Journal of the Society of Korea Industrial and Systems Engineering
Vol, 30, No. 1, pp.33-40, March 2007.

부분적인 서비스 보호와 부정적인 고객을 고려한 대기행렬 모형

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Queueing System with Negative Customers and Partial Protection of Service

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A multi-server queueing system with finite buffer is considered. The input flow is the BMAP (Batch Markovian Arrival Process). The service time has the PH (Phase) type distribution. Customers from the BMAP enter the system according to the discipline of partial admission. Besides ordinary (positive) customers, the Markovian flow (MAP) of negative customers arrives to the system. A negative customer can delete an ordinary customer in service if the state of its PH-service process belongs to some given set. In opposite case the ordinary customer is considered to be protected of the effect of negative customers. The stationary distribution and the main performance measures of the considered queueing system are calculated.

Keywords : Batch Markovian Arrival Process, Phase Type Service Time Distribution, Negative Customers, and Partial Service Protection

1. Introduction

In real telecommunication and computer systems, customers may leave the system without full service by various reasons such as impatience of customers, deleting files infected by computer viruses, the accidental fails of servers, etc.

Such situations can be modeled in terms of queues or networks with negative customers. The notion of a negative customer was introduced by E. Gelenbe in [7]. A negative customer has the effect of a signal that induces an ordinary customer (or a batch of customers), if any, to leave the system immediately. Since their introduction, queues and networks with negative customers (so called G-queues and G-networks) were studied by many authors. The detailed overviews of G-queues and G-networks can be found in the surveys by [2, 3]. Among the last years publications we mention the papers ([1, 5, 6, 11]) where G-queues with Markovian flows or/and phase type or general service time distribution are considered. In [5, 6], the controlled systems of the *GI/PH/1* and the *BMAP/SM/1* type are investigated. The pa-

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※ This research has been supported by Sangji University Research Fund 2005 while Prof. Chesoong Kim was a Visiting Scholar in the Department of Industrial Engineering, University of Washington, USA.

per in [11] deals with the *M/G/1* queue with two types of service and two disciplines of removing positive customers. As the most close related publications to the present paper, we mention the papers ([1, 9]) where multi-server queues with negative customers are considered. In [1] the *M/M/N* retrial queue is investigated where a negative customer deletes a batch of secondary customers of random size. The stationary distribution is calculated and the system behavior in overload regime is studied. The subject of [9] is the sojourn time distribution in the *MMCPP/GE/c* queue with negative customers.

In the present paper, we consider the model *BMAP/PH/N/R* with the *MAP* flow of negative customers. A negative customer can delete an ordinary customer in service if at the moment of the negative customer arrival the state of the *PH* service process of the ordinary customer belongs to some definite set.

In opposite case an ordinary customer is considered to be protected of the effect of negative customers. To resolve conflicts arising at arrival epochs in consequence of finite number of places in the system and unlimited group size in the input flow, we consider the discipline of partial admission to the system. We calculate the stationary distribution and the main performance characteristics of the system such as the probabilities of a positive customer loss in consequence of absence of free places in the system or /and because of the effect of negative customers. We give the simple numerical examples which illustrate the behavior of these probabilities depending on the ordinary and negative customer arrival rates and service rate.

The rest of the paper is organized as follows. In section 2, the mathematical model is described. In section 3, the process of the system states is defined and the stable algorithm for calculating the stationary distribution of the system states is presented. The formulas for unsuccessful service probabilities are derived in section 4. Section 5 contains the numerical examples.

2. Model Descriptions

We consider an N-server queueing system with a buffer of size $R \le \infty$. Ordinary (positive) customers arrive to the system according to the *BMAP*. To the present day, the *BMAP* (*Batch Markovian Arrival Process*)(see, [12]) is the most popular mathematical model for the telecommunication networks traffic because it catches the typical features of this traffic such as correlation and burstiness. In the *BMAP*, the batch arrivals are directed by an irreducible continuous time Markov chain v_t, $t \ge 0$ (the underlying process) with the state space $\{0, 1, \cdots, W\}$. Sojourn time of process v_t in the state v is exponentially distributed with the parameter λ_v, $\lambda_v \ge 0$, $v = \overline{0, W}$. After this time expires, the process v_t either jumps to the state r, $r = \overline{0, W}$, $v \ne r$, without generating arrival with probability $p_0(v, r)$, or the process v_t jumps to the state r, $r = \overline{0, W}$, with generating a batch arrival of size k with probability $p_k(v, r)$,

$$\sum_{r=0, r \ne v}^{W} p_0(v, r) + \sum_{k=1}^{\infty} \sum_{r=0}^{W} p_k(v, r) = 1, \quad v = \overline{0, W}$$

