FEYNMAN INTEGRAL, ASPECT OF DOBRakov INTEGRAL, I

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ABSTRACT. This paper is the first in a series in which we consider bilinear integration with respect to measure-valued measure. We use the integration techniques to establish generalized Egorov theorem and Vitali theorem.

1. Introduction

The measure-valued measures $V_\varphi$ were introduced in [13] and studied in relation to a measure-valued Feynman-Kac formula. For a given complex Borel measure $\varphi : B(\mathbb{R}) \to \mathbb{C}$ on $\mathbb{R}$, the measure-valued measures $V_\varphi$ is defined as follows. The space of all continuous functions $\omega : [0, t] \to \mathbb{R}$ is denoted by $C([0, t])$. It is given with the uniform norm. If $X_s : C([0, t]) \to \mathbb{R}$ denotes evolution at time $0 \leq s \leq t$, then for the cylinder set $E = \{ X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n \}$ in $C([0, t])$ with $0 \leq t_1 < \cdots < \cdots < t_n \leq t$ and Borel sets $B_1, \ldots, B_n$, the complex Borel measure $V_\varphi(E)$ is defined by the formula

$$
(V_\varphi(E))(B) = \frac{1}{\sqrt{(2\pi(t-t_n))\cdots(2\pi t_1)}} \int_B \int_{B_n} \cdots \int_{B_1} \int_{\mathbb{R}} e^{-\frac{|x_n-x_{n-1}|^2}{2(t-t_n)}} \\
\times e^{-\frac{|x_{n-1}-x_{n-2}|^2}{2(t_{n-1}-t_n)}} \cdots e^{-\frac{|x_2-x_1|^2}{2(t_2-t_1)}} e^{-\frac{|x_1-x|^2}{2t_1}} d\varphi(x) dx_1 \cdots dx_n d\xi
$$

(1)

for each Borel subset $B$ of $\mathbb{R}$. Clearly $V_\varphi$ is closely related to Wiener measure and the complex valued measure $V_\varphi(\cdot)(B)$ may be viewed as Wiener measure with an initial distribution $\varphi$ subject to the condition $\{X_t \in B\}$. Another way of looking at the measure valued measure $V_\varphi$ is to take the semigroup $S(t) : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$, $t \geq 0$, defined on the space $\mathcal{M}(\mathbb{R})$ of Borel measures on $\mathbb{R}$.

Received January 12, 2006; Revised February 1, 2006.

2000 Mathematics Subject Classification. Primary 81S40, 58D30; Secondary 46G10, 28B05.

Key words and phrases. measure-valued measure, Dobrakov integral.

This work was supported by the Post-doctoral Fellowship Program of Korea Science & Engineering Foundation (KOSEF).

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$\mathbb{R}$ by $S(0) = Id$ and
\[
[S(t)\mu](B) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{|\xi - \mu|^2}{4t}} d\mu(x) d\xi, \quad B \in \mathcal{B}(\mathbb{R}), \quad \mu \in \mathcal{M}(\mathbb{R})
\]
for $t > 0$ and $Q(B)\mu = \chi_B \cdot \mu$ for all $B \in \mathcal{B}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$. If $M^t$ denotes the operator valued set function associated with the $(S, Q)$-process [7], then for each cylinder set $E$ we have $M^t(E)\varphi = V_{\varphi}(E \cap C([0, t]))$. If we replace $\mathcal{M}(\mathbb{R})$ by the Hilbert space $L^2(\mathbb{R})$ and $S$ by
\[
[S_F(t)](\psi)(\xi) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i\frac{\xi - \mu}{2t}} \psi(x) dx, \quad \psi \in L^2(\mathbb{R}), \quad t > 0,
\]
where the integral is understood in the sense of mean-square convergence, then we obtain the operator valued set functions $M^t_F$ associated with the Feynman path integral. Roughly speaking, for each $t \geq 0$, the bounded linear operator $S_F(t)$ is equal to $S(it)$ applied to complex measures with square-integrable densities with respect to Lebesgue measure on $\mathbb{R}$.

2. Convergence theorem for measure-valued measures

In this paper we denote the inner product of two elements $a, b$ in a Banach space to $(a, b)$ or even $ab$.

Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces. The space of all complex measures defined on $\mathcal{E}$ with the total variation norm is denoted by $\mathcal{M}(\mathcal{E})$. The space of nonnegative elements of $\mathcal{M}(\mathcal{E})$ is written as $\mathcal{M}_+(\mathcal{E})$. The variation of a scalar measure $\mu$ is written as $|\mu|$. Let $\mathcal{X}$ be a Banach space. For any $\mathcal{M}(\mathcal{E})$-valued measure $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$, the operator valued measure $m^{\mathcal{X}} : \mathcal{B} \to \mathcal{L}(\mathcal{X}, \mathcal{M}(\mathcal{E}, \mathcal{X}))$ be defined by
\[
m^{\mathcal{X}}(B)x = xm(B), \quad x \in \mathcal{X}, \quad B \in \mathcal{B}.
\]
Here $\mathcal{M}(\mathcal{E}, \mathcal{X})$ is the space of $\mathcal{X}$-valued measures on $\mathcal{E}$ equipped with the semivariation norm defined by
\[
||n|| = \sup\{|(n, x')(\Sigma) : x' \in \mathcal{X}', ||x'|| \leq 1\}, \quad n \in \mathcal{M}(\mathcal{E}, \mathcal{X}).
\]
We also write this as $||n||_{\mathcal{M}(\mathcal{E}, \mathcal{X})}$. Although the semivariation of an $\mathcal{X}$-valued measure is always finite, for every infinite dimensional Banach space $\mathcal{X}$, there is an $\mathcal{X}$-valued measure $n$ whose total variation
\[
||n||_v = \sup\{\sum_j ||n(E_j)||\}
\]
is infinite. The supremum is over all finite partitions $\{E_j\}$ of $\Sigma$ by elements of $\mathcal{E}$. The space of $\mathcal{X}$-valued measures on $\mathcal{E}$ with finite variation and equipped with the variation norm is denoted by $\mathcal{M}_v(\mathcal{E}, \mathcal{X})$. The variation of $n \in \mathcal{M}_v(\mathcal{E}, \mathcal{X})$ is written as $v_{\mathcal{X}}(n) : \mathcal{E} \to [0, \infty)$. Our aim is to integrate $\mathcal{X}$-valued functions
with respect to the \( \mathcal{M}(\mathcal{E}) \)-valued measure \( m \); the integral takes its values in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \). We also need to consider the \( \mathcal{X} \)-semivariation of \( m \) on \( \mathcal{B} \):

\[
\beta_{\mathcal{X}}(m)(B) = \sup\{\| \sum_{j=1}^{n} x_j m(B_j \cap B) \|_{\mathcal{M}(\mathcal{E}, \mathcal{X})} \}.
\]

The supremum is taken over all \( x_j \in \mathcal{X} \) with \( \|x_j\| \leq 1 \) and all finite partitions \( \{B_j\} \) of \( \Omega \). The \( \mathcal{X} \)-semivariation \( \beta_{\mathcal{X}}(m) \) of \( m \) is identical to the semivariation of the operator valued measure \( m^\mathcal{X} \) in the sense of Dobrakov. It can happen that \( \beta_{\mathcal{X}}(m) \) has only the values 0 or \( \infty \). An \( \mathcal{X} \)-valued function \( f : \Omega \to \mathcal{X} \) is called a simple function if for some \( n \in \mathbb{N} \), there exist vectors \( x_j \in \mathcal{X} \) and sets \( B_j \in \mathcal{B} \) for \( j = 1, 2, \ldots, n \) such that \( f = \sum_{j=1}^{n} x_j \chi_{B_j} \). For an \( \mathcal{X} \)-valued simple function \( f = \sum_{j=1}^{n} x_j \chi_{B_j} \) and for \( B \in \mathcal{B} \), we define the integral

\[
\int_{B} f \otimes dm = \sum_{j=1}^{n} x_j m(B \cap B_j) \in \mathcal{M}(\mathcal{E}, \mathcal{X}).
\]

We also write this \( \int_{B} f dm^\mathcal{X} \). A standard argument ensures that the \( \mathcal{X} \)-valued measure \( \int_{B} f \otimes dm \) is well-defined. The \( \mathcal{X} \)-semivariation \( \beta_{\mathcal{X}}(m) \) of \( m \) is extended from sets to functions \( f : \Omega \to \mathcal{X} \) by setting

\[
\beta_{\mathcal{X}}(m)(f) = \beta_{\mathcal{X}}(m)(|f|) = \sup\{\| \int_{B} s \otimes dm \|_{\mathcal{M}(\mathcal{E}, \mathcal{X})} \},
\]

where the supremum is taken for all \( \mathcal{X} \)-valued \( \mathcal{B} \)-simple functions \( s \) with \( \|s(\omega)\|_{\mathcal{X}} \leq \|f(\omega)\|_{\mathcal{X}} \) for \( m \)-almost all \( \omega \in \Omega \). In the notation of Dobrakov, we have \( \beta_{\mathcal{X}}(m) = m^\mathcal{X} \). We set

\[
\mathcal{L}_1(\beta_{\mathcal{X}}(m)) = \{ f : \Omega \to \mathcal{X} \text{ is } m \text{-measurable and } \beta_{\mathcal{X}}(m)(f) < \infty \}
\]

\[
\mathcal{B}_{\mathcal{X}}(\mathcal{B}) = \{ f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m)) | f \text{ is uniformly bounded} \}
\]

\[
\mathcal{L}_1(\mathcal{B}_{\mathcal{X}}, \beta_{\mathcal{X}}(m)) = \mathcal{B}_{\mathcal{X}}(\mathcal{B}).
\]

