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# A Note on Robust Combinatorial Optimization Problem*

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**ABSTRACT**

In [1], robust combinatorial optimization problem is introduced, where a positive integer $\Gamma$ is used to control the degree of robustness. The proposed algorithm needs solutions of $n+1$ nominal problems. In this paper, we show that the number of problems needed reduces to $n+1-\Gamma$.

Keywords: Combinatorial Optimization, Robust Optimization

## 1. Introduction

In [1], Bertsimas and Sim proposed a robust combinatorial optimization problem where uncertainty on the objective coefficients is considered. For the details of more general robust discrete optimization and its applications, refer to [1, 2, 3]. Formally, the robust combinatorial optimization is defined as follows [1].

Let $X \subseteq \{0,1\}^n$ be a set of feasible solutions for a combinatorial optimization problem whose decision variables are binary. For each $j \in N = \{1, \cdots, n\}$, the cost of the item $j$ takes a value in $[c_j, c_j + d_j]$, where $d_j \geq 0$. Then the problem is formulated as follows:

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$$Z^* = \min\{c^T x + \max_{\{K | K \subseteq N, |K| \leq \Gamma\}} \sum_{j \in K} d_j x_j \mid x \in X\} \quad (1)$$

In (1), $\Gamma$ is a given integer which is used to control the robustness of the solution [1], where $1 \leq \Gamma \leq n$.

We assume that the indices are ordered in such that $d_1 \geq d_2 \geq \cdots \geq d_n$ and also define $d_{n+1} = 0$. Bertsimas and Sim [1] proved that the problem (1) can be solved by solving at most $n+1$ nominal problems which is summarized as follows:

$$Z^* = \min_{l=1,\ldots,n+1} G^l, \quad (2)$$

where for $l = 1, \cdots, n+1$:

$$G^l = \Gamma d_l + \min\{c^T x + \sum_{j \in N_l} (d_j - d_l) x_j \mid x \in X\}, \quad (3)$$

where $N_l = \{j \in N \mid j \leq l\}$. The proof is based on the duality of linear programming, see [1]. The result is very useful in that we can solve combinatorial optimization with data uncertainty on the objective coefficients by solving a few numbers of nominal problems. So, if an ordinary problem can be solved in a polynomial time, its robust version also can be solved in a polynomial time, for example, the robust shortest path problem.

However, one can easily note that if $\Gamma = n$, we need to solve only one problem. So we can conjecture the number of problems needed may be reduced and it should depend on $\Gamma$. In the next section, we show that this is actually true and the number of problems needed is $n - \Gamma + 1$.

## 2. Improvement of the Algorithm

In this section, we will use pure combinatorial arguments. Let $\mathrm{K} \subseteq 2^N$ be the set of feasible solutions for a given combinatorial optimization problem. For $K \in \mathrm{K}$, let us define $\Gamma(K)$ as follows:

$$\Gamma(K) = K, \text{ if } |K| \leq \Gamma$$

$$\Gamma(K) \subset K, \text{ such that } |\Gamma(K)| = \Gamma \text{ and } \max_{j \in \Gamma(K)} \{j\} < \min_{j \in K \setminus \Gamma(K)} \{j\}, \text{ if } |K| \geq \Gamma + 1 \quad (4)$$

Also define $v(K) = \sum_{j\in K} c_j + \sum_{j\in\Gamma(K)} d_j$. Then the problem (1) can be restated as follows:

$$Z^* = \min\{v(K) \mid K \in \mathrm{K}\} \tag{5}$$

For $l = 1, \cdots, n+1$, let us define

$$v_l(K) = \Gamma d_l + \sum_{j\in K} c_j + \sum_{j\in K_l} (d_j - d_l) \tag{6}$$

where $K_l = K \cap N_l = \{j \in K \mid j \le l\}$.

Now we can prove the main result. First, we need two lemmas.

