].$$

By using the monotonicity of ω_1, ω_2 , and asymptotic equilibrium of (2.4), we have

$$\begin{aligned} |v(n) - v(n_1)| & \leq \left| \sum_{s=n_1}^{n-1} R(n, s + 1) \left[F(s, x(s)) + \sum_{\sigma=n_0}^s G(s, \sigma, x(\sigma)) \right] \right| \\ & \leq M \sum_{s=n_1}^{n-1} \left[\omega_1(s, |x(s)|) + \sum_{\sigma=n_0}^s \omega_2(s, \sigma, |x(\sigma)|) \right] \\ & \leq M \sum_{s=n_1}^{n-1} \left[\omega_1(s, u(s)) + \sum_{\sigma=n_0}^s \omega_2(s, \sigma, u(\sigma)) \right] \\ & \leq u(n) - u(n_1), \end{aligned}$$

for any $n \geq n_1 \geq n_0$. Since $u(n)$ has the Cauchy property, $v(n)$ converges to a vector $v_\infty(n_0, x_0)$ as $n \rightarrow \infty$. Hence the solution $x(n)$ of (2.1) tends to a finite limit vector ξ as $n \rightarrow \infty$, i.e., $\xi = R_\infty x_0 + v_\infty(n_0, x_0)$.

By the same method of Medina in [9] we can be prove that the converse asymptotic relationship holds. □

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