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KUCERA GROUP OF CIRCULAR UNITS IN FUNCTION FIELDS

Jaehyun Ahn and Hwanyup Jung

Abstract. Let $\mathbb{A} = \mathbb{F}_q[T]$ be the polynomial ring over a finite field $\mathbb{F}_q$ and $k = \mathbb{F}_q(T)$ its field of fractions. Let ℓ be a fixed prime divisor of $q-1$. Let J be a finite set of monic irreducible polynomials $P \in \mathbb{A}$ with $\deg P \equiv 0 \pmod{\ell}$. In this paper we define the group C_K of circular units in $K = k(\{\sqrt[\ell]{P} : P \in J\})$ in the sense of Kucera [4] and compute the index of C_K in the full unit group $\mathcal{O}_K^*$.

1. Introduction

Let $\mathbb{A} = \mathbb{F}_q[T]$ be the polynomial ring over a finite field $\mathbb{F}_q$ and $\mathbb{A}^+ = \{N \in \mathbb{A} : N \text{ is monic}\}$. Let $k = \mathbb{F}_q(T)$ be the field of fractions of $\mathbb{A}$. Let k^{ab} be a fixed algebraic closure of k. For any non-constant $N \in \mathbb{A}$, we denote by Λ_N the set of zeros of $\rho_N(X)$ in k^{ab}, where ρ denotes the Carlitz module of $\mathbb{A}$. Then Λ_N is isomorphic to $\mathbb{A}/N\mathbb{A}$ as $\mathbb{A}$-modules. The field $k(\Lambda_N)$ which is obtained from k by adjoining Λ_N is called the N-th cyclotomic function field over k. Let us fix a generator λ_N of Λ_N. The extension $k(\Lambda_N)/k$ is a finite abelian extension and its Galois group G_N is isomorphic to $(\mathbb{A}/N\mathbb{A})^*$. Explicitly there is an isomorphism $\sigma : (\mathbb{A}/N\mathbb{A})^* \xrightarrow{\sim} G_N, A \mapsto \sigma_A$, where $\sigma_A(\lambda_N) = \rho_A(\lambda_N)$. Under this isomorphism, $I_\infty = \{\sigma_c : c \in \mathbb{F}_q\}$ is the inertia group of ∞. The fixed field $k(\Lambda_N)^+$ of I_∞ is called the maximal real subfield of $k(\Lambda_N)$. For more details on the Carlitz module and the cyclotomic function fields, we refer to the Rosen's Book [6].

Let ℓ be a fixed prime divisor of $q-1$. For any irreducible $P \in \mathbb{A}^+$ with $\deg(P) \equiv 0 \pmod{\ell}$, it is well known that $k(\sqrt[\ell]{P})$ is contained in $k(\Lambda_P)^+$. Hence we may regard such monic irreducible polynomial P as an analog of the prime number p with $p \equiv 1 \pmod 4$. Let J be a finite set of monic irreducible polynomials P such that $\deg P \equiv 0 \pmod{\ell}$. Let $K = k(\{\sqrt[\ell]{P} : P \in J\})$ and $F = k(\sqrt[\ell]{D})$, where $D = \prod_{P \in J} P$. Let us denote by $\mathcal{O}_K$ (resp. $\mathcal{O}_F$) the integral closure of $\mathbb{A}$ in K (resp. in F). Let $\mathcal{O}_K^*$ and $\mathcal{O}_F^*$ be the group of units of $\mathcal{O}_K$

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and $\mathcal{O}_F$, respectively. In this paper, following Kucera's idea [4], we define the groups C_K and C_F of circular units of K and F, respectively. We also give a bases B_K and B_F of $C_K/\mathbb{F}_q^*$ and $C_F/\mathbb{F}_q^*$, respectively. By using these bases, we calculate the indices $[\mathcal{O}_K^* : C_K]$ and $[\mathcal{O}_F : C_F]$.