The closure is taken in the norm \( \beta_{\mathcal{X}}(m) \) defined on \( \mathcal{L}_1(\beta_{\mathcal{X}}(m)) \). With modulo \( m \)-null functions, \( \mathcal{L}_1(\beta_{\mathcal{X}}(m)) \) becomes a Banach space \( \mathcal{L}_1(\beta_{\mathcal{X}}(m)) \). As noted above, the \( \mathcal{X} \)-semivariation \( \beta_{\mathcal{X}}(m) : \mathcal{B} \to [0, \infty] \) may take only the values 0 and \( \infty \), so it can happen that \( \mathcal{L}_1(\beta_{\mathcal{X}}(m)) \) just consists of the zero function. Let \( M \subset \mathcal{M}(\mathcal{E}, \mathcal{X})^* \). Let \( M_1 \) denote the set of all elements \( \mu \) of \( M \) such that \( \|\mu\| \leq 1 \) and suppose that \( M_1 \) is norming for \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \). Then for any \( \mu \in M \) we define \( \mu \circ m^\mathcal{X} : \mathcal{B} \to \mathcal{X}^\ast \) by \( \langle x, \mu \circ m^\mathcal{X}(B) \rangle = \langle m^\mathcal{X}(B)x, \mu \rangle \) for \( x \in \mathcal{X} \) and \( B \in \mathcal{B} \). The set

\[
\mathcal{M}_1(m^\mathcal{X}) = \{ v_{\mathcal{X}^*}(\mu \circ m^\mathcal{X}) : \mu \in M_1 \}
\]

consists of measures on \( \mathcal{B} \) with values in \([0, \infty]\). Because

\[
\| \sum_{j=1}^{n} \langle x_j m(B_j \cap B), \mu \rangle \| \leq \| \sum_{j=1}^{n} x_j m(B_j \cap B) \|_{\mathcal{M}(\mathcal{E}, \mathcal{X})}
\]
for each \( \mu \in M_1 \), it follows that \( M_1(m^X) \) consists of finite measures if the \( X \)-semivariation \( \beta_X(m)(\Omega) \) on \( \Omega \) is finite. Because \( M_1 \) is a norming set for \( M(\mathcal{E}, X) \), we have
\[
(2) \quad \beta_X(m)(B) = \sup_{\nu \in M_1(m^X)} \nu(B), \quad B \in \mathcal{B}.
\]

By Hahn-Banach theorem, \( M = \text{sim}(\mathcal{E}) \otimes X^* \) is a dense subset of \( M(\mathcal{E}, X)^* \), so \( M_1 \) is norming for \( M(\mathcal{E}, X) \). In this section we take \( M = \text{sim}(\mathcal{E}) \otimes X^* \). In the case that \( \Sigma \) is a locally compact Hausdorff space and \( \mathcal{E} \) is the Borel \( \sigma \)-algebra of \( \Sigma \), another choice is \( M = C_0(\Sigma) \otimes X^* \) where \( C_0(\Sigma) \) is the set of all continuous functions on \( \Sigma \) vanishing at infinity. We note that for any finite dimensional Banach space \( X \), there exist measurable spaces \( (\Sigma, \mathcal{E}), (\Omega, \mathcal{B}) \) and a measure-valued measure \( m : \mathcal{B} \to M(\mathcal{E}) \) such that \( \beta_X(m)(\Omega) = \infty \). Suppose that \( \beta_X(m)(\Omega) < \infty \). By virtue of the equality (2), the condition that \( M_1(m^X) \) is uniformly countable additive is equivalent to Dobrakov’s condition that \( \beta_X(m) \) is continuous, that is, if \( B_n \downarrow \emptyset \), then \( \beta_X(m)(B_n) \to 0 \) as \( n \to \infty \). For \( X = c_0 \), the classical Banach space, there exist vector measures \( m \) for which \( \beta_{c_0}(m)(\Omega) < \infty \) but \( \beta_{c_0}(m) \) is not continuous.

