Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 241–246

NOTES ON A NON-ASSOCIATIVE ALGEBRA WITH EXPONENTIAL FUNCTIONS II

Seul Hee Choi

Abstract. For the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$, if $M = \{\partial\}$, then

$$Der_{non}(\mathbf{F}[e^{\pm x}]_M)$$

of the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$ is found in the paper [15]. For $M = \{\partial, \partial^2\}$, we find $Der_{non}(\mathbf{F}[e^{\pm x}]_M))$ of the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$ in this paper. We show that there is a non-associative algebra which is the direct sum of derivation invariant subspaces.

1. Preliminaries

Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the non-negative integers and the integers, respectively. Let A be an associative algebra and $M = \{ \delta \mid \delta \text{ is a mapping from } A \text{ to itself} \}$. The evaluation algebra $A_M = \{ a\delta \mid a \in A,\ \delta \in M\}$ with the obvious addition and the multiplication $*$ is defined as follows:

$$a_1\delta_1 * a_2\delta_2 = a_1\delta_1(a_2)\delta_2$$

for any $a_1\delta_1, a_2\delta_2 \in A_M$ [1], [2], [3], [10]. For A_M, if $M = \{id\}$, then the ring $A_M = A$ where id is the identity map of A. Note that $A_M = \langle A_M, +, *\rangle$ is not an associative ring generally. Using the commutator $[,]$ of A_M, we can define the semi-Lie ring $A_{M[,]} = \langle A_M, +, [,]\rangle$ [1]. If the Jacobi identity holds in $A_{M[,]}$, then $A_{M[,]}$ is a Lie ring [13]. Generally, A_M is not a Lie ring, because of the Jacobi identity. Let $\mathbf{F}[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_n}]$ be a ring in the formal power series ring $\mathbf{F}[[x_1, x_2, \ldots, x_n]]$ [5], [6]. If we take the subalgebra $\mathbf{F}[e^{\pm x}]$ in $\mathbf{F}[[x_1, x_2, \ldots, x_n]]$ and the map $M = \{\partial, \partial^2\}$, the we have the simple evaluation algebra $\mathbf{F}[e^{\pm x}]_M$ [4], [8], [9], [11], [12]. We know that $\mathbf{F}[e^{\pm x}]_{\{\partial,\partial^2\}} = \mathbf{F}[e^{\pm x}]_{\{\partial\}} \oplus \mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ such that $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ and $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ are simple [1], [2], [15].

Received April 5, 2006.
2000 *Mathematics Subject Classification.* 17B40, 17B56.
Key words and phrases. simple, Witt algebra, graded, radical homogeneous equivalent component, order, derivation invariant.

2. Derivations of $\mathbf{F}[e^{\pm x}]_M$

From now on, M denotes the set $\{\partial, \partial^2\}$. For any element $e^{ax}\partial^u$ of $\mathbf{F}[e^{\pm x}]_M$, we define degree $deg(e^{ax}\partial^u)$ as a where $u \in \{1,2\}$. Through out the paper, using the degree, for any element l in $\mathbf{F}[e^{\pm x}]_M$, the element l can be written as follows:

$$\begin{aligned} l &= C(a_1,1)e^{a_1x}\partial + C(a_2,1)e^{a_2x}\partial + \cdots + C(a_r,1)e^{a_rx}\partial \\ &\quad + C(b_1,2)e^{b_1x}\partial^2 + C(b_2,2)e^{b_2x}\partial^2 + \cdots + C(b_s,2)e^{b_sx}\partial^2 \end{aligned}$$

such that $a_1 > \cdots > a_r$ and $b_1 > \cdots > b_s$ with appropriate scalars. Thus we can define the order of elements of $\mathbf{F}[e^{\pm x}]_M$ obviously.

Note 1. Let $I = \{1,2\}$. For any basis element $e^{mx}\partial^i$, $i \in I$, of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$, if we define $\mathbf{F}$-additive linear map D_c of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ as follows:

$$D_c(e^{mx}\partial^i) = cme^{mx}\partial^i \tag{1}$$

then D_c can be linearly extended to a derivation of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ where $c \in \mathbf{F}$.

