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Fuzzy Convergence Approach Spaces

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Abstract

In this paper, we define that a fuzzy convergence approach limit and a fuzzy approach Cauchy structure on X. And we investigate the relations between the category **CAP** and the category **FCAP**. And we show that the categories $\mathbf{FCAP_{RC}}$ and $\mathbf{uFACHY_{cpl}}$ are isomorphic.

Key words : fuzzy convergence approach limit, u-fuzzy approach Cauchy structure

1. Introduction

In 1989, R. Lowen [5], introduce the category **AP** of approach spaces which contains the categories **TOP** and **MET** as full subcategories. In [7], E. Lowen and R. Lowen embedded **AP** in the quasitopos **CAP** of convergence approach spaces. On the other hand the concept of fuzzy set was introduced by Zadeh. And fuzzy topological and its generalizations are later studied by several authors, one of which, developed by a theory of fuzzy topological space has been developed in various directions. A notion of convergence of prefilters, called fuzzy limitierung, is defined by K. C. Min [4] in 1989 as a generalization of a Q-neighborhood system of a fuzzy point in a fuzzy topological space. In this paper we define the category **FCAP** of fuzzy approach convergence limit spaces and show that the category **FCAP** is topological category and the category **CAP** is a bicoreflective subcategory of the category **FCAP**. And we define the category **FACHY** of fuzzy approach Cauchy spaces. And we show that the categories $\mathbf{FCAP_{RC}}$ and $\mathbf{uFACHY_{cpl}}$ are isomorphic.

2. Preliminaries

Let X be a set. A *fuzzy set* in X is a membership function μ_A from X to $I = [0, 1]$. We say that B *includes* A if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$and write $A \subseteq B$.

A *fuzzy point* $p = (x_0, \lambda)$ in X is a fuzzy set in X defined by $\mu_p(x) = \lambda$ for $x = x_0 (0 < \lambda \leq 1)$ and $\mu_p(x) = 0$ for $x \neq x_0$. We call x_0 the support of p and λ the *value* of p. We say that $p = (x_0, \lambda)$ is *quasi-coincident* with a fuzzy set A, denoted by $p\,q\,A$, if $\mu_A(x_0) > 1 - \lambda$ and p *belongs* to A, denoted by $p \in A$, if $\mu_A(x_0) > \lambda$.

For any set X, let $\mathcal{B}(X)$ be the collection of all prefilters on X and $\mathcal{H}(X)$ be the set of all fuzzy points in X.

Definition 2.1. ([4]) A *fuzzy limitierung* $\triangle$ is a map from $\mathcal{H}$ into $\mathcal{P}(\mathcal{B}(X))$ satisfying the following axioms: for each $x_\lambda \in \mathcal{H}(X)$,
(L0) if $\mathcal{F} \in \triangle(x_\lambda)$, then $\underline{\alpha} \in \mathcal{F}$ for every $\alpha > 1 - \lambda,\ \lambda \in [0, 1]$,
(L1) $\dot{x_\lambda} = \{A \in I^X : x_\lambda q A\} \in \triangle(x_\lambda)$,
(L2) if $\mathcal{F} \in \triangle(x_\lambda)$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in \triangle(x_\lambda)$,
(L3) if $\mathcal{F}, \mathcal{G} \in \triangle(x_\lambda)$, then $\mathcal{F} \cap \mathcal{G} \in \triangle(x_\lambda)$.

The pair $(X, \triangle)$ is called a *fuzzy limit space*.