**Definition 2.1.** ([9, Definition 1.5]) Let \( m : \mathcal{B} \to M(\mathcal{E}) \) be a vector measure. A function \( f : \Omega \to X \) is said to be \( m \)-integrable in \( M(\mathcal{E}, X) \) if there exist \( X \)-valued \( \mathcal{B} \)-simple function \( f_j, \ j \in \mathbb{N} \), such that \( f_j \to f \) pointwisely \( m \)-almost everywhere as \( j \to \infty \) and \( \{ \int_B f_j \otimes dm \} \) converges in \( M(\mathcal{E}, X) \) for each \( B \in \mathcal{B} \). Let \( \int_B f \otimes dm \) denote this limit.

The above limit is well defined and independent of the approximating sequence ([7, Lemma 4.1.4]). The set function \( B \to \int_B f \otimes dm, \ B \in \mathcal{B} \), is \( \sigma \)-additive in \( M(\mathcal{E}, X) \) by the Vitali-Hahn-Saks theorem ([1, Theorem 1.5.6]). Clearly the map \( (f, m) \to \int f \otimes dm \) is bilinear in the obvious sense. Also, for the case \( X = \mathbb{C} \), a function \( f : \Omega \to \mathbb{C} \) is \( m \)-integrable (as defined above) if and only if it is \( m \)-integrable in the sense of vector measures defined in I. Klíma and G. Knowles [12]. If the \( X \)-semivariation \( \beta_X(m) \) of \( m \) is \( \sigma \)-finite on the set \( \{ f \neq 0 \} \), then \( f \) is \( m \)-integrable if and only if it is \( m^X \)-integrable in the sense of Dobrakov and in this case,
\[
\int_B f \otimes dm = \int_B f dm^X
\]
for every \( B \in \mathcal{B} \) [9, Definition 1.5]. If \( \beta_X(m) \) is finite and continuous, then integral coincide with the Bartle bilinear integral.

We are now in a position to state our general convergence theorem for measure-valued measures. In the next section, we see how they can be simplified under the additional assumption of order boundedness.

**Theorem 2.2.** (Egorov) Let \( (\Sigma, \mathcal{E}) \), \( (\Omega, \mathcal{B}) \) be measurable spaces and \( X \) a Banach space. Suppose that \( m : \mathcal{B} \to M(\mathcal{E}) \) is an \( M(\mathcal{E}) \)-valued measure for which \( \beta_X(m)(\Omega) < \infty \) and \( \beta_X(m) \) is continuous. Let \( f_n, f : \Omega \to X, n \in \mathbb{N} \) be \( m \)-measurable functions such that \( f_n \to f \), \( m \)-a.e. Then
a) for any \( \varepsilon > 0 \), there is a set \( B \in \mathcal{B} \) such that \( \beta_X(m)(B^c) < \varepsilon \) and \( f_n \to f \) uniformly on \( B \).

b) \( f_n \to f, \beta_X(m) \)-measure.

**Proof.** Since \( \beta_X(m) \) is continuous, the set \( \mathcal{M}_1(m^X) \) is uniformly \( \sigma \)-additive on \( \mathcal{B} \). The Bartle-Dunford-Schwartz theorem and equation (2) shows that there is a positive, finite and \( \sigma \)-additive measure \( \lambda \) on \( \mathcal{B} \) such that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( F \in \mathcal{B} \) with \( \lambda(F) < \delta \), \( \beta_X(m(F)) < \varepsilon \). Hence by the Egorov theorem for \( \lambda \), for this \( \delta > 0 \), there is \( B \in \mathcal{B} \) such that \( \lambda(B^c) < \delta \) and \( f_n \to f \) uniformly on \( B \), and \( f_n \to f \) in \( \beta_X(m) \)-measure. Thus we have \( \beta_X(m)(B^c) < \varepsilon \) and \( f_n \to f \) in \( \beta_X(m) \)-measure.

**Theorem 2.3.** (Vitali) Let \( (\Sigma, \mathcal{E}), (\Omega, \mathcal{B}) \) be measurable spaces and \( \mathcal{X} \) a Banach space. Suppose that \( m : \mathcal{B} \to \mathcal{M}(\mathcal{E}) \) is an \( \mathcal{M}(\mathcal{E}) \)-valued measure for which \( \beta_X(m)(\Omega) < \infty \) and \( \beta_X(m) \) is continuous. Let \( \langle f_n \rangle \) be a sequence from \( L^1(\beta_X(m)) \) with \( f_n \) \( m \)-integrable functions and let \( f : \Omega \to \mathcal{X} \) be \( m \)-measurable. Assume that

a) \( f_n \to f \) in \( \beta_X(m) \)-measure or

a') \( f_n \to f, m \text{-a.e.} \)

b) \( \lim_{\beta_X(m)(A) \to 0} \beta_X(m)(f_n \chi_A) = 0 \), uniformly for \( n \in \mathbb{N} \).

c) For all \( \varepsilon > 0 \), there is a set \( A_\varepsilon \in \mathcal{B} \) with \( \beta_X(m)(A_\varepsilon) < \infty \), such that \( \beta_X(m)(f_n \chi_{\Omega - A_\varepsilon}) < \varepsilon \), for all \( n \in \mathbb{N} \).