2. Kucera group of circular units

Fix a prime divisor ℓ of $q-1$. Let J be a finite set of monic irreducible polynomials $P \in \mathbb{A}$ with $\deg P \equiv 0 \pmod{\ell}$. It is well known ([1, Lemma 3.2]) that $k(\sqrt[\ell]{P})$ is contained in $k(\Lambda_P)^+$ for any $P \in J$. For any subset S of J let (by convention, an empty product is 1)

$$D_S = \prod_{P\in S} P, \quad \lambda_S = \lambda_{D_S}, \quad \Lambda_S = \Lambda_{D_S}, \quad K_S = k(\{\sqrt[\ell]{P} : P \in S\}).$$

For each $P \in J$, we denote by σ_P a fixed generator of $\mathrm{Gal}(K_J/K_{J\setminus\{P\}})$. Following Kucera ([4, Sect.2]), let us further define

$$\varepsilon_S = \begin{cases} 1 & \text{if} \quad S = \emptyset, \\ \dfrac{1}{\sqrt[\ell]{P}} N_{k(\Lambda_S)/K_S}(\lambda_S) & \text{if} \quad S = \{P\}, \\ N_{k(\Lambda_S)/K_S}(\lambda_S) & \text{if} \quad |S| > 1. \end{cases}$$

It is easy to see that ε_S is a unit of $\mathcal{O}_{K_S}$.

For any irreducible $P \in \mathbb{A}^+$ and $A \in \mathbb{A}$ with $P \nmid A$, we denote by $(A/P)_\ell$ the ℓ-th power residue symbol, i.e., $(A/P)_\ell$ is the unique element of $\mathbb{F}_q^*$ such that

$$A^{\frac{q^{\deg P}-1}{\ell}} \equiv (A/P)_\ell \pmod{P}.$$

Lemma 2.1. *Let S be any subset of J and $P \in S$. Then*

$$N_{K_S/K_{S\setminus\{P\}}}(\varepsilon_S) = \begin{cases} (-1)^{\ell-1} & \textit{if} \quad S = \{P\}, \\ (Q/P)_\ell \cdot \varepsilon_{\{Q\}}^{1-\mathrm{Frob}^{-1}(P,K_{\{Q\}})} & \textit{if} \quad S = \{P,Q\}, P \neq Q, \\ \varepsilon_{S\setminus\{P\}}^{1-\mathrm{Frob}^{-1}(P,K_{S\setminus\{P\}})} & \textit{if} \quad |S| > 2, \end{cases}$$

where $\mathrm{Frob}(P, K_{S\setminus\{P\}})$ *denotes the Frobenius automorphism of P in $K_{S\setminus\{P\}}$.*

Proof. At first, suppose that $S = \{P\}$. Since $N_{k(\sqrt[\ell]{P})/k}(\sqrt[\ell]{P}) = (-1)^{\ell-1}P$ and $N_{k(\Lambda_P)/k}(\lambda_P) = P$, we have

$$\begin{aligned} N_{K_S/K_{S\setminus\{P\}}}(\varepsilon_S) &= N_{k(\sqrt[\ell]{P})/k}(\sqrt[\ell]{P})^{-1} N_{k(\Lambda_P)/k}(\lambda_P) \\ &= (-1)^{\ell-1}P^{-1}P = (-1)^{\ell-1}. \end{aligned}$$

Let us suppose that $|S| > 1$. Then by Lemma 2.3 in [3],

$$\begin{aligned} N_{K_S/K_{S\setminus\{P\}}}(\varepsilon_S) &= N_{k(\Lambda_{S\setminus\{P\}})/K_{S\setminus\{P\}}}\big(N_{k(\Lambda_S)/k(\Lambda_{S\setminus\{P\}})}(\lambda_S)\big) \\ &= N_{k(\Lambda_{S\setminus\{P\}})/K_{S\setminus\{P\}}}\big(\lambda_{S\setminus\{P\}}^{1-\mathrm{Frob}^{-1}(P,k(\Lambda_{S\setminus\{P\}}))}\big) \\ &= N_{k(\Lambda_{S\setminus\{P\}})/K_{S\setminus\{P\}}}(\lambda_{S\setminus\{P\}})^{1-\mathrm{Frob}^{-1}(P,K_{S\setminus\{P\}})}. \end{aligned}$$