Definition 2.2. ([8]) A fuzzy Cauchy space (X, s) is said to be *complete* if for each $\mathfrak{F} \in s$, there exists $x_\lambda \in \mathcal{H}(X)$ such that $\mathfrak{F} \in \triangle_s(x_\lambda)$, where $\mathfrak{F} \in \triangle_s(x_\lambda) \iff \mathfrak{F} \cap \dot{x_\lambda} \in s,\ l(\mathfrak{F}) \geq \lambda,\ l(\mathfrak{F}) = \sup\{\lambda \in (0, 1] | \mathfrak{F} \supseteq \bigcap_{x \in X} \dot{x_\lambda}\}$

Definition 2.3. ([6]) A *convergence approach limit* on a set X is a map $\lambda : \mathcal{F}(\mathcal{X}) \to [0, \infty]^X$ fulfilling:

(CAL1) $\lambda(\dot{x})(x) = 0$ for all $x \in X$,

(CAL2) if $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$ and $\mathcal{F} < \mathcal{G}$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{G})$,

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(CAL3) $\lambda(\mathcal{F} \cap \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$.

Definition 2.4. ([6]) For any convergence approach limit spaces (X, λ) and (Y, λ'), a map $f : X \to Y$ is called a *contraction*, if $\lambda'(f(\mathcal{F}))(f(x)) \leq \lambda(\mathcal{F})(x)$ for all $\mathcal{F} \in \mathcal{F}(X)$ and $x \in X$.

Let **CAP** denote the category of all convergence approach spaces and all contractions.

Definition 2.5. ([6]) An *ultra-approach Cauchy structure* γ on a set X is a map $\gamma : \mathcal{F}(X) \to [0, \infty]$ fulfilling:

(ACHY1) $\gamma(\dot{x}) = 0$ for all $x \in X$,

(ACHY2) if $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$ and $\mathcal{F} \subset \mathcal{G}$, then $\gamma(\mathcal{F}) \geq \gamma(\mathcal{G})$,

u-(ACHY3) if $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$ and there exists $\mathcal{F} \vee \mathcal{G}$, then $\gamma(\mathcal{F} \cap \mathcal{G}) = \gamma(\mathcal{F}) \vee \gamma(\mathcal{G})$.

The pair (X, γ) is called an *ultra-approach Cauchy space*.

Definition 2.6. ([6]) For any ultra-approach Cauchy spaces (X, γ) and (Y, γ'), a map $f : X \to Y$ is called a *contraction*, if $\gamma'(f(\mathcal{F})) \leq \gamma(\mathcal{F})$ for all $\mathcal{F} \in \mathcal{F}(X)$.

Let **uACHY** denote the category of all ultra-approach Cauchy spaces and all contractions.

3. The category FCAP and the category FACHY

Definition 3.1. A *fuzzy convergence approach limit* on a set X is a map $\Lambda : \mathcal{B}(X) \to [0, \infty]^{\mathcal{H}(X)}$ fulfilling:

(FCAL1) $\Lambda(\dot{x_\lambda})(x_\lambda) = 0$ for all $x_\lambda \in \mathcal{H}(X)$,

(FCAL2) if $\mathfrak{F}, \mathfrak{G} \in \mathcal{F}(X)$ and $\mathfrak{F} < \mathfrak{G}$, then $\Lambda(\mathfrak{F})(x_\lambda) \geq \Lambda(\mathfrak{G})(x_\lambda)$,

(FCAL3) $\Lambda(\mathfrak{F} \cap \mathfrak{G})(x_\lambda) = \Lambda(\mathfrak{F})(x_\lambda) \vee \Lambda(\mathfrak{G})(x_\lambda)$ for all $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$.

(X, Λ) is called a *fuzzy convergence approach space* .

Definition 3.2. For any fuzzy convergence approach spaces (X, Λ) and (Y, Λ'), a map $f : X \to Y$ is called a *contraction*, if $\Lambda'(f(\mathfrak{F}))(f(x_\lambda)) \leq \Lambda(\mathfrak{F})(x_\lambda)$ for all $\mathfrak{F} \in \mathcal{B}(X)$.

Let **FCAP** denote the category of all fuzzy convergence approach spaces and all contractions.