Then \( f \in L^1(\beta_X(m)) \) and \( \beta_X(m)(f_n - f) \to 0 \). Furthermore, the function \( f \) is \( m \)-integrable in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \) and \( \int_B f_n \otimes dm \to \int_B f \otimes dm \) in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \), uniformly for \( B \in \mathcal{B} \) as \( n \to \infty \). Conversely, if \( f \in L^1(\beta_X(m)) \) and \( \beta_X(m)(f_n - f) \to 0 \) as \( n \to \infty \), then conditions a) and b) are satisfied.

**Proof.** By Egorov theorem, a') implies a). Assume conditions a), b) and c) satisfied. To show that \( \langle f_n \rangle \) is a Cauchy sequence in \( L^1(\beta_X(m)) \), let \( \varepsilon > 0 \) and let \( A_\varepsilon \in \mathcal{B} \) be a set satisfying condition c). By condition b) there is a \( \delta > 0 \) for all \( A \in \mathcal{B} \) if \( \beta_X(m)(A) < \delta \) then \( \beta_X(m)(f_n \chi_A) < \varepsilon \) for all \( n \in \mathbb{N} \). By a), there exists \( N_\varepsilon \) such that if

\[
\begin{align*}
B_{n,m} &= \{ s \in A_\varepsilon : ||f_n(s) - f_m(s)||_\mathcal{X} > \varepsilon / \beta_X(m)(A_\varepsilon) \}, \\
\end{align*}
\]

then for all \( n,m \geq N_\varepsilon \), \( B_{n,m} \in \mathcal{B} \) and \( \beta_X(m)(B_{n,m}) < \delta \). Then for \( n,m \geq N_\varepsilon \)

\[
\begin{align*}
\beta_X(m)(f_n - f_m) &\leq \beta_X(m)(f_n - f_m)\chi_{B_{n,m}} + \beta_X(m)(f_n - f_m)\chi_{A_\varepsilon - B_{n,m}} \\
&+ \beta_X(m)(f_n - f_m)\chi_{\Omega - A_\varepsilon} \\
&\leq \beta_X(m)(f_n \chi_{B_{n,m}}) + \beta_X(m)(f_m \chi_{B_{n,m}}) + \beta_X(m)(f_n \chi_{A_\varepsilon - B_{n,m}}) \\
&+ \beta_X(m)(f_n \chi_{\Omega - A_\varepsilon}) + \beta_X(m)(f_m \chi_{\Omega - A_\varepsilon}) \\
&\leq 5\varepsilon.
\end{align*}
\]

Hence \( \langle f_n \rangle \) is a Cauchy sequence in \( L^1(\beta_X(m)) \). In particular, \( \int_B f_n \otimes dm \), \( n = 1, 2, \ldots \), converges for each \( B \in \mathcal{B} \). Since \( L^1(\beta_X(m)) \) is complete, there a
\( m \)-measurable function \( g \) in \( L_1(\beta_\mathcal{X}(m)) \) such that \( \beta_\mathcal{X}(m)(f_n - g) \to 0 \). Then \( f_n \to g \) in \( \beta_\mathcal{X}(m) \)-measure. From a) \( f = g \) \( \beta_\mathcal{X}(m) \)-a.e., \( f \in L_1(\beta_\mathcal{X}(m)) \) and \( \beta_\mathcal{X}(m)(f_n - f) \to 0 \). To check that \( f \) is \( m \)-integrable in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \), we need to exhibit a sequence \( (s_k) \) of \( \mathcal{X} \)-valued \( \mathcal{E} \)-simple functions such that \( s_k \to f \) \( m \)-a.e. as \( k \to \infty \) and the indefinite integrals \( \int s_k \otimes dm \), \( k = 1, 2, \ldots \), are uniformly bounded and uniformly countably additive in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \). This follows from ([9, Theorem 2.6]), where it is also shown that \( \int_B f_n \otimes dm \to \int_B f \otimes dm \) in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \), uniformly for \( B \in \mathcal{B} \) as \( n \to \infty \). Conversely, assume that \( f \in L_1(\beta_\mathcal{X}(m)) \) and \( \beta_\mathcal{X}(m)(f_n - f) \to 0 \). Then \( f_n \to f \) in \( \beta_\mathcal{X}(m) \)-measure, so a) is satisfied. To prove b), assume \( f \in L_1(\beta_\mathcal{X}(m)) \). Let \( \varepsilon > 0 \) and let \( N \in \mathbb{N} \) be such that for every \( n \geq N \) \( \beta_\mathcal{X}(m)(f_n - f) < \varepsilon / 2 \). Then for all \( A \in \mathcal{B} \) and \( n \geq N \)

\[
\beta_\mathcal{X}(m)(f_n \chi_A - f \chi_A) < \varepsilon / 2
\]

that is,

\[
\beta_\mathcal{X}(m)(f_n \chi_A) \leq \beta_\mathcal{X}(m)(f \chi_A) + \varepsilon / 2.
\]