Lemma 2.1. *Let D be a derivation of $\mathbf{F}[e^{\pm x}]_M$. $D(\partial) = 0$ if and only if $D(\partial^2) = 0$.*

Proof. Let D be the derivation of $\mathbf{F}[e^{\pm x}]_M$. Let us assume that $D(\partial) = 0$. Since ∂ is the left (multiplicative) identity of $e^x\partial$, we have that $\partial * D(e^x\partial) = D(e^x\partial)$. Assume that $D(e^x\partial) = \sum_a c_{a,1}e^{ax}\partial + \sum_b c_{b,2}e^{bx}\partial^2$, where $c_{a,1}, c_{b,2} \in \mathbf{F}$. Since ∂ is the left (multiplicative) identity of $e^x\partial$, we prove that $D(e^x\partial) = c_{1,1}e^x\partial + c_{1,2}e^x\partial^2$. Since $D(e^{-x}\partial * e^x\partial) = 0$, we have that $D(e^{-x}\partial) = -c_{1,1}e^{-x}\partial - c_{1,2}e^{-x}\partial^2$. Since ∂^2 is the left (multiplicative) identity of $e^{-x}\partial$, we have that $D(\partial^2) = 0$.

Conversely, let us assume that $D(\partial^2) = 0$. Similarly as $D(\partial) = 0$ case, we can prove that if $D(\partial^2) = 0$, then $D(\partial) = 0$ easily. This completes the proof of the lemma. □

Lemma 2.2. *For any $D \in Der_{non}(\mathbf{F}[e^{\pm x}]_M)$, D is the derivation D_c in Note 1 for appropriate scalar.*

Proof. Let D be a derivation of $\mathbf{F}[e^{\pm x}]_M$. Since ∂ is in the right annihilator of ∂^2, we have that $D(\partial) = c_1\partial + c_2\partial^2$ for $c_1, c_2 \in \mathbf{F}$. Similarly, we have that $D(\partial^2) = c_3\partial + c_4\partial^2$ for $c_3, c_4 \in \mathbf{F}$. Since ∂ is a left (multiplicative) identity of $e^x\partial$, we put $D(e^x\partial)$ as follows:

$$D(e^x\partial) = C(a_1,1)e^{a_1x}\partial + \#_1 + C(b_1,2)e^{b_1x}\partial^2 + \#_2, \tag{2}$$

where $e^{a_1x}\partial$ and $e^{b_1x}\partial^2$ are appropriate maximal elements of $D(e^x\partial)$, $\#_1$ and $\#_2$ are the sums of its remaining terms with non-zero coefficients respectively.

Since ∂ is a left (multiplicative) identity of $e^x\partial$, we have that

$$\begin{aligned} & c_1e^x\partial + c_2e^x\partial + a_1C(a_1,1)e^{a_1x}\partial + \#_3 + b_1C(b_1,2)e^{b_1x}\partial^2 + \#_4 \\ (3) \qquad = \; & C(a_1,1)e^{a_1x}\partial + \#_1 + C(b_1,2)e^{b_1x}\partial^2 + \#_2, \end{aligned}$$

where $\#_3$ and $\#_4$ are the sums of its remaining terms of the left side. This implies that $b_1 = 1$ and we have the following two cases, $a_1 \neq 1$ and $a_1 = 1$. If $a_1 \neq 1$, then the equality (3) does not hold. Thus we have that $a_1 = b_1 = 1$ and $c_1 = -c_2$, i.e., $D(\partial) = c_1\partial - c_1\partial^2$. Similarly, by considering the minimal elements of $D(e^x\partial)$ with ∂ and ∂^2, we can prove that $D(e^x\partial) = C(1,1)e^x\partial + C(1,2)e^x\partial^2$ and $D(\partial^2) = c_3\partial - c_3\partial^2$. By $D(e^x\partial * e^{-x}\partial) = -D(\partial) = c_1\partial^2 - c_1\partial$, we have that

$$\{C(1,1)e^x\partial + C(1,2)e^x\partial^2\} * e^{-x}\partial + e^x\partial * D(e^{-x}\partial) = c_1\partial^2 - c_1\partial.$$

This implies that $e^x\partial * D(e^{-x}\partial) = c_1\partial^2 - c_1\partial + C(1,1)\partial - C(1,2)\partial$. So we have that

$$D(e^{-x}\partial) = -c_1e^{-x}\partial^2 + c_1e^{-x}\partial - C(1,1)e^{-x}\partial + C(1,2)e^{-x}\partial. \tag{4}$$

By (4), we can prove that

$$\begin{aligned} & D(e^{-x}\partial) * e^x\partial \\ = \; & \{-c_1e^{-x}\partial^2 + c_1e^{-x}\partial - C(1,1)e^{-x}\partial + C(1,2)e^{-x}\partial\} * e^x\partial \\ (5) \qquad = \; & -c_1\partial + c_1\partial - C(1,1)\partial + C(1,2)\partial \\ = \; & -C(1,1)\partial + C(1,2)\partial. \end{aligned}$$