Thus the lemma is proved if $|S| > 2$. If $S = \{P, Q\}$, then we have

$$N_{k(\Lambda_Q)/k(\sqrt[\ell]{Q})}(\lambda_Q) = \sqrt[\ell]{Q} \cdot \varepsilon_{\{Q\}}.$$

Thus it suffices to show that $(\sqrt[\ell]{Q})^{1-\mathrm{Frob}^{-1}(P,k(\sqrt[\ell]{Q}))} = (Q/P)_\ell$. Write

$$(\sqrt[\ell]{Q})^{\mathrm{Frob}(P,k(\sqrt[\ell]{Q}))} = c \cdot \sqrt[\ell]{Q}$$

for some ℓ-th root of unity $c \in \mathbb{F}_q^*$. Then

$$c \equiv (\sqrt[\ell]{Q})^{q^{\deg P}-1} = Q^{\frac{q^{\deg P}-1}{\ell}} \pmod{\mathfrak{P}}$$

for any prime ideal $\mathfrak{P}$ of $\mathcal{O}_{k(\sqrt[\ell]{Q})}$ lying above P. Since c and $Q^{\frac{q^{\deg P}-1}{\ell}}$ are in k, we have $c \equiv Q^{\frac{q^{\deg P}-1}{\ell}} \pmod{P}$. Thus $c = (Q/P)_\ell$, and so

$$(\sqrt[\ell]{Q})^{1-\mathrm{Frob}^{-1}(P,k(\sqrt[\ell]{Q}))} = (Q/P)_\ell.$$

This completes the proof. □

Let $K = K_J$, $F = k(\sqrt[\ell]{D})$ with $D = D_J$ and $G_K = \mathrm{Gal}(K/k)$, $G_F = \mathrm{Gal}(F/k)$. Let C_K be the subgroup of $\mathcal{O}_K^*$ generated by $\mathbb{F}_q^*$ and by $\{\varepsilon_S^\sigma : S \subseteq J, \sigma \in G_K\}$. Let $\eta = N_{K/F}(\varepsilon_J) \in \mathcal{O}_F^*$ and let C_F be the subgroup of $\mathcal{O}_F^*$ generated by $\mathbb{F}_q^*$ and $\{\eta^\sigma : \sigma \in G_F\}$.

Define $B_K = \{\varepsilon_S^\sigma : \emptyset \neq S \subseteq J, \sigma = \prod_{P\in S} \sigma_P^{a_P}, 0 \leq a_P \leq \ell - 2\}$ and $B_F = \{\eta^{\tau^i} : 0 \leq i \leq \ell - 2\}$, where $G_F = \langle \tau \rangle$. Then we have

Lemma 2.2. *$\mathbb{F}_q^* \cup B_K$ and $\mathbb{F}_q^* \cup B_F$ are systems of generators of C_K and C_F, respectively.*

Proof. Let $\sigma = \prod_{P\in J} \sigma_P^{a_P} \in G_K$ with $0 \leq a_P \leq \ell - 1$. Note that if $P \notin S$, then $\varepsilon_S^{\sigma_P} = \varepsilon_S$. Thus $\varepsilon_S^\sigma = \varepsilon_S^{\prod_{P\in S} \sigma_P^{a_P}}$. Hence it suffices to show that $\varepsilon_S^{\sigma_P^{\ell-1}}$ is generated by $\mathbb{F}_q^* \cup B_K$ for any $P \in S \subseteq J$. It easily follows from Lemma 2.1 and induction on $|S|$. Since $N_{F/k}(\eta) = \prod_{i=0}^{\ell-1} \eta^{\tau^i} \in \mathbb{F}_q^*$, $\mathbb{F}_q^* \cup B_F$ is a system of generators of C_F. □