But since \( f \in L_1(\beta_\mathcal{X}(m)) \), there is a \( \delta_0 > 0 \) such that \( A \in \mathcal{B} \) and \( \beta_\mathcal{X}(m)(A) < \delta_0 \) then \( \beta_\mathcal{X}(m)(f \chi_A) < \varepsilon / 2 \). Then for \( n \geq N \) and \( \beta_\mathcal{X}(m)(A) < \delta_0 \), \( \beta_\mathcal{X}(m)(f_n \chi_A) < \varepsilon \). For \( n \leq N \), we can find \( \delta_1 > 0 \) such that \( A \in \mathcal{B} \) and \( \beta_\mathcal{X}(m)(A) < \delta_1 \) then \( \beta_\mathcal{X}(m)(f_n \chi_A) < \varepsilon \) for all \( n \leq N \). If we take \( \delta = \inf\{ \delta_0, \delta_1 \} \), then for any \( A \in \mathcal{B} \) with \( \beta_\mathcal{X}(m)(A) < \delta \), \( \beta_\mathcal{X}(m)(f_n \chi_A) < \varepsilon \) for all \( n \in \mathbb{N} \). \( \square \)

3. Order bounded measures

Let \( (\Sigma, \mathcal{E}), (\Omega, \mathcal{B}) \) be measurable spaces. A measure-valued measure \( m : \mathcal{B} \to \mathcal{M}(\mathcal{E}) \) is said to be positive if it takes its values in the space \( \mathcal{M}_+(\mathcal{E}) \) of nonnegative measures. We say that \( m \) is order bounded if there exists a positive vector measure, \( n \), for which \( n(A) \geq |m(A)| \) holds for all \( A \in \mathcal{B} \). In the present context, this is equivalent to saying that \( m \) has order bounded range in the Banach lattice \( \mathcal{M}(\mathcal{E}) \), and see ([7, Lemma 4.4.4]). The smallest positive measure \( |m|_\mathcal{M} \) satisfying this requirement is called the modulus of \( m \). It is associated with the modulus of the regular linear map from the Banach lattice \( L_\infty(\Omega, \mathcal{B}) \) to the Banach lattice \( \mathcal{M}(\mathcal{E}) \) defined by integration with respect to \( m \). We write \( |m|_\mathcal{M} \geq |n|_\mathcal{M} \) and say that \( m \) dominates \( n \) if \( |m|_\mathcal{M}(A) \geq |n|_\mathcal{M}(A) \) for all \( A \in \mathcal{B} \). As for the case of \( L^p \)-valued measure [10], the convergence theorems described in Section 2 can be simplified considerably for order bounded measure-valued measures, such as the measures \( V_\phi \) described in Section 1.

Example 3.1. Let \( \phi \in \mathcal{M}(\mathcal{B}(\mathbb{R})) \). Then \( |V_\phi(B)| \leq V_{|\phi|}(B) \) for all \( B \in \mathcal{B} \). Moreover, \( |V_\phi|_\mathcal{M} = V_{|\phi|} \).

The modulus \( |m|_\mathcal{M} \) is easily described.

Lemma 3.2. Let \( m : \mathcal{B} \to \mathcal{M}(\mathcal{E}) \) be a measure-valued measure and let \( \mu \) be the variation of the additive set function

\[
B \times E \mapsto (m(B))(E), \quad B \in \mathcal{B}, E \in \mathcal{E}.
\]
Then \( m \) is order bounded if and only if \( \mu(\Omega \times \Sigma) < \infty \). If \( m \) is order bounded, then the modulus \( |m|_{\mathcal{M}} : \mathcal{B} \to \mathcal{M}_+(\mathcal{E}) \) is given by \( (|m|_{\mathcal{M}}(B))(E) = \mu(B \times E) \), for all \( B \in \mathcal{B} \), \( E \in \mathcal{E} \).