Proposition 3.3. For any family $((X_j, \Lambda_j))_{j \in J}$ of fuzzy approach convergence spaces and a source $(X \xrightarrow{f_j} X_j)_{j \in J}$, if we define a map $\Lambda : \mathcal{B}(X) \to [0, \infty]^{\mathcal{H}(X)}$ on X by $\Lambda(\mathfrak{F})(x_\lambda) = \sup_{j \in J} \Lambda_j(f_j(\mathfrak{F}))(f_j(x_\lambda))$, then Λ is an initial fuzzy convergence approach limit on X.

Proof. (FCAP1) $\Lambda(\dot{x_\lambda})(x_\lambda) = 0$, since for all $j \in J$, $\Lambda_j(f_j(\dot{x_\lambda}))(f(x_\lambda)) = 0$.
(FCAP2) If $\mathfrak{F} < \mathfrak{G}$, then for all $j \in J$, $\Lambda_j(f_j(\mathfrak{F}))(f(x_\lambda)) \geq \Lambda_j(f_j(\mathfrak{G}))(f(x_\lambda))$ and hence $\Lambda(\mathfrak{F})(x_\lambda) \geq \Lambda(\mathfrak{G})(x_\lambda)$.
(FCAP3) $\Lambda(\mathfrak{F} \cap \mathfrak{G})(x_\lambda)$
$= \sup \Lambda_j(f_j(\mathfrak{F} \cap \mathfrak{G}))(f_j(x_\lambda))$
$= \sup \Lambda_j(f_j(\mathfrak{F}) \cap f_j(\mathfrak{G}))(f_j(x_\lambda))$
$= \sup(\Lambda_j(f_j(\mathfrak{F}))(f_j(x_\lambda)))$
$\vee \Lambda_j(f_j(\mathfrak{G}))(f_j(x_\lambda)))$
$= \Lambda(\mathfrak{F})(x_\lambda) \vee \Lambda(\mathfrak{G})(x_\lambda)$.
The remains are obvious, by construction of Λ on X. □

Theorem 3.4. The category **FCAP** is topological.

Proof. **FCAP** admits initial structures by Proposition 3.3. And for any set X, the collection of all fuzzy convergence approach spaces on X is a set. On any singleton set, there exists precisely one fuzzy convergence approach limit. Thus a fuzzy convergence approach limit has smallness properties. In all, the category **FCAP** of fuzzy convergence approach spaces is topological category. □

Definition 3.5. An T-*fuzzy approach Cauchy structure* on a set X is a map $\Gamma : \mathcal{B}(X) \to [0, \infty]$ fulfilling:

(FACHY1) $\Gamma(\dot{x_\lambda}) = 0$ for all $x_\lambda \in \mathcal{H}(X)$,

(FACHY2) if $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$ and $\mathfrak{F} \subset \mathfrak{G}$, then $\Gamma(\mathfrak{F}) \geq \Gamma(\mathfrak{G})$,

(T-FACHY3) if $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$ and there exists $\mathfrak{F} \vee \mathfrak{G}$, then $\Gamma(\mathfrak{F} \cap \mathfrak{G}) \leq \Gamma(\mathfrak{F}) * \Gamma(\mathfrak{G})$ where a * b=T(a, b).

The pair (X, Γ) is called a T-*fuzzy approach Cauchy space*. And replace (T-FACHY3) by the following (uFACHY3), (X, Γ) is called an *ultra fuzzy approach Cauchy space*. (uFACHY3) If $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$ and there exists $\mathfrak{F} \vee \mathfrak{G}$, then $\Gamma(\mathfrak{F} \cap \mathfrak{G}) = \Gamma(\mathfrak{F}) \vee \Gamma(\mathfrak{G})$.

Definition 3.6. For any T-(u-)-fuzzy approach Cauchy spaces (X, Γ) and (Y, Γ'), a map $f : X \to Y$ is called a *contraction*, if $\Gamma'(f(\mathfrak{F})) \le \Gamma(\mathfrak{F})$ for all $\mathfrak{F} \in \mathcal{B}(X)$.