Introduce into consideration matrices D_k, $k \ge 0$, which elements are defined as :

$$(D_0)_{v,v} = \lambda_v, \quad v = \overline{0, W}$$
$$(D_0)_{v,r} = \lambda_v p_0(v, r), \quad r = \overline{0, W}, \quad v \ne r$$
$$(D_k)_{v,r} = \lambda_v p_k(v, r), \quad k \ge 1, \quad v, \quad r = \overline{0, W}$$

So, the input flow is completely defined by the set of matrices D_k, $k \ge 0$, or their generating function $D(z) = \sum_{k=0}^{\infty} D_k z^k$, $|z| \le 1$. The matrix $D(1)$ is an irreducible generator of the chain v_t, $t \ge 0$. The average intensity λ(fundamental rate) of the *BMAP* is defined as $\lambda = \vec{\theta}(D(z))'|_{z=1}\vec{e}$ and the intensity λ_b of batch arrivals is defined as $\lambda_b = \vec{\theta}(-D_0)\vec{e}$. Here and in the sequel $\vec{e}$ is a column vector of appropriate size consisting of 1's, $\vec{\theta}$ is the solution to the equations $\vec{\theta}D(1) = \vec{0}$, $\vec{\theta}\vec{e} = 1$. The variation coefficient c_{var} of intervals between batch arrivals is given by $(c_{\text{var}})^2 = 2\lambda_b\vec{\theta}(-D_0)^{-1}\vec{e} - 1$ while the correlation coefficient c_{cor} of intervals between successive batch arrivals is calculated as $c_{cor} = (\lambda_b\vec{\theta}(-D_0)^{-1}(D(1) - D_0)(-D_0)^{-1}\vec{e} - 1)/(c_{\text{var}})^2$. For more information about the *BMAP*, its properties, partial cases, usefulness in telecommunication networks modeling and related research, see [4].

The system under consideration has finite waiting space. Due to possibility of group arrivals, it can occur that there are free places in the system at arrival epoch, however the number of these places is less than the number of customers in a group. In such situation the acceptance of customers to the system is realized according to the partial admission discipline (only a part of the group corresponding to the number of free places is allowed to enter the system while the rest of the group is lost).

It is assumed that all servers are identical and independent of each other. Service time of a customer by a server has *PH* type distribution with irreducible representation (β, S). It means the following. Service time is interpreted as the time until the continuous time Markov chain m_t, $t \geq 0$, with the state space (set of phases) $\{1, \cdots, M+1\}$ reaches the absorbing state (phase) $M+1$. Transitions of the chain m_t, $t \geq 0$ within the state space $\{1, \cdots, M\}$ are defined by the sub-generator S while the intensities of transitions into the absorbing state are defined by the vector $S_0 = -S\vec{e}$. At the service beginning epoch, the state of the process m_t, $t \geq 0$ is chosen within the state space $\{1, \cdots, M\}$ according to the probabilistic row vector β. It is assumed that the matrix $S+S_0\beta$ is an irreducible one. For more information about the *PH* type distribution, see [13]. Negative customers arrive to the system according to the *MAP*. The *MAP* is defined by the state space $\{0, 1, \cdots, V\}$ of underlying process η_2, $t \geq 0$ and by the matrix generating function $H(z) = H_0 + H_1 z$, $|z| \leq 1$. We suppose that there exist a set of phases of service process which are protected of the effect of negative customers. Without loss of generality we define this set as $\{j+1, \cdots, M\}$. A negative customer with equal probability goes to any busy server. If the state of the PH service process of an ordinary customer, which occupies the server-target, belongs to the set $\{1, \cdots, j\}$ the negative customer deletes the ordinary customer being in service.

In the opposite case, the ordinary customer is considered to be protected of the effect of negative customers. In such case the negative customer leaves the system without any effect.