Another way of viewing a measure-valued measure is as a bimeasure. In general, the variation \( \mu \) defined in Lemma 3.2 need not be \( \sigma \)-additive. A sufficient condition guaranteeing the \( \sigma \)-additive of \( \mu \) is that \( m \) is regular in each variable. We denote the scalar measure \( B \mapsto (m(B))(E) \), \( B \in \mathcal{B} \), by \( m_E \) for each \( E \in \mathcal{E} \).

**Proposition 3.3.** Let \( m : \mathcal{B} \to \mathcal{M}_+(\mathcal{E}) \) be a positive measure-valued measure. Then

\[
v_{\mathcal{M}(\mathcal{E})}(m)(B) = \beta_X(m)(B) = v_{\mathcal{L}(X,\mathcal{M}(\mathcal{E},\mathcal{X}))}(m^X)(B) = m_\Sigma(B)
\]

for all \( B \in \mathcal{B} \).

**Proof.** i)

\[
v(m)(B) = \sup \sum_{j=1}^{n} \|m(B_j \cap B)\| = \sup \left( \sum_{j=1}^{n} m(B_j \cap B)(\Sigma) \right) = m(B)(\Sigma) = m_\Sigma(B).
\]

ii)

\[
\beta_X(m)(B) = \sup_{\{x_j\} \{B_j\}} \|\sum_{j=1}^{n} x_j m(B_j \cap B)\|_{\mathcal{M}(\mathcal{E},\mathcal{X})} = \sup_{\{x_j\} \{B_j\}} \sup_{\{E_k\} \{\sigma^*\} \{B_j\}} \left| \sum_{j,k} c_k(x_j, x^*)(m(B_j \cap B))(E_k) \right| = \sup_{\{x_j\} \{B_j\}} \sup_{\{E_k\} \{\sigma^*\} \{B_j\}} \left| \sum_{j,k} c_k(x_j, x^*) \mu((B_j \cap B) \times E_k) \right| = \sup_{j,k} \left| \langle x_j, x^* \rangle \mu((B_j \cap B) \times E_k) \right| = \sup_{j} \left| \langle x_j, x^* \rangle \mu((B_j \cap B) \times \Sigma) \right| = \sup_{j} \left| \langle x_j, x^* \rangle m_\Sigma(B_j \cap B) \right| = m_\Sigma(B)
\]
where \( c_k \in \mathbb{C} \).

\[
v(m^\mathcal{X})(B) = \sup \sum_{j=1}^{n} \|m(B_j \cap B)\|_{\mathcal{L}(\mathcal{X}, \mathcal{M}(\mathcal{E}, \mathcal{X}))}
= \sup \sum_{j=1}^{n} \sup_{\|x\| \leq 1} \|xm(B_j \cap B)\|_{\mathcal{M}(\mathcal{E}, \mathcal{X})}
= \sup \sum_{j=1}^{n} \sup_{\|x\| \leq 1, \|x^*\| \leq 1} \left| \sum_{k} (x, x^*)c_k m(B_j \cap B)(E_k) \right|
= m_\Sigma(B).
\]

**Corollary 3.4.** Let \( m : \mathcal{B} \rightarrow \mathcal{M}(\mathcal{E}) \) be an order bounded measure-valued measure with modulus \( \|m\|_\mathcal{M} \). Let \( f : \Omega \rightarrow \mathcal{X} \) be an \( (\|m\|_\mathcal{M})^\Sigma \)-Bochner integrable function. Then \( f \) is \( m_E \)-Bochner integrable for each \( E \in \mathcal{E} \) and \( m \)-integrable in \( \mathcal{M}_E(\mathcal{E}, \mathcal{X}) \). The equality

\[
\left( \int_B f \otimes dm \right)(E) = \int_B f dm_E
\]

holds for all \( B \in \mathcal{B} \) and \( E \in \mathcal{E} \). Moreover,

\[
(v_{\mathcal{M}_E(\mathcal{E}, \mathcal{X})}) \left( \int_B f \otimes dm \right)(B) \leq \int_B \|f\|_{\mathcal{X}} dm_{\mathcal{M}(\mathcal{E}, \mathcal{X})}.
\]

**Proof.** [3, Theorem 6].

For order bounded measure-valued measures \( m \), the \( m \)-integrability of strongly measurable \( \mathcal{X} \)-valued functions is equivalent to their Pettis integrability with respect to an associated scalar measure.