Let $\mathbf{FACHY}_T$ denote the category of all T-fuzzy approach Cauchy spaces and contractions. Let **uFACHY** denote the category of all ultra fuzzy approach Cauchy spaces and contractions.

Consider

$$G : \mathbf{CAP} \longrightarrow \mathbf{FCAP}$$
$$(X, \eta) \mapsto (X, \Lambda_\eta)$$
$$f \mapsto f$$

Define a $\Lambda_\eta(\mathfrak{F})(x_\lambda) = \inf\{\alpha | \mathfrak{F} \supseteq \mathcal{G}^\lambda_\mathcal{F}$ for some $\mathcal{F}$ such that $\eta(\mathcal{F})(x) \le \alpha\}$ where $0 \le \alpha \le \infty$, and $\mathcal{G}^\lambda_\mathcal{F}$ is generated by the basis $\{A \in I^X | \mu_A(\mathcal{F}) \to \mu_A(x)$ in I_r, $\mu_A(x) > 1 - \lambda\}\}$.

Proposition 3.7. Λ_η is a fuzzy convergence approach limit.

Proof. (FCAP1) and (FCAP2) are trivial, since $\eta(\dot{x})(x) = 0 \le \alpha$ and $\eta(\mathfrak{F})(x) \ge \eta(\mathfrak{G})(x)$ when $\mathfrak{F} \subset \mathfrak{G}$. To show that (FCAP3), by (FCAP2), clearly $\Lambda_\eta(\mathfrak{F} \cap \mathfrak{G})(x_\lambda) \le \Lambda_\eta(\mathfrak{F})(x_\lambda) \vee \Lambda_\eta(\mathfrak{G})(x_\lambda)$ and let $\Lambda_\eta(\mathfrak{F})(x_\lambda) = \beta_1$, $\Lambda_\eta(\mathfrak{G})(x_\lambda) = \beta_2$. Then $\eta(\mathcal{F})(x) \le \beta_1$, and $\eta(\mathcal{G})(x) \le \beta_2$ and hence $\eta(\mathcal{F})(x) \vee \eta(\mathcal{G})(x) \le \beta_1 \vee \beta_2$. Thus $\Lambda_\eta(\mathfrak{F} \cap \mathfrak{G})(x_\lambda) \le \beta_1 \vee \beta_2$. □

Proposition 3.8. For any **CAP**-spaces (X, η) and (Y, η'), if $f : (X, \eta) \to (Y, \eta')$ is a contraction, then $f : (X, \Lambda_\eta) \to (Y, \Lambda_{\eta'})$ is a contraction.

Proof. For any $\mathfrak{F} \in \mathcal{B}(X)$, let $\Lambda_\eta(\mathfrak{F})(x_\lambda)$
$= \inf\{\alpha | \mathfrak{F} \subseteq \mathcal{G}^\lambda_\mathcal{F}$ for some $\mathcal{F}$ such that $\eta(\mathcal{F})(x) \le \alpha\}$
$= \beta$ and then $\eta(\mathcal{F})(x) \le \beta$. Since $\eta'(f(\mathcal{F}))(f(x))$
$\le \eta(\mathcal{F})(x) \le \beta$ and $f(\mathfrak{F}) \supseteq f(\mathcal{G}^\lambda_\mathcal{F})$
$= \mathcal{G}^\lambda_{f(\mathcal{F})}$, and thus $\Lambda(f(\mathfrak{F}))(f(x_\lambda))$
$\le \beta$. □

Consider

$$H : \mathbf{FCAP} \longrightarrow \mathbf{CAP}$$
$$(X, \Lambda) \mapsto (X, \eta_\Gamma)$$
$$f \mapsto f$$

Define a $\eta_\Lambda(\mathcal{F})(x)$
$= \inf\{\alpha | \Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \le \alpha\}$ for each $\lambda \in (0, 1]$, $0 \le \alpha \le \infty$, $\mathfrak{F}^\lambda_\mathcal{F}$ prefilter is generated by
$\{B \in I^X | \mu_B(\mathcal{F}) \to \mu_B(x),\ \mu_B > 1 - \lambda\}$.