3. Stationary Distributions

The process of the system states is described in terms of the irreducible continuous-time Markov chain

$$\xi_t = \left\{i_t, v_t, \eta_t, m_t^{(1)}, \cdots, m_t^{(\min\{i_t, N\})}\right\}, t \geq 0,$$

where i_t is the number of customers in the system, v_t and η_t are the states of the *BMAP* and the *MAP* underlying processes respectively, $m_t^{(r)}$ is the state of the *PH* service process on the r^{th} busy server at time t (the busy servers are numerated in order of their occupying, i.e., the server, which begins service, is appointed the maximal number among all busy servers; when some server finishes the service or becomes free as a result of a negative customer effect, the rest busy severs are enumerated in the above manner), $m_t^{(r)} = \overline{1, M}$, $v_t = \overline{0, W}$, $\eta_t = \overline{0, V}$, $r = \overline{1, \min\{i_t, N\}}$, $i_t = \overline{0, N+R}$. We suppose that the states of the ξ_t, $t \geq 0$, are enumerated in lexicographic order.

In the sequel we will use the following denotations :

- I_a is an identity matrix of size a, $I_0 = 1$
- I is an identity matrix of appropriate dimension;
- $\overline{I_m}$ is a diagonal matrix of size M having zeroes as the first j diagonal entries and 1's as the rest of diagonal;
- $\vec{e}_\alpha(\vec{0}_a)$ is a column vector of size a, consisting of 1's (zeroes);
- $\hat{\vec{e}}_M$ is a column vector of size M, having 1's as the first j entries and zeroes as the rest entries;
- $\otimes$ and $\oplus$ are the symbols of Kronecker product and sum of matrices, see, e.g., (Graham, 1981);

$$A^{\otimes l} = \underbrace{A \otimes \cdots \otimes A}_{l},\ l \geq 1,\ A^{\otimes 0} = 1$$

$$A^{\otimes l} = \sum_{m=0}^{l-1} I_{M^m} \otimes A \otimes I_{M^{l-m-1}},\ l \geq 1,$$

$$\mathrm{H}^{(i)} = (1/i) I_{\overline{W}} \otimes H_1 \otimes \hat{\vec{e}}_M^{\oplus i},\ i = \overline{1, N}$$

$$\mathrm{H} = (1/N) I_{\overline{W}} \otimes H_1 \otimes (\hat{\vec{e}}_M \beta)^{\oplus N}$$

$$\overline{H}^{(i)} = (1/i) H_1 \otimes \overline{I}_M^{\oplus i},\ i = \overline{1, N}$$

$$\overline{W} = W+1\ ;\ \overline{V} = V+1$$

Lemma 1. *Infinitesimal generator A of the Markov chain* ξ_t, $t \geq 0$ *has the block structure* $A = (A_{i,j})_{i,j=\overline{0, N+R}}$, *where the non-zero blocks have form*

$$A_{i,i-1} = \begin{cases} I_{\overline{W}\overline{V}} \otimes S_0^{\oplus i} + \mathrm{H}^{(i)}, & i = \overline{1, N} \\ I_{\overline{W}\overline{V}} \otimes (S_0 \beta)^{\oplus N} + \mathrm{H}, & i = \overline{N+1, N+R} \end{cases}$$

$$A_{i,i} = \begin{cases} D_0 \oplus H(1),\ i = 0 \\ D_0 \oplus (H_0 \oplus S^{\oplus \min\{i, N\}} + \overline{\mathrm{H}}^{(\min\{i, N\})}),\ i = \overline{1, N+R-1} \\ D(1) \oplus (H_0 \oplus S^{\oplus N} + \overline{\mathrm{H}}^{(N)},\ i = N+R) \end{cases}$$

$$A_{i,i+k} = \begin{cases} D_k \otimes I_{\overline{V}M} \otimes \beta^{\otimes \min\{k, N-i\}},\ k = \overline{1, N+R-i-1},\ i = \overline{0, N} \\ D_k \otimes I_{\overline{V}M^N},\ k = \overline{1, N+R-i-1},\ i = \overline{N+1, N+R-1} \\ \left(D(1) - \sum_{l=0}^{k-1} D_l\right) \otimes I_{\overline{V}M^i} \otimes \beta^{\otimes (N-i)},\ k = N+R-1,\ i = \overline{0, N} \\ \left(D(1) - \sum_{l=0}^{k-1} D_l\right) \otimes I_{\overline{V}M^N},\ k = N+R-1,\ i = \overline{N+1, N+R-1} \end{cases}$$

The lemma is proved by analyzing the intensities of the Markov chain ξ_t, $t \geq 0$ transitions. Since all customers at any of the servers are either serviced or removed by negative customers one by one and the processes of servicing and negative customer arrival are Markov, the transition fundamental rates that correspond to the release of two or more servers are equal to zero. As such fundamental rates are set by the matrices $A_{i,j}$, $j \leq i-1$, $i \geq 0$, then, all these matrices are zero.