**Proposition 3.5.** Let \( m : \mathcal{B} \rightarrow \mathcal{M}(\mathcal{E}) \) be an order bounded measure-valued measure with modulus \( \|m\|_\mathcal{M} \). A strongly \( m \)-measurable function \( f : \Omega \rightarrow \mathcal{X} \) is \( m \)-integrable in \( \mathcal{M}(\mathcal{E}, \mathcal{X}) \) if and only if it is Pettis \( (\|m\|_\mathcal{M})^\Sigma \)-integrable in \( \mathcal{X} \). In this case, \( f \) is Pettis \( m_E \)-integrable in \( \mathcal{X} \) for each \( E \in \mathcal{E} \) and the equality

\[
\left( \int_B f \otimes dm \right)(E) = \int_B f dm_E
\]

holds for all \( B \in \mathcal{B} \) and \( E \in \mathcal{E} \).

**Proof.** To show this proposition, it is sufficient to show that

\[
\left( \int_B f \otimes dm \right)(E) = \int_B f(t)\mu(dt \times E).
\]

Since all simple functions are \( m^\mathcal{X} \)-integrable, above equality is true for simple \( f_n \). As general proof method of integral, it is satisfying the equality for the function \( f \) such that simple functions \( f_n \) converges to \( f \) \( m^\mathcal{X} \)-a.e. And \( \{(f_n m^\mathcal{X})(\cdot)(E) : E \in \mathcal{E}, n = 1, 2, \ldots\} \) is uniformly countably additive if and
only if \( \{ \int |(f_n, x^*)| \mu(dt \times E) : \|x^*\| \leq 1, n = 1, 2, \ldots \} \) is uniformly countably additive if and only if \( f \) is \( m_\Sigma \)-Pettis integrable. \( \square \)

**Remark 3.6.** If \( m \) is positive and \( \beta_X(m)(f) < \infty \), then \( \int_\Omega \|f\|_X dm_\Sigma < \infty \). See [4].

**Proposition 3.7.** Let \( m : B \to \mathcal{M}(E) \) be an order bounded measure-valued measure. Then for any Banach space \( X \), we have \( \beta_X(m)(\Omega) < \infty \) and \( \beta_X(m) \) is continuous.

**Proof.** We have

\[
\beta_X(m)(E) = \sup\{ \sum_{j,k} \langle x_k m(B_k \cap E)(B_j), x_j^* \rangle : \|x_k\|, \|x_j^*\| \leq 1 \}
\]

\[
\leq (|m|_E)(\Sigma)
\]

\[
\leq (|m|_E)_\Sigma(E),
\]

so \( \beta_X(m)(\Omega) \) is finite. Because \( |m|_E : B \to \mathcal{M}_+(E) \) is a measure, it follows that \( \beta_X(m) \) is continuous. \( \square \)

For order bounded measure-valued measures, we have the following simplified versions of the convergence theorem of Section 2. We state them without proof.

**Theorem 3.8.** (Egorov) Let \((\Sigma, E), (\Omega, B)\) be measurable spaces and \( X \) a Banach space. Suppose that \( m : B \to \mathcal{M}(E) \) is an order bounded \( \mathcal{M}(E) \)-valued measure. Let \( f_n, f : \Omega \to X, n \in \mathbb{N} \) be \( m \)-measurable functions such that \( f_n \to f \), \( m \)-a.e. Then a) for any \( \varepsilon > 0 \), there is a set \( B \in \mathcal{B} \) such that \( \beta_X(m)(B^c) < \varepsilon \) and \( f_n \to f \) uniformly on \( B \). b) \( f_n \to f, \beta_X(m) \)-measure.

**Theorem 3.9.** (Vitali) Let \((\Sigma, E), (\Omega, B)\) be measurable spaces and \( X \) a Banach space. Suppose that \( m : B \to \mathcal{M}(E) \) is an order bounded \( \mathcal{M}(E) \)-valued measure with modulus \(|m|_E\). Let \( (f_n) \) be a sequence from \( \mathcal{L}_1(\beta_X(m)) \) with \( f_n \) Pettis \(|m|_E\)-integrable functions and let \( f : \Omega \to X \) be \( m \)-measurable. Assume that a) \( f_n \to f \) in \( \beta_X(m) \)-measure or a') \( f_n \to f \), \( m \)-a.e. b) \( \lim_{n \to \infty} \beta_X(m)(f_n) = 0, \) uniformly for \( n \in \mathbb{N} \). Then \( f \in \mathcal{L}_1(\beta_X(m)) \) and \( \beta_X(m)(f_n - f) \to 0 \). Furthermore, the function \( f \) is \( m \)-integrable in \( \mathcal{M}(E, X) \) and \( \int_B f \otimes dm \to \int_B f \otimes dm \in \mathcal{M}(E, X) \), uniformly for \( B \in \mathcal{B} \) as \( n \to \infty \). Conversely, if \( f \in \mathcal{L}_1(\beta_X(m)) \) and \( \beta_X(m)(f_n - f) \to 0 \) as \( n \to \infty \), then conditions a) and b) are satisfied.

References


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