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ON NUMBER OF WAYS TO SHELL THE k-DIMENSIONAL TREES

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Abstract. Which spheres are shellable?[2]. We present one of them which is the k-tree with n-labeled vertices. We found that the number of ways to shell the k-dimensional trees on n-labeled vertices is

$$\frac{n!}{(k+1)!}(nk-k^2-k+1)!_k.$$

1. Introduction

A tree is a connected graph that has no cycles. A tree also can be defined as a single vertex is a tree, and a tree with $n+1$ vertices is any graph obtained by joining a new vertex to any one vertex in a tree with n vertices. A k-dimensional tree(k-tree for short) can be defined analogously starting with a complete graph of order k, a k-tree with $n+1$ vertices is obtained by joining a new vertex to any k vertices already joined in a k-tree with n vertices. Harary [3] defined a pure n-complex as a finite n-dimensional simplicial complex in which every k-simplex with $k < n$ is contained in an n-simplex. A 2-tree is a simply connected, acyclic 2-complex [4, 5]. Also a k-tree can be considered as a pure k-complex. An r-cell(called a simplex) of a k-tree is a complete subgraph of $r+1$ vertices. Thus, 0-cells, 1-cells, and 2-cells are vertices, edges, and K_3(which is called a 2-dimensional edge including inside). A $(k-1)$-cell is a k-tree, and a k-tree with $n+1$ vertices is obtained when a k-cell is added to a k-tree with n vertices and has precisely a $(k-1)$-cell in common with it. A pure(no isolated vertices) finite simplicial complex is said to be shellable if its maximal cells can be ordered F_1, $F_2, \ldots, F_n$ in such a way that $F_k \bigcap (\bigcup_{i=1}^{k-1} F_i)$ is a nonempty union of maximal proper cell of F_k for $k = 2, 3, \ldots, n$. Counting the number of such orderings of maximal cells is called the number of ways to shell a pure finite simplicial complex. Note that, by the construction of k-trees, it is clearly shellable. Note that Beineke and Pippert [1] found the number of k-trees with n vertices which

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is

$$\binom{n}{k}\{k(n-k)+1\}^{n-k-2}$$

which reduces to Cayley's formula n^{n-2} when $k = 1$. In this paper we deal with the number of ways to shell the k-trees with n-labeled vertices. It turns out to be

$$\frac{n!}{(k+1)!}(nk-k^2-k+1)!_k.$$

In case of $k = 1$, this sequence has the absolute catalogue number A010796 of J. A. Sloane [6].

2. Recursive k-trees

A tree T_n rooted on the vertex labeled 1 with n vertices labeled $1, 2, \ldots, n$ is a *recursive tree* if $n = 1$ or if $n \geq 2$ and T_n is obtained by joining the n^{th} vertex to one vertex of some recursive tree T_{n-1}. There are $(n-1)!$ recursive trees T_n and a tree with n labeled vertices is recursive if and only if the labels of the vertices in the path from the 1st vertex to the k^{th} vertex of the tree form an increasing subsequence of $\{1, 2, \ldots, k\}$ for $k = 1, 2, \ldots, n$. A recursive k-tree T_n^k can be analogously defined as follows.