Let $\widehat{G}_K$ and $\widehat{G}_F$ be the character group of G_K and G_F with values in $\mathbb{C}^*$, respectively. For any $\chi \in \widehat{G}_K$, let $S_\chi = \{P \in J : \chi(\sigma_P) \neq 1\}$. Let ζ_ℓ be a fixed a primitive ℓ-th root of unity in $\mathbb{C}^*$. For any $P \in S_\chi$, the integer $a_{\chi,P}$ is defined by $\chi(\sigma_P) = \zeta_\ell^{a_{\chi,P}}$ with $1 \leq a_{\chi,P} \leq \ell - 1$. For any $\chi \in \widehat{G}_K$, we define $\varepsilon_\chi = \varepsilon_{S_\chi}^{\sigma_\chi}$ with $\sigma_\chi = \prod_{P\in S_\chi} \sigma_P^{(\ell-1-a_{\chi,P})}$. For any $1 \neq \psi \in \widehat{G}_F$, the integer a_ψ is defined by $\psi(\tau) = \zeta_\ell^{a_\psi}$ with $1 \leq a_\psi \leq \ell - 1$. Let $\eta_\psi = \eta^{\tau_\psi}$ with $\tau_\psi = \tau^{(\ell-1-a_\psi)}$. Then we have

Proposition 2.3. *$B_K = \{\varepsilon_\chi : 1 \neq \chi \in \widehat{G}_K\}$ and $B_F = \{\eta_\psi : 1 \neq \psi \in \widehat{G}_F\}$.*

Proof. For any $\varepsilon_S^\sigma \in B_K$ with $\emptyset \neq S \subseteq J, \sigma = \prod_{P\in S} \sigma_P^{a_P}$ and $0 \leq a_P \leq \ell - 2$, there exists a unique $\chi \in \widehat{G}_K$ such that $\chi(\sigma_P) = \zeta_\ell^{(\ell-1-a_P)}$ if $P \in S$ and $\chi(\sigma_P) = 1$ if $P \notin S$. Thus $\varepsilon_S^\sigma = \varepsilon_\chi$. Therefore we have $B_K = \{\varepsilon_\chi : 1 \neq$

$\chi \in \widehat{G}_K\}$. For any $1 \neq \psi \in \widehat{G}_F$, we have $0 \leq \ell - 1 - a_\psi \leq \ell - 2$. Thus $B_F = \{\eta_\psi : 1 \neq \psi \in \widehat{G}_F\}$. □

3. Computation of $[\mathcal{O}_K^* : C_K]$ and $[\mathcal{O}_F^* : C_F]$

In this subsection, we follow Kucera's argument in [4, Sect.3] to compute the unit-indices $[\mathcal{O}_K^* : C_K]$ and $[\mathcal{O}_F^* : C_F]$.

Any place of K lying above the place ∞ of k is called an *infinite place* of K. Fix an infinite place ∞_K of K. Then $\{\sigma^{-1}\infty_K : \sigma \in G_K\}$ is the set of all infinite places of K. Let R'_K be the regulator of the set B_K. Then

$$R'_K = \left|\det\left(\mathrm{ord}_{\infty_K}(\varepsilon_\chi^\sigma)\right)_{\substack{1\neq\sigma\in G_K \\ 1\neq\chi\in\widehat{G}_K}}\right| = \frac{1}{[K:k]}\left|\det\left(\mathrm{ord}_{\infty_K}(\varepsilon_\chi^\sigma)\right)_{\substack{\sigma\in G_K \\ \chi\in\widehat{G}_K}}\right|.$$

Here we note that $\varepsilon_{\chi_0}^\sigma = 1$ for trivial character χ_0. For any $\chi, \psi \in \widehat{G}_K$, let the elements $a_{\chi,\psi}$ be defined by the following matrix identity:

$$\left(a_{\chi,\psi}\right)_{\chi,\psi\in\widehat{G}_K} = \left(\chi(\sigma)\right)_{\substack{\chi\in\widehat{G}_K \\ \sigma\in G_K}} \left(\mathrm{ord}_{\infty_K}(\varepsilon_\psi^\sigma)\right)_{\substack{\sigma\in G_K \\ \psi\in\widehat{G}_K}}.$$

For any $\chi \in \widehat{G}_K$, let $\overline{\chi} \in \widehat{G}_K$ be the inverse of χ (i.e., $\overline{\chi}(\sigma) = \chi(\sigma)^{-1}$ for any $\sigma \in G_K$).