The matrices $A_{i,i-1}$, $i=\overline{1, N+R}$, set the transition fundamental rates that lead to the release of one of the occupied servers. The release of one of the servers with a non-zero probability can occur as a result of servicing final of a customer at this server (the matrix of corresponding transition fundamental rates of the chain ξ_t, $t \geq 0$, is set by the expression $I_{\overline{W}\overline{N}} \otimes S_0^{\oplus i}$, if the buffer is empty and by the expression $I_{\overline{W}\overline{V}} \otimes (S_0 \beta)^{\oplus N}$ if in the buffer there is at least one customer), or as a result of the removal of a customer under servicing by a negative customer (the matrix of corresponding fundamental rates has the form $(1/i) I_{\overline{W}} \otimes H_1 \otimes \hat{e}_M^{\oplus i}$, $i=\overline{1, N}$, if the buffer is empty or the form $(1/N) I_{\overline{W}} \otimes H_1 \otimes (\hat{e}_M \beta)^{\oplus N}$ if there are customers in the buffer).

The off-diagonal elements of the matrix $A_{i,i}$ define the transition fundamental rates of the Markov chain ξ_t, $t \geq 0$ that do not cause the number change i of the customers in the system. In case $i=0$, such transitions with a nonzero probability occur as a result of rapid changes of the process manager v_t, $t \geq 0$, of the input flow that is not accompanied by a customer generation or as a result of any rapid changes of the process manager η_t, $t \geq 0$, of the flow of negative customers.

The corresponding fundamental rates are the off-diagonal elements of the matrix $D_0 \oplus H(1)$. In case $1 \leq i \leq N+R-1$, transitions caused by the phase change of PH servicing at any of $\min\{i, N\}$ occupied servers are added but the transitions of the process η_t, $t \geq 0$ accompanied by a negative customer generation are not taken into consideration. In this case, transition fundamental rates of the process ξ_t, $t \geq 0$, are set by the off-diagonal elements of the matrices $D_0 \oplus H_0 \oplus S^{\oplus \min\{i, N\}}$, $1 \leq i \leq N+R-1$. With $i=N+R$, we also take into consideration the transitions specified by rapid changes of the process v_t, $t \geq 0$, which are accompanied by the generation of customers, as such transitions do not lead to the change of the number of customers in the system. In this case, transition fundamental rates of the chain under study ξ_t, $t \geq 0$ are set by off-diagonal elements of the matrix $D_0 \oplus H_0 \oplus S^{\oplus N}$. Finishing explanations, concerning the form of the matrices $A_{i,i}$, we note that off-diagonal elements of these matrices are the fundamental rates of the Markov chain output ξ_t, $t \geq 0$, from corresponding states, taken with the opposite signs. Further, the matrices $A_{i,i+k}$ *set the transition fundamental rates of the chain under* study that correspond to the increase in the number of customers in the system from i to $i+k$. When analyzing such transitions, the discipline of customer admission to the system is very important. In case of the partial admission discipline, transitions that lead to the increase in the number of customers in the system from i to $i+k$ occur with a non-zero probability as a result of the following events : 1) the arrival of a customer group of the size k if at the arrival moment the number of unoccupied positions in the system is larger than k (the fundamental rates of corresponding transitions are set by the matrix $D_k \otimes I_{\overline{V}M^i} \otimes \beta^{\otimes \min\{k, N-i\}}$ if there are $\min\{k, N-i\}$ unoccupied servers and by the matrix $D_k \otimes I_{\overline{V}M^N}$ if all servers are occupied); 2) the arrival of a customer group of the size larger than or equal to k, if at the arrival moment the number of unoccupied positions in the system is equal to k (the fundamental rates of corresponding transitions are set by the matrix $\sum_{l=k}^{\infty} D_l \otimes I_{\overline{V}M^i} \otimes \beta^{\otimes (N-i)}$ if there are i unoccupied servers and by the matrix $\sum_{l=k}^{\infty} D_l \otimes I_{\overline{V}M^N}$ if there are no unoccupied servers). Guiding by these arguments and taking into consideration that $\sum_{l=k}^{\infty} D_l = D(1) - \sum_{l=0}^{k-1} D_l$, we obtain the desired expression for the matrices $A_{i,i+k}$.