On the other hand, by $D(e^{-x}\partial * e^x\partial) = D(\partial) = c_1\partial - c_1\partial^2$, we have that $D(e^{-x}\partial) * e^x\partial + e^{-x}\partial * \{C(1,1)e^x\partial + C(1,2)e^x\partial^2\} = c_1\partial - c_1\partial^2$. This implies that

$$D(e^{-x}\partial) * e^x\partial = c_1\partial - c_1\partial^2 - C(1,1)e^x\partial - C(1,2)e^x\partial^2. \tag{6}$$

By comparing (5) with (6), we have that $c_1 = 0$ and $C(1,2) = 0$. This implies that

$$D(\partial) = 0, \tag{7}$$

$$D(e^x\partial) = C(1,1)e^x\partial, \tag{8}$$

$$D(e^{-x}\partial) = -C(1,1)e^{-x}\partial. \tag{9}$$

Thus by induction $p \in \mathbf{Z}$ of $e^{px}\partial$, we can prove that $D(e^{px}\partial) = pC(1,1)e^{px}\partial$. Since ∂ is the left multiplicative identity of $e^x\partial^2$, we have that $D(e^x\partial^2) = c_5e^x\partial + c_6e^x\partial^2$ for $c_5, c_6 \in \mathbf{F}$. By $D(e^{-x}\partial * e^x\partial^2) = D(\partial^2)$, we prove that $c_3 = c_5$ and $c_6 = C(1,1) - c_3$, i.e., $D(e^x\partial^2) = c_3e^x\partial + (C(1,1) - c_3)e^x\partial^2$. By (9), we have that $D(e^{-2x}\partial) = -2C(1,1)e^{-2x}\partial$. By $D(e^{-2x}\partial * e^x\partial^2) = D(e^{-x}\partial^2)$, we also have that $D(e^{-x}\partial^2) = -C(1,1)e^{-x}\partial^2$. By $D(e^x\partial^2 * e^{-x}\partial^2) = D(\partial^2)$, we have that $c_3 = 0$. This implies that

$$D(\partial^2) = 0, \tag{10}$$

$$D(e^x\partial^2) = C(1,1)e^x\partial^2, \tag{11}$$

$$D(e^{-x}\partial^2) = -C(1,1)e^{-x}\partial^2. \tag{12}$$

Thus by induction $p \in \mathbf{Z}$ of $e^{px}\partial^2$, we can prove that $D(e^{px}\partial^2) = pC(1,1)e^{px}\partial^2$. Thus D can be linearly extended to the derivation D_c which is defined in Note 1 by putting $c = C(1,1)$. Therefore we have proven the lemma. □

Theorem 2.1. *$Der_{non}(\mathbf{F}[e^{\pm x}]_M)$ is generated by D_c in Note 1.*

Proof. The proof of the theorem is straightforward by Lemma 2.2 and Note 1, so omit its details. □

Corollary 2.1. *$Dim(Der_{non}(\mathbf{F}[e^{\pm x}]_M)) = 1$.*

Proof. The proof of the corollary is straightforward by Theorem 2.1. □

Corollary 2.2. *For any D in $Der_{non}(\mathbf{F}[e^{\pm x}]_M)$, $D(\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}) \subset \mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$, $1 \leq u \leq 2$, i.e., $\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$ is derivation invariant.*

Proof. The proof of the corollary is straightforward by Theorem 2.1. □

Proposition 2.1. *There is no isomorphism from $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ to $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ as non-associative algebras.*

Proof. Let us assume that there is an isomorphism θ from $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ to $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ as non-associative algebras. We can prove that $\theta(\partial) = c\partial^2$ for $c \in \mathbf{F}^\bullet$ easily. Since ∂ is the left identity of $e^x\partial$, we have that $c\partial^2 * \theta(e^x\partial) = \theta(e^x\partial)$. Since $\mathbf{F}$ is a field of characteristic zero, we have that $c = 1$ and either $\theta(e^x\partial) = d_1e^x\partial^2$ or $\theta(e^x\partial) = d_2e^{-x}\partial^2$ for $d_1, d_2 \in \mathbf{F}^\bullet$. Let us assume that $\theta(e^x\partial) = d_1e^x\partial^2$. By $\theta(e^{-x}\partial * e^x\partial) = \partial^2$, we also have that $\theta(e^{-x}\partial) = d_1^{-1}e^{-x}\partial^2$. From the inequality $\theta(e^x\partial * e^{-x}\partial) \neq -\partial^2$ we can derive a contradiction. In the same way, if we assume that $\theta(e^x\partial) = d_2e^{-x}\partial^2$, we can also derive a contradiction. These contradictions show that there is no algebra isomorphism from $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ to $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$. We have proven the proposition. □

Corollary 2.3.

$$Hom_{non}(\mathbf{F}[e^{\pm x}]_{\{\partial\}}, \mathbf{F}[e^{\pm x}]_{\{\partial^2\}}) = \{0\},$$

where $Hom_{non}(\mathbf{F}[e^{\pm x}]_{\{\partial\}}, \mathbf{F}[e^{\pm x}]_{\{\partial^2\}})$ is the set of all non-associative algebra homomorphisms from $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ to $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ and 0 is the zero map.