Proposition 3.9. η_Λ is a convergence approach limit on X.

Proof. (CAP1) and (CAP2) are trivial, since $\Lambda(\mathfrak{F}^\lambda_{\dot{x}})(x_\lambda) = \Lambda(\dot{x_\lambda})(x_\lambda) = 0 \le \alpha$ and $\Lambda(\mathfrak{F}^\lambda_\mathcal{G})(x_\lambda) \le \Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda)$ when $\mathcal{F} \subset \mathcal{G}$. To show that (CAP3), first $\eta_\Lambda(\mathcal{F} \cap \mathcal{G})(x) \le \eta_\Lambda(\mathcal{F})(x) \vee \eta_\Lambda(\mathcal{G})(x)$, by (CAP2). And let $\eta_\Lambda(\mathcal{F})(x) = \beta_1$, $\eta_\Lambda(\mathcal{G})(x) = \beta_2$ and then $\Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \le \beta_1$, $\Lambda(\mathfrak{F}^\lambda_\mathcal{G})(x_\lambda) \le \beta_2$. Then $\Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \vee \Lambda(\mathfrak{F}^\lambda_\mathcal{G})(x_\lambda) \le \beta_1 \vee \beta_2$. But $\Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \vee \Lambda(\mathfrak{F}^\lambda_\mathcal{G})(x_\lambda) = \Lambda(\mathfrak{F}^\lambda_\mathcal{F} \cap \mathfrak{F}^\lambda_\mathcal{G})(x_\lambda) = \Lambda(\mathfrak{F}^\lambda_{\mathcal{F}\cap\mathcal{G}})(x_\lambda)$. Thus $\eta_\Lambda(\mathcal{F} \cap \mathcal{G})(x) \le \beta_1 \vee \beta_2$. □

Proposition 3.10. For any **FCAP**-spaces (X, Λ) and (Y, Λ'), if $f : (X, \Lambda) \to (Y, \Lambda')$ is a contraction, then $f : (X, \eta_\Lambda) \to (Y, \eta_{\Lambda'})$ is a contraction.

Proof. For any $\mathcal{F} \in \mathcal{F}(X)$, let $\eta_\Lambda(\mathcal{F})(x) = \inf\{\alpha | \Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \le \alpha\} = \beta$ and then $\Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \le \beta$. But $\Lambda'(f(\mathfrak{F}^\lambda_\mathcal{F}))(f(x_\lambda)) \le \Lambda(\mathfrak{F}^\lambda_\mathcal{F})(x_\lambda) \le \alpha\}$), and thus $\eta_{\Lambda'}(f(\mathcal{F}))(f(x)) \le \beta$. □

Proposition 3.11. (1) For any convergence approach limit η on a set X, $\eta = \eta_{\Lambda_\eta}$.
(2) For any fuzzy convergence approach limit Λ on a set X, $\Lambda \le \Lambda_{\eta_\Lambda}$.

Theorem 3.12. The category **CAP** is a bicoreflective subcategory of the category **FCAP**.

For any **uFACHY**-space (X, Γ), define a map $\Lambda_\Gamma : \mathcal{B}(X) \to [0, \infty]^{\mathcal{H}(X)}$ by $\Lambda_\Gamma(\mathfrak{F})(x_\lambda) = \Gamma(\mathfrak{F} \cap \dot{x_\lambda})$. Then we can easily check the following propositions.

Proposition 3.13. For any **uFACHY**-space (X, Γ), the pair (X, Λ_Γ) is a fuzzy convergence approach space.

Proposition 3.14. If (X, Γ), $(Y, \Gamma') \in |\mathbf{uFACHY}|$ and $f : (X, \Gamma) \to (Y, \Gamma')$ is a contraction, then $f : (X, \Lambda_\Gamma) \to (Y, \Lambda_{\Gamma'})$ is a contraction.

Therefore we have a functor $F : \mathbf{uFACHY} \to \mathbf{FCAP}$.

We introduce the following conditions.

(R1) If $\mathfrak{F}$ and $\mathfrak{G} \in \mathcal{B}(X)$ are such that $\mathfrak{F} \vee \mathfrak{G}$ exists then
$\inf_{x_\lambda \in \mathcal{H}(X)} \Lambda(\mathfrak{F} \cap \mathfrak{G})(x_\lambda)$
$\le \inf_{x_\lambda \in \mathcal{H}(X)} \Lambda(\mathfrak{F})(x_\lambda) \vee \inf_{x_\lambda \in \mathcal{H}(X)} \Lambda(\mathfrak{G})(x_\lambda)$.

(R2) For any $\mathfrak{F} \in \mathcal{B}(X)$ and $x_\lambda \in \mathcal{H}(X)$,
$\inf_{x_\lambda \in \mathcal{H}(X)} \Gamma(\mathfrak{F} \cap \dot{x_\lambda})(y_{\lambda'}) = \Lambda(\mathfrak{F})(x_\lambda)$.

(C) For any $\mathfrak{F} \in \mathcal{B}(X)$, there exists a minimal point $y_\lambda \in \mathcal{H}(X)$ such that $\Lambda(\mathfrak{F})(y_\lambda) \le \Lambda(\mathfrak{F})(x_{\lambda'})$ for all

$x_{\lambda'} \in \mathcal{H}(X)$.

Let $\mathbf{FCAP_R}$ denote the full subcategory of the category $\mathbf{FCAP}$ consisting of all fuzzy convergence approach spaces satisfying (R1) and (R2) and the full subcategory of the category $\mathbf{FCAP_R}$ of all fuzzy convergence approach spaces fulfilling in addition (C) will be denoted by $\mathbf{FCAP_{R\,C}}$.

Proposition 3.15. If (X, Γ) is a **uFACHY**-space, then the pair (X, Λ_Γ) satisfies (R1) and (R2).

Proof. (R1) For any **uFACHY**-space (X, Γ) and $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$ such that $\mathfrak{F}\vee\mathfrak{G}$ exists, $\Lambda_\Gamma(\mathfrak{F}\cap\mathfrak{G})(x_\lambda) = \Gamma(\mathfrak{F}\cap\mathfrak{G}\cap\dot{x_\lambda}) \leq \Gamma(\mathfrak{F}\cap\mathfrak{G}\dot{x_\lambda}\cap\dot{y_{\lambda'}}) = \Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \vee \Gamma(\mathfrak{G}\cap\dot{y_{\lambda'}})$. Then

$$\inf_{x_\lambda, y_{\lambda'} \in \mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\mathfrak{G}\cap\dot{x_\lambda}\cap\dot{y_{\lambda'}})$$
$$= \inf(\Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \vee \Gamma(\mathfrak{G}\cap\dot{y_{\lambda'}}))$$
$$= \inf_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \vee \inf_{y_{\lambda'}\in\mathcal{H}(X)} \Gamma(\mathfrak{G}\cap\dot{y_{\lambda'}})$$
$$= \inf_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \vee \inf_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{G}\cap\dot{x_\lambda})$$
$$= \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda_\Gamma(\mathfrak{F})(x_\lambda) \vee \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda_\Gamma(\mathfrak{G})(x_\lambda).$$

(R2) Let $\mathfrak{F} \in \mathcal{B}(X)$ and $x_\lambda \in \mathcal{H}(X)$, then $\inf_{y_{\lambda'}\in\mathcal{H}(X)} \Lambda_\Gamma(\mathfrak{F}\cap\dot{x_\lambda})(y_{\lambda'})$
$\leq \Lambda_\Gamma(\mathfrak{F}\cap\dot{x_\lambda})(x_\lambda) = \Gamma(\mathfrak{F}\cap\dot{x_\lambda})$
$= \Lambda_\Gamma(\mathfrak{F})(x_\lambda)$.
And converse follows from the inequality
$\inf_{y_{\lambda'}\in\mathcal{H}(X)} \Lambda_\Gamma(\mathfrak{F}\cap\dot{x_\lambda})(y_{\lambda'})$
$= \inf_{y_{\lambda'}\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda}\cap\dot{y_{\lambda'}}) \geq \Gamma(\mathfrak{F}\cap\dot{x_\lambda})$
$= \Lambda_\Gamma(\mathfrak{F})(x_\lambda)$. □

For any $(X, \Lambda) \in \mathbf{FCAP_R}$ and $\mathfrak{F} \in \mathcal{B}(X)$, define a map
$\Gamma_\Lambda : \mathcal{B}(X) \to [0, \infty]$ by $\Gamma_\Lambda(\mathfrak{F})$
$= \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{F})(x_\lambda)$

Proposition 3.16. For any (X, Λ) in $\mathbf{FCAP_R}$, (X, Γ_Λ) is an **uFACHY**-space.

Proof. (FACHY1)
$\Gamma_\Lambda(\dot{x_\lambda}) = \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\dot{x_\lambda})(x_\lambda) = 0$.
(FACHY2) If $\mathfrak{F} \subset \mathfrak{G}$, then $\Gamma_\Lambda(\mathfrak{F})$
$= \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{F})(x_\lambda)$
$\geq \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{G})(x_\lambda) = \Gamma_\Lambda(\mathfrak{G})$.
(uFACHY3) For any $\mathfrak{F}, \mathfrak{G} \in \mathcal{B}(X)$ such that $\mathfrak{F}\vee\mathfrak{G}$ exists,
$\Gamma_\Lambda(\mathfrak{F}\cap\mathfrak{G}) = \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{F}\cap\mathfrak{G})(x_\lambda)$
$\leq \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{F})(x_\lambda) \vee \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda(\mathfrak{G})(x_\lambda)$
$= \Gamma_\Lambda(\mathfrak{F}) \vee \Gamma_\Lambda(\mathfrak{G})$, by (R1). And by (FACHY2), converse inequality holds. □

Proposition 3.17. For any fuzzy convergence approach spaces (X, Λ) and (Y, Λ') satisfying (R1) and (R2), if a map $f : (X, \Lambda) \to (Y, \Lambda')$ is a contraction, then the map $f : (X, \Gamma_\Lambda) \to (Y, \Gamma_{\Lambda'})$ is a contraction.

Proof. For any $\mathfrak{F} \in \mathcal{B}(X)$, $\Gamma_{\Lambda'}(f(\mathfrak{F})) = \inf_{y_{\lambda'}\in\mathcal{H}(Y)} \Lambda'(f(\mathfrak{F}))(y_{\lambda'})$
$\leq \inf_{x_{\lambda'}\in\mathcal{H}(X)} \Lambda'(f(\mathfrak{F}))(f(x_{\lambda'}))$
$\leq \inf_{x_{\lambda'}\in\mathcal{H}(X)} \Lambda(\mathfrak{F})(x_{\lambda'}) = \Gamma_\Lambda(\mathfrak{F})$, hence it follows. □

Therefore we have an embedding

$$G : \mathbf{FCAP_R} \longrightarrow \mathbf{uFACHY}$$
$$(X, \Lambda) \mapsto (X, \Gamma_\Lambda)$$
$$f \mapsto f$$

Proposition 3.18. (1) For any **uFACHY**-structure Γ on $X, \Gamma \leq \Gamma_{\Lambda_\Gamma}$.
(2) For any $\mathbf{FCAP_R}$ -structure Λ, $\Lambda = \Lambda_{\Gamma_\Lambda}$.

Proof. (1) $\Gamma_{\Lambda_\Gamma}(\mathfrak{F}) = \inf_{x_\lambda\in\mathcal{H}(X)} \Lambda_\Gamma(\mathfrak{F})(x_\lambda)$
$= \inf_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \geq \Gamma(\mathfrak{F})$.
(2) $\Lambda_{\Gamma_\Lambda}(\mathfrak{F})(x_\lambda) = \Gamma_\Lambda(\mathfrak{F}\cap\dot{x_\lambda}) = \inf_{y_{\lambda'}\in\mathcal{H}(X)} \Lambda(\mathfrak{F}\cap\dot{x_\lambda})(y_{\lambda'}) = \Lambda(\mathfrak{F})(x_\lambda)$, by (R2). □

Theorem 3.19. The category $\mathbf{FCAP_R}$ is a bicoreflective subcategory of the category **uFACHY**.

Definition 3.20. A **FACHY**-space (X, Γ) is said to be *complete* (for short, **cpl**) if for any $\mathfrak{F} \in \mathcal{B}(X)$, $\min_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda})$ exists and it holds that $\Gamma(\mathfrak{F}) = \min_{x_\lambda\in\mathcal{H}(X)} \Gamma(\mathfrak{F}\cap\dot{x_\lambda})$.

The full subcategory of **uFACHY** consisting of all complete **uFACHY**-spaces shall be denoted by $\mathbf{FACHY_{cpl}}$.

Proposition 3.21. If (X, Γ) is a complete **uFACHY**-space, then the pair (X, Λ_Γ) satisfies the property (C).

Proof. For any $\mathfrak{F} \in \mathcal{B}(X)$ and $x_\lambda \in \mathcal{H}(X)$, $\Lambda_\Gamma(\mathfrak{F})(x_\lambda) = \Gamma(\mathfrak{F}\cap\dot{x_\lambda}) \geq \Gamma(\mathcal{F}\cap\dot{y_{\lambda'}}) = \Lambda_\Gamma(\mathfrak{F})(y_{\lambda'})$ for some $y_{\lambda'}$, by completeness. So (X, Λ_Γ) fulfills (C). □

Proposition 3.22. For any $(X, \Lambda) \in |\mathbf{FCAP_{R\,C}}|$, the pair (X, Γ_Λ) is a complete **uFACHY**-space.

Proof. Take any $\mathfrak{F} \in \mathcal{B}(X)$ and $x_\lambda \in \mathcal{H}(X)$. Then $\Gamma_\Lambda(\mathfrak{F} \cap \dot{x_\lambda}) = \inf_{y_\lambda \in \mathcal{H}(X)} \Lambda(\mathfrak{F} \cap \dot{x_\lambda})(y_{\lambda'}) = \Lambda(\mathfrak{F})(x_\lambda)$, by (R2). And by (C), we have $y_{\lambda'} \in \mathcal{H}(X)$ such that $\Lambda(\mathfrak{F})(x_\lambda) \geq \Lambda(\mathfrak{F})(y_{\lambda'})$ for all $x_\lambda \in \mathcal{H}(X)$. So there exists $\min \Gamma_\Lambda(\mathfrak{F} \cap \dot{x_\lambda})$ and it holds that $\min_{x_\lambda \in \mathcal{H}(X)} \Gamma(\mathfrak{F} \cap \dot{x_\lambda})$
$= \min_{x_\lambda \in \mathcal{H}(X)} \Lambda(\mathfrak{F})(x_\lambda) = \Gamma_\Lambda(\mathfrak{F})$. □

Theorem 3.23. The categories $\mathbf{uFACHY_{cpl}}$ and $\mathbf{FCAP_{R\,C}}$ are isomorphic.

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