Lemma 3.1. *Let $\chi, \psi \in \widehat{G}_K$. Then*

$$a_{\chi,\psi} = \begin{cases} [K:k] & \text{if } \chi = 1 \text{ and } \psi = 1, \\ 0 & \text{if } \chi = 1 \text{ and } \psi \neq 1 \text{ or} \\ & \quad \chi \neq 1 \text{ and } \psi = 1, \\ 0 & \text{if } S_\chi \not\subseteq S_\psi, \\ (q-1)[K:K_{S_\chi}]\overline{\chi}(\sigma_\psi)L_k(0,\chi) & \text{if } S_\chi = S_\psi, \chi \neq 1, \psi \neq 1, \end{cases}$$

where $L_k(s,\chi)$ is the Artin L-function associated to χ.

Proof. We only prove the case that $\chi \neq 1, \psi \neq 1$ and $S_\chi = S_\psi$. Since the conductor of χ is D_{S_χ}, we can consider χ as a character of $\mathrm{Gal}(K_{S_\chi}/k)$. Note that

$$\sum_{\sigma\in G_K} \chi(\sigma)\cdot \mathrm{ord}_{\infty_K}((\sqrt[\ell]{P})^\sigma) = \frac{\mathrm{ord}_{\infty_K}(P)}{\ell}\sum_{\sigma\in G_K}\chi(\sigma) = 0$$

for any $P \in J$. Let $l_K : K^* \to \mathbb{Q}[G_K]$ be the logarithm map defined by $l_K(x) = \sum_{\sigma\in G_K}\mathrm{ord}_{\infty_K}(x^\sigma)\sigma^{-1}$ for any $x \in K^*$. Then we have

$$\begin{aligned} a_{\chi,\psi} &= \sum_{\sigma\in G_K}\chi(\sigma)\cdot\mathrm{ord}_{\infty_K}(N_{k(\Lambda_{S_\chi})/K_{S_\chi}}(\lambda_{S_\chi})^{\sigma_\psi\sigma}) \\ &= \overline{\chi}(\sigma_\psi)\sum_{\sigma\in G}\chi(\sigma)\cdot\mathrm{ord}_{\infty_K}(N_{k(\Lambda_{S_\chi})/K_{S_\chi}}(\lambda_{S_\chi})^{\sigma}) \\ &= \overline{\chi}(\sigma_\psi)\overline{\chi}(l_K(N_{k(\Lambda_{S_\chi})/K_{S_\chi}}(\lambda_{S_\chi}))) = (q-1)[K:K_{S_\chi}]\overline{\chi}(\sigma_\psi)L_k(0,\chi), \end{aligned}$$

where the last equality follows from the equation (3.4) in [2]. □

To calculate the determinant of $(a_{\chi,\psi})_{\chi,\psi\in\widehat{G}}$, we needs a simple lemma. Let A and B be $m\times m$ and $n\times n$ matrices, respectively. We define $A*B$ as the $mn\times mn$ matrix

$$\begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix} \text{ with } A=(a_{ij}).$$

The operation $*$ is associative and $\det(A*B)=\det(B*A)$. Moreover, we have the following which can be easily shown from the properties of determinant.

Lemma 3.2. $\det(A*B)=\det(A)^n\det(B)^m$.

Let $h(K)$ and $h(\mathcal{O}_K)$ be the divisor class number of K and the ideal class number of $\mathcal{O}_K$, respectively. We compute the absolute value of determinant of $(a_{\chi,\psi})_{\chi,\psi\in\widehat{G}_K}$.

Lemma 3.3. $\left|\det\left(a_{\chi,\psi}\right)_{\chi,\psi\in\widehat{G}_K}\right| = (q-1)^{([K:k]-1)}\ell^{\frac{|J|\ell^{|J|}}{2}}\cdot h(K)$.

Proof. Let us fix some linear ordering $\prec$ on $\widehat{G}_K$ such that

(i) $S_\chi\subseteq S_\psi\Rightarrow\chi\prec\psi$,
(ii) if $\chi\prec\psi\prec\varphi$ and $S_\chi=S_\varphi$, then $S_\chi=S_\psi=S_\varphi$,

for any $\chi,\psi,\varphi\in\widehat{G}_K$. From Lemma 3.1, we see that $(a_{\chi,\psi})_{\chi,\psi\in\widehat{G}_K}$ is a upper block-triangular matrix with respect to $\prec$ and

$$\begin{aligned} &\left|\det\left(a_{\chi,\psi}\right)_{\chi,\psi\in\widehat{G}_K}\right| \\ &= (q-1)^{[K:k]-1}[K:k]|\det(\mathbb{B}_K)|\prod_{1\neq\chi\in\widehat{G}_K}[K:K_{S_\chi}]\prod_{1\neq\chi\in\widehat{G}_K}L_k(0,\chi) \\ &= (q-1)^{[K:k]-1}|\det(\mathbb{B}_K)|h(K)\prod_{\chi\in\widehat{G}_K}[K:K_{S_\chi}], \end{aligned} \tag{3.1}$$

where $\mathbb{B}_K$ is the $(\ell^{|J|}-1)\times(\ell^{|J|}-1)$ block-diagonal matrix with blocks $\mathbb{B}_S=\left(\overline{\chi}(\sigma_\psi)\right)_{\substack{\chi,\psi\in\widehat{G}_K\\ S_\chi=S_\psi=S}}$ for each $\emptyset\neq S\subseteq J$. To compute the $\det(\mathbb{B}_K)$, it suffices to compute the determinant of each block $\mathbb{B}_S$. Consider a block $\mathbb{B}_S$ with $|S|=1$, say $S=\{P\}$. The block is of the form

$$\mathbb{B}_{\{P\}}=\begin{pmatrix} \vdots & \vdots & & \vdots \\ 1 & \chi(\sigma_P) & \cdots & \chi(\sigma_P^{\ell-2}) \\ \vdots & \vdots & & \vdots \end{pmatrix}_{\chi\in\widehat{G}_K,S_\chi=\{P\}}.$$

Enlarge this matrix to

$$\tilde{\mathbb{B}}_{\{P\}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ \chi(\sigma_P^{\ell-1}) & 1 & \chi(\sigma_P) & \cdots & \chi(\sigma_P^{\ell-2}) \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

by adding $(1 \cdots 1)$ to the first row, which corresponds to $\chi = 1$, and

$$^t(\cdots \chi(\sigma_P^{\ell-1}) \cdots)_\chi$$

to the first column. The matrix $\tilde{\mathbb{B}}_{\{P\}}$ can be considered as $\big(\chi(\sigma)\big)_{\substack{\sigma \in H \\ \chi \in \widehat{H}}}$ with $H = \langle \sigma_P \rangle$ by exchanging rows and columns if necessary. Thus, as in the proof of [4, Theorem 1], we have

$$\left|\det\left(\tilde{\mathbb{B}}_{\{P\}}\right)\right| = |H|^{\frac{|H|}{2}} = \ell^{\frac{\ell}{2}}.$$

On the other hand, by adding all the other columns to the first column in $\tilde{\mathbb{B}}_{\{P\}}$, we have

$$\det\left(\tilde{\mathbb{B}}_{\{P\}}\right) = \ell \cdot \det\left(\mathbb{B}_{\{P\}}\right).$$