Proof. Since $\mathbf{F}[e^{\pm x}]_{\{\partial\}}$ and $\mathbf{F}[e^{\pm x}]_{\{\partial^2\}}$ are simple, the proof of the corollary is straightforward by Schur's lemma and Proposition 2.1. □

Corollary 2.4. *There is no automorphism θ of $\mathbf{F}[e^{\pm x}]_M$ such that $\theta(\partial) = c\partial^2$ for any $c \in \mathbf{F}^\bullet$.*

Proof. The proof of the corollary is straightforward by Proposition 2.1 [7]. □

Note 2. If $M = \{\partial, \ldots, \partial^n\}$, then we have the following decomposition of subalgebras (not just as vector subspaces) of $\mathbf{F}[e^{\pm x}]_M$

$$\mathbf{F}[e^{\pm x}]_M = \bigoplus_{1 \leq u \leq n} \mathbf{F}[e^{\pm x}]_{\{\partial^u\}},$$

where $\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$, $1 \leq u \leq n$, are simple [1], [2], [15]. By Theorem 2.1, we know that each subalgebra $\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$, $1 \leq u \leq n$, of $\mathbf{F}[e^{\pm x}]_M$ is derivation invariant.

Proposition 2.2. *For $n > 1$, the matrix ring $M_n(\mathbf{F})$ is not embedded in the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$.*

Proof. The proof of the proposition is standard, so let us omit it. □

Question. For any $D \in Der_{non}(\mathbf{F}[e^{\pm x}]_M)$ (resp. $\theta \in Aut_{non}(\mathbf{F}[e^{\pm x}]_M)$), does $D(\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}) \subset \mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$ (resp. $\theta(\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}) \subset \mathbf{F}[e^{\pm x}]_{\{\partial^u\}}$) hold where $1 \leq u \leq n$?

References

[1] M. H. Ahmadi, K. B. Nam, and J. Pakinathan, *Lie admissible non-associative algebras*, Algebra Colloq. **12** (2005), no. 1, 113–120.

[2] S. H. Choi and K. B. Nam, *The derivation of a restricted Weyl type non-associative algebra*, Hadronic J. **28** (2005), no. 3, 287–295.

[3] ———, *Derivation of symmetric non-associative algebra I*, Algebras Groups Geom. **22** (2005), no. 3, 341–352.

[4] ———, *Derivations of a restricted Weyl type algebra I*, Accepted, Rocky Mountain Journal of Mathematics, 2005.

[5] T. Ikeda, N. Kawamoto, and K. B. Nam, *A class of simple subalgebras of Generalized Witt algebras*, Proceedings of the International Conference in 1998 at Pusan (Eds. A. C. Kim), Walter de Gruyter Gmbh Co. KG (2000), 189–202.

[6] V. G. Kac, *Description of filtered Lie algebra with which graded Lie algebras of Cartan type are associated*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 800–834.

[7] N. Kawamoto, A. Mitsukawa, K. B. Nam, and M. O. Wang, *The automorphisms of generalized Witt type Lie algebras*, J. Lie Theory **13** (2003), no. 2, 573–578.

[8] I. Kaplansky, *The Virasoro algebra*, Comm. Math. Phys. **86** (1982), no. 1, 49–54.

[9] K. B. Nam, *On some non-associative algebras using additive groups*, Southeast Asian Bull. Math. **27** (2003), no. 3, 493–500.

[10] K. B. Nam and S. H. Choi, *On the derivations of non-associative Weyl-type algebras*, Appear, Southeast Asian Bull. Math., 2005.

[11] K. B. Nam, Y. Kim, and M. O. Wang, *Weyl-type non-associative algebras I*, Advances in algebra towards millenninum problems, SAS Publishers (2005), 147–155.

[12] K. B. Nam and M. O. Wang, *Notes on some non-associative algebras*, J. Appl. Algebra Discrete Struct. **1** (2003), no. 3, 159–164.

[13] A. N. Rudakov, *Groups of automorphisms of infinite-dimensional simple Lie algebras*, Math. USSR-Izv. **3** (1969), 707–722.

[14] R. D. Schafer, *Introduction to nonassociative algebras*, Dover, 1995.

[15] M. O. Wang, J. G. Hwang, and K. S. Lee, *Some results on non-associative algebras*, Bull. Korean Math. Soc., Accepted, 2006.

Department of Mathematics
University of Jeonju
Chon-ju 560-759, Korea
E-mail address: chois@www.jj.ac.kr