**Lemma 1.** For $K \in \mathrm{K}$ with $|K| \le \Gamma$,

$$\begin{aligned} &v(K) = v_{n+1}(K) \quad \text{and} \\ &v(K) \le v_l(K), \text{ for all } 1 \le l \le n \end{aligned} \tag{7}$$

**Proof.** Since $|K| \le \Gamma$, we have $\Gamma(K) = K$. Hence $v(K) = v_{n+1}(K)$ holds. Consider a case with $l$, where $1 \le l \le n$. Then

$$\begin{aligned} v_l(K) &= \Gamma d_l + \sum_{j\in K} c_j + \sum_{j\in K_l} (d_j - d_l) \\ &= \sum_{j\in K} c_j + \sum_{j\in K_l} d_j + (\Gamma - |K_l|) d_l \ge v(K). \end{aligned}$$

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**Lemma 2.** For $K \in \mathrm{K}$ with $|K| \ge \Gamma + 1$,

$$\begin{aligned} &v(K) = v_{l^*}(K) \quad \text{and} \\ &v(K) \le v_l(K), \text{ for all } l \ne l^*, \end{aligned} \tag{8}$$

where $l^* = \max\{j \mid j \in \Gamma(K)\}$.

**Proof.** First note that when $l = l^*$, $K_{l^*} = \Gamma(K)$ and so

$$v_{l^*}(K) = \Gamma d_{l^*} + \sum_{j\in K} c_j + \sum_{j\in K_{l^*}} (d_j - d_{l^*}) = v(K).$$

Consider the case where $l < l^*$. In this case, $K_l \subset \Gamma(K)$, $|K_l| < \Gamma$ and so

$$\begin{aligned} v_l(K) &= \Gamma d_l + \sum_{j\in K} c_j + \sum_{j\in K_l} (d_j - d_l) \\ &= \sum_{j\in K} c_j + \sum_{j\in K_l} d_j + (\Gamma - |K_l|) d_l \ge v(K). \end{aligned}$$

Now let $l > l^*$. In this case, $\Gamma(K) \subseteq K_l$ and so

$$v_l(K) = \Gamma d_l + \sum_{j\in K} c_j + \sum_{j\in K_l} (d_j - d_l)$$
$$= \sum_{j\in K} c_j + \sum_{j\in \Gamma(K)} d_j + \sum_{j\in K_l \setminus \Gamma(K)} (d_j - d_l) \geq v(K).$$

■

By using the above two lemmas, we can prove the main result as presented below.

**Proposition 1.** The problem (5) (and also the problem (1)) can be solved by solving the $n - \Gamma + 1$ nominal problems:

$$Z^* = \min_{l=\Gamma,\cdots,n-1,n+1} G^l, \tag{9}$$

where for $l = \Gamma, \cdots, n-1, n+1$:

$$\begin{aligned} G^l &= \min\{v_l(K) = \Gamma d_l + \sum_{j\in K} c_j + \sum_{j\in K_l} (d_j - d_l) \mid K \in \mathrm{K}\} \\ &= \Gamma d_l + \min\{c^T x + \sum_{j\in N_l} (d_j - d_l) x_j \mid x \in X\} \end{aligned} \tag{10}$$

**Proof.** First, note that from lemmas 1 and 2,

$$Z^* = \min\{v(K) \mid K \in \mathrm{K}\} = \min_{l=1,\cdots,n+1} \{v_l(K) \mid K \in \mathrm{K}\}. \tag{11}$$

In Lemma 2, since $|K| \geq \Gamma + 1$, we should have $l^* \geq \Gamma$. So in (10), we can ignore the cases where $l < \Gamma$. Now if $l^* = n$ in Lemma 2, $\Gamma(K) = K_n = K$. So the case is covered by Lemma 1. Hence we have

$$Z^* = \min_{l=1,\cdots,n+1} \{v_l(K) \mid K \in \mathrm{K}\} = \min_{l=\Gamma,\cdots,n-1,n+1} \{v_l(K) \mid K \in \mathrm{K}\}.$$

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In [1], robust approximation algorithms were also presented which needed $n+1$ nominal approximation problems. By using the same method, we can also show that the number of problems can be reduced to $n - \Gamma + 1$.

Finally, we want to mention that the argument used in this note is different from that in [1]. Only combinatorial argument is used here and so it can be viewed as another approach to obtain the result.

## References


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