Thus $\left|\det\left(\mathbb{B}_{\{P\}}\right)\right| = \ell^{\frac{\ell}{2}-1}$. For the block $\mathbb{B}_S$ with $|S| = d \geq 2$, say $S = \{P_1, P_2, \ldots, P_d\}$, $\mathbb{B}_S$ becomes $\mathbb{B}_{\{P_1\}} * \cdots * \mathbb{B}_{\{P_d\}}$ by exchanging rows and columns if necessary. Thus, by applying Lemma 3.2 repeatedly, we have

$$|\det(\mathbb{B}_S)| = \ell^{(\frac{\ell}{2}-1)(\ell-1)^{d-1}d}.$$

Therefore

$$\begin{aligned} |\det(\mathbb{B}_K)| &= \prod_{\emptyset \neq S \subseteq J} \ell^{(\frac{\ell}{2}-1)(\ell-1)^{|S|-1}|S|} = \prod_{d=1}^{|J|} \left(\ell^{(\frac{\ell}{2}-1)(\ell-1)^{d-1}d}\right)^{\binom{|J|}{d}} \\ &= \ell^{(\frac{\ell}{2}-1)\sum_{d=1}^{|J|}\binom{|J|}{d}(\ell-1)^{d-1}d} = \ell^{|J|(\frac{\ell}{2}-1)\ell^{|J|-1}}. \end{aligned} \tag{3.2}$$

For any $S \subseteq J$, there are $(\ell-1)^{|S|}$ different characters $\chi \in \widehat{G}_K$ with $S_\chi = S$. A simple computation shows that

$$\prod_{\chi \in \widehat{G}_K} [K : K_{S_\chi}] = \ell^{\sum_{\chi \in \widehat{G}_K}(|J|-|S_\chi|)} = \ell^{|J|\ell^{|J|-1}}. \tag{3.3}$$

By combining (3.1), (3.2) and (3.3), the lemma follows. □

When we speak about a basis of a group of units we always have in mind a basis of non-torsion part. Consider the set B_K of Lemma 2.2. Then

Theorem 3.4. *B_K is a basis of C_K. Moreover*

$$[\mathcal{O}_K^* : C_K] = \frac{(q-1)^{[K:k]-1}}{[K:k]} \cdot h(\mathcal{O}_K).$$

Proof. As in the proof of [4, Theorem 1], we have

$$\left|\det\left(\chi(\sigma)\right)_{\substack{\sigma\in G_K\\ \chi\in\widehat{G}_K}}\right| = |G_K|^{\frac{|G_K|}{2}} = \ell^{\frac{|J|\ell^{|J|}}{2}}.$$

Thus, by Lemma 3.3, we have

$$R'_K = \frac{1}{[K:k]}\cdot\frac{\left|\left(a_{\chi,\psi}\right)_{\chi,\psi\in\widehat{G}_K}\right|}{\left|\det\left(\chi(\sigma)\right)_{\sigma\in G_K,\chi\in\widehat{G}_K}\right|} = \frac{(q-1)^{[K:k]-1}}{[K:k]}\cdot h(K)\neq 0.$$

Hence R'_K is linearly independent, and so B_K is a basis of C_K. Moreover $[\mathcal{O}_K^*:C_K] = R'_K/R(\mathcal{O}_K)$, where $R(\mathcal{O}_K)$ is the regulator of $\mathcal{O}_K$, and $h(K) = h(\mathcal{O}_K)R(\mathcal{O}_K)$. Thus the theorem follows. □

The computation of the index $[\mathcal{O}_F^*:C_F]$ is much the same as the one of $[\mathcal{O}_K^*:C_K]$. We only state the result leaving the proof to the reader.

Theorem 3.5. *B_F is a basis of C_F. Moreover*

$$[\mathcal{O}_F^*:C_F] = \frac{(q-1)^{[F:k]-1}}{[F:k]}\cdot h(\mathcal{O}_F).$$

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JAEHYUN AHN
DEPARTMENT OF MATHEMATICS
CHUNGNAM NATIONAL UNIVERSITY
DAEJEON 305-764, KOREA
E-mail address: jhahn@cnu.ac.kr

HWANYUP JUNG
DEPARTMENT OF MATHEMATICS EDUCATION
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU 361-763, KOREA
E-mail address: hyjung@chungbuk.ac.kr