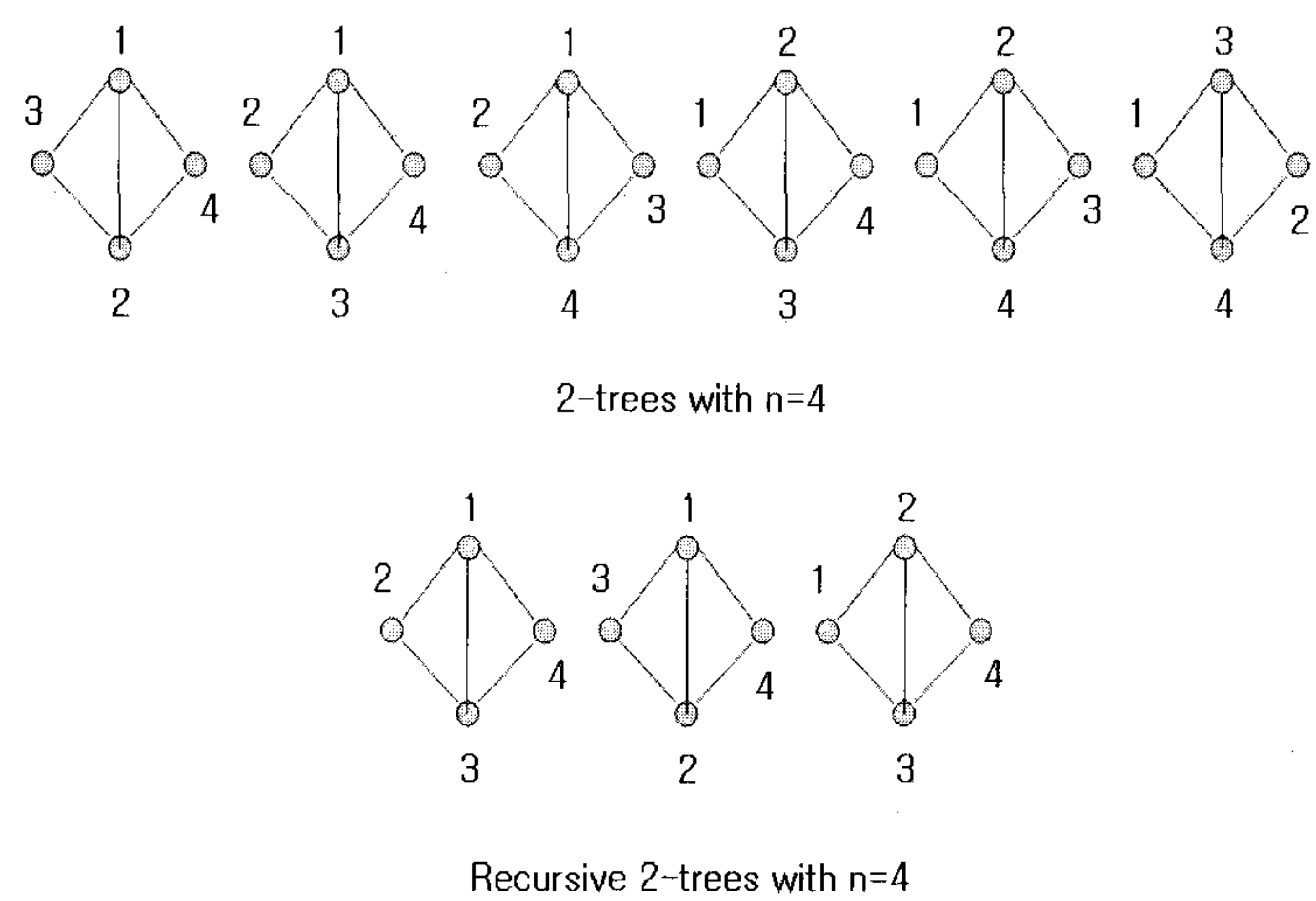

FIGURE 1. k-trees and recursive k-trees with $n = 4$.

Definition 1. A *recursive k-tree* T_n^k rooted on the complete graph of order k with n labeled vertices is a recursive k-tree if $n = k$ or if $n \geq k+1$ and T_n^k is obtained by joining the n^{th} vertex to each of k vertices on a $(k-1)$-cell of some recursive tree T_{n-1}^k with labeling in this order.

Figure 1 shows that 2-trees and recursive 2-trees with $n = 4$. The number of recursive k-trees T_n^k with n vertices will be computed in the Lemma 2. If n is a non-negative integer, then define its k^{th} factorial as

$$n!_k := n(n-k)(n-2k)\cdots$$

where $n!_k = 0$ when $n < k$ and $(n - ik) = 1$ if $(n - ik) \leq 1$ for some i when $n \geq k$. The following lemma is stated in [1].

Lemma 1. *The number of r-cells in a k-tree with n-vertices is*

$$\binom{k}{r+1} + (n-k)\binom{k}{r}$$

Proof. Note that $0 \leq r \leq k$, and r-cell is complete graph of order $r+1$. Since a recursive k-tree starting with a complete graph of order k, there are $\binom{k}{r+1}$ r-cells in the K_k, from which the k-tree started. When we add each vertex to a k-tree, this new vertex is joined to any k vertices already joined in a k-tree. Hence we can have new $\binom{k}{r}$ r-cells for each $(n-k)$ vertices. □

By the previous Lemma, the number of $(k-1)$-cells in a k-tree with n vertices is

$$1 + (n-k)\binom{k}{k-1} = nk - k^2 + 1$$

that will be useful in the next Lemma.

Lemma 2. *The number of recursive k-trees T_n^k with n vertices is*

$$((n-1)k - k^2 + 1)!_k = (nk - k^2 - k + 1)!_k.$$

Proof. Let us prove inductively on the number of vertices. When $n = 1$, then $k = 1$ so the number of recursive trees is $0! = 1$. Let us