Let $p(i, v, \eta, m^{(1)}, \cdots, m^{(\min\{i, N\})})$, $v=\overline{0, W}$, $\eta=\overline{0, V}$, $m_t^{(r)}=\overline{1, M}$, $r=\overline{1, \min\{i, N\}}$, $i=\overline{0, N+R}$ be the steady state probabilities of the Markov chain ξ_t, $t \geq 0$ and $\vec{p}_i$ be the row vector of these probabilities corresponding to the state i of the first component of the chain, $i=\overline{0, N+R}$. The vectors $\vec{p}_i$, $i=\overline{0, N+R}$ satisfy Chapman-Kolmogorov's equations

$$\sum_{j=0}^{N+R} \vec{p}_i A_{i,j} = 0, \ j=\overline{0, N+R}, \ \sum_{j=0}^{N+R} \vec{p}_i \vec{e} = 1 \quad \cdots\cdots\cdots (1)$$

The rank of the system is equal to $\overline{W}\overline{V}((M^{N+1}-1/M-1)+RM^{N})$ and can be very large. For example, in case $\overline{W}=2$, $\overline{V}=2$, $M=2$, $N=4$, $R=4$ the rank is equal to 380, in case $N=5$ it is equal to 768, etc. Thus, the direct solution of system (1) can be time and resource consuming. Fortunately, the matrix A has special, upper-block hessenbergian structure. It allows to develop the stable effective procedure for solving the system. Such a procedure is presented in the following statement.

Theorem 1. *The stationary probability vectors* $\vec{p}_i$, $i=\overline{0, N+R}$ *are calculated as follows;*

$$\vec{p}_l=\vec{p}_0 F_l,\ l=\overline{1, N+R}$$

where the matrices F_l *are calculated recurrently :*

$$F_l=\left(\overline{A}_{0,l}+\sum_{i=1}^{l-1}F_i\overline{A}_{i,l}(-\overline{A}_{l,l})^{-1}\right),\ l=\overline{1, N+R-1}$$

$$F_{N+R}=\left(A_{0,N+R}+\sum_{i=1}^{N+R-1}F_i A_{i,N+R}(-A_{N+R,N+R})^{-1}\right)$$

The matrices $\overline{A}_{i,l}$ *are calculated from the backward recursion*

$$\overline{A}_{i,N+R}=A_{i,N+R}+\overline{A}_{i,N+R+1}G_{N+R},\ i=\overline{0, N+R}$$

$$\overline{A}_{i,l}=A_{i,l}+\overline{A}_{i,l+1}G_l,\ i=\overline{0, l},$$
$$l=N+R-1,\ N+R-2, \cdots, 0$$

The matrices G_i, $i=\overline{0, N+R-1}$ *are calculated from the backward recursion*

$$G_i=\left(-A_{i+1,i+1}-\sum_{l=1}^{N+R-i-1}A_{i+1,i+1+l}G_{i+1}G_{i+l-1}\cdots G_{i+1}\right)^{-1}$$
$$A_{i+1,i},\ i=N+R-1,\ N+R-2, \cdots 0$$

The vector $\vec{p}_0$ *is calculated as the unique solution of the system:*

$$\vec{p}_0\overline{A}_{0,0}=0,\ \vec{p}_0\left(\sum_{l=1}^{N+R}F_i\vec{e}+\vec{e}\right)=1$$

The proof of the theorem is analogous to the proof of theorem 1 in [10].

The given algorithm operates with the matrices of which size does not exceed the value $\overline{W}\overline{N}M^N$. The stability of the algorithm is explained by the fact that all matrices included into recursion are non-negative.

4. Performance Measures

Having the stationary distribution $\vec{p}_i$, $i=\overline{0, N+R}$ been calculated we can find a number of stationary performance characteristics of the considered system. The most important of them are the loss probability and the probability of breaking a service by negative customers. We call these probabilities as unsuccessful service probabilities and define them in the next statement.

Theorem 2. *A collection of unsuccessful service probabilities is evaluated as follows.*

- Probability that an arbitrary customer will be lost because of an absence of free places in the system

$$P_{loss}=\frac{1}{\lambda}\sum_{i=0}^{N+R}\vec{p}_i\sum_{k=N+R-i+1}^{\infty}kD_k\otimes I_{\overline{V}M^{\min\{i,N\}}}\vec{e} \quad (2)$$

- Probability that an arbitrary customer will be deleted by a negative customer

$$P_{break}=\frac{1}{\lambda}\sum_{i=1}^{N+R}\vec{p}_i\left(i_{\overline{W}}\otimes\frac{1}{\min\{i, N\}}H_1\otimes\vec{e}_M^{\,\hat{\oplus}\min\{i, N\}}\right)\vec{e} \quad (3)$$

- Probability that an arbitrary customer will be lost or deleted by a negative customer

$$P_{failure}=P_{loss}+P_{break} \quad (4)$$

- Probability that an arbitrary customer entering the system, will be deleted by a negative customer

$$\hat{P}_{break}=(1-P_{loss})^{-1}P_{break} \quad (5)$$

Proof. According to the formula of the total probability, the probability P_{loss} is calculated as

$$P_{loss}=1-\sum_{i=0}^{N+R-1}\sum_{k=1}^{\infty}P_k P_i^{(k)}R^{(i,k)} \quad (6)$$

where P_k is a probability that an arbitrary customer arrives in a batch consisting of k customers, $P_i^{(k)}$ is a probability to see i customers in the system at the epoch of the k size batch arrival, $R^{(i,k)}$is a probability that an arbitrary customer will not be loss conditional it arrives in a batch consisting of k customers and see i customers in the system.

It can be shown that

$$P_i^{(k)} = \frac{\vec{p}_i D_k \otimes I_{V_M^{\min\{i,N\}}} \vec{e}}{\vec{\theta} D_k \vec{e}}, \quad i=\overline{0, N+R-1}, \quad k \geq 1 \cdots\cdots (7)$$

$$P_k = \frac{k\vec{\theta} D_k \vec{e}}{\vec{\theta} \sum_{l=}^{\infty} l D_l \vec{e}} = k\frac{\vec{\theta} D_k \vec{e}}{\lambda}, \quad k \geq 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

$$R^{(i,k)} = \begin{cases} 1, k \leq N+R-i \\ \frac{N+R-i}{k}, k > N+R-i, \ i=\overline{0, N+R-1} \end{cases} \cdots\cdots (9)$$

By substituting (7)~(9) into (6) after some algebra we get (2). Expression (3) for P_{break} stems from the following formula [5] :

$$P_{break} = \frac{1}{\lambda} \sum_{i=1}^{N+R} \vec{p}_i \left(i_{\overline{W}} \otimes \frac{1}{\min\{1, N\}} H_1 \otimes (I - \tilde{I}_M)^{\oplus \min\{i,N\}} \right) \vec{e} \quad (10)$$

The sum in (10) is the average intensity of the flow of negative customers which meet the system non-empty and go to a busy server with service process being on phases $1, 2, \cdots, j$. Every of such negative customer deletes an ordinary customer in a service. So, the sum under consideration gives the intensity of ordinary customers which leave the system being deleted by negative customers. The ratio of this intensity to the average intensity of input flow gives the probability P_{break}. Using the relation $(I - \tilde{I}_M)^{\oplus \min\{i,N\}} \vec{e} = \overset{\triangle}{e}{}_M^{\oplus \min\{i,N\}} \vec{e}$ we reduce formula (10) to form (3). Formula (4) for the probability $P_{failure}$ is evident. And, finally, the expressions for conditional probability $\hat{P}_{break}$ is derived in trivial way by using the formula for conditional probability.

5. Numerical Examples

In our example, we present the results of four experiments. The goal of the first experiment is to analyze the influence of the *BMAP* intensity on the probabilities of unsuccessful service. In the second experiment we are interested in how the service rate impacts on the probabilities of the unsuccessful service. In the third experiment the dependence of unsuccessful service probabilities of the negative customer arrival rate is shown. And in the fourth experiment unsuccessful service probabilities for the case of unprotected service are compared with the corresponding probabilities for the case when one of two phases of a service is assumed to be protected of the effect of negative customers.

We suppose that the *PH* service process is characterized by the vector $\vec{\beta} = (1, 0)$ and the matrix $S = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$. It means that the service time has Erlangian distribution of order 2 with the service rate $\mu = 1$ and squared variation coefficient $(c_{var})^2 = 0.5$.

We consider *BMAP* having the fundamental rate $\lambda = 5$ and geometric distribution (with parameter $q = 0.8$) of a number of customers in a batch. The matrices D_k are defined as $D_k = Dq^{k-1}(1-q)/(1-q^{10})$, $k = \overline{1, 10}$ and then normalized to get the rate $\lambda = 5$,

$$D_0 = \begin{pmatrix} -25.5398 & 0.393329 & 0.361199 \\ 0.14515 & -2.2322 & 0.200007 \\ 0.295961 & 0.3874445 & -1.752618 \end{pmatrix}$$

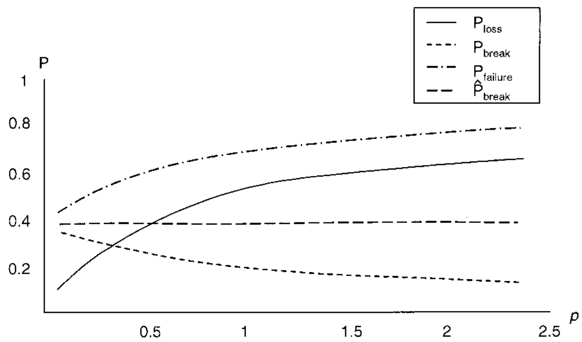
$$D = \begin{pmatrix} 24.24212 & 0.466868 & 0.076323 \\ 0.034097 & 1.666864 & 0.186082 \\ 0.009046 & 0.255481 & 0.804685 \end{pmatrix}$$

The *BMAP* has the squared variation coefficient $(c_{var})^2 = 4$ and the correlation coefficient $c_{cor} = 0.3$. The *MAP* of the negative customers is described by the matrices

$$H_0 = \begin{pmatrix} -15.732675 & 0.606178 & 0.592394 \\ 0.517817 & -2.289675 & 0.467885 \\ 0.597058 & 0.565264 & -1.959664 \end{pmatrix}$$

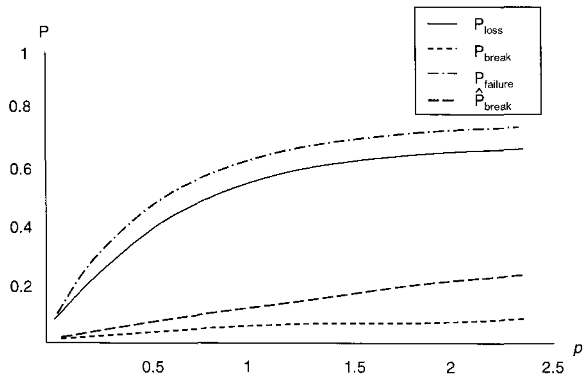
$$H_1 = \begin{pmatrix} 14.15020 & 0.302098 & 0.081805 \\ 0.107066 & 1.0.2280 & 0.164627 \\ 0.085830 & 0.197946 & 0.513566 \end{pmatrix}$$

The *MAP* has the rate $h = 5$, the squared variation coefficient $(c_{var})^2 = 2$ and the correlation coefficient $c_{cor} = 0.2$. We investigate the systems with $N = 4$ servers having $R = 4$ waiting positions. In the first experiment we vary the *BMAP* fundamental rate by multiplying the matrices D_k, $k = \overline{0, 10}$ by some positive number. In this way any desired value of the system load $\rho = \lambda/(N\mu)$ can be obtained. The dependence of unsuccessful service probabilities of the system load is shown on <Figure 1>.



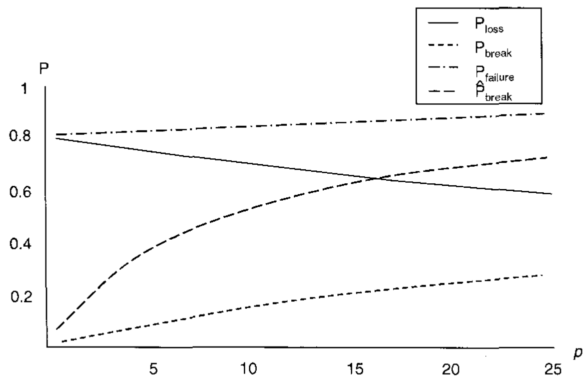
<Figure 1> Dependence of unsuccessful service probabilities of the system load (*BMAP* intensity is changed)

In the second experiment we vary the service rate by multiplying the matrix S by some positive number. In this case the dependence of unsuccessful service probabilities of the system load is shown on <Figure 2>.



<Figure 2> Dependence of unsuccessful service probabilities of the system load (*PH* intensity is changed)

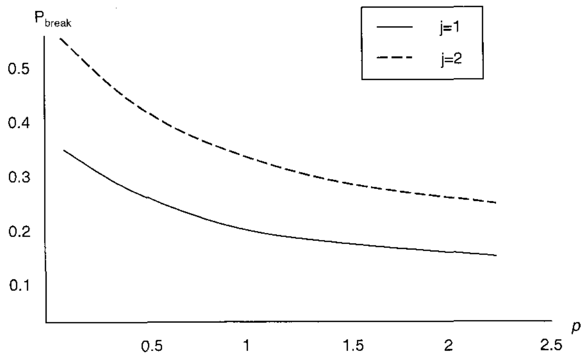
In the third experiment we vary the negative customers arrival rate by multiplying the matrices H_0, H_1 by some positive number. The dependence of unsuccessful service probabilities of the negative customer arrival rate h is shown on <Figure 3>.



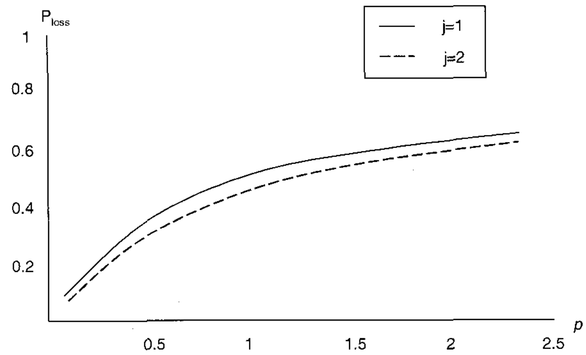
<Figure 3> Dependence of unsuccessful service probabilities of the negative *MAP* intensity

In the last experiment we compare unsuccessful service probabilities for the case $j=1$ (the second phase is protected of the effects of negative customers) and for the case $j=2$ (the service process has no protected phases).

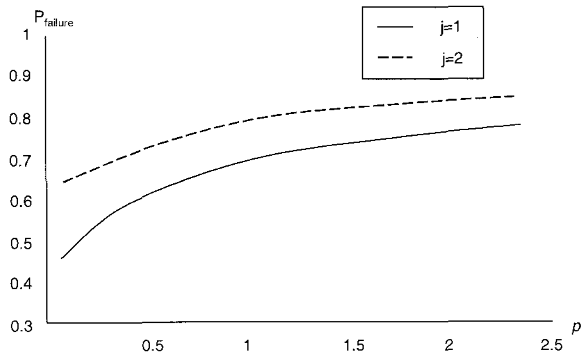
The behavior of these probabilities depending on the system load is shown on <Figure 4>~<Figure 6>. Here the variation of the system load is realized by means of the *BMAP* intensity variation.



<Figure 4> Dependence of Probability P_{break} of the system load in the cases $j=1$ and $j=2$



<Figure 5> Dependence of Probability P_{loss} of the system load in the cases $j=1$ and $j=2$



<Figure 6> Dependence of Probability $P_{failure}$ of the system load in the cases $j=1$ and $j=2$

6. Conclusion

We have considered the *BMAP/PH/N/R* G-queue with partial admission discipline and partial protection of service. We calculated the stationary distribution of the system and the collection of unsuccessful service probabilities including the probability to lose a customer due to the absence of free places in the system and the probability of service interruption because of the negative customer effect. The numerical examples illustrating the behavior of these probabilities depending on the system parameters are presented.

The results can be used for the estimation of the influence of such negative events as computer viruses, servers' breakdowns, and etc. on the servicing quality in real computer systems and telecommunications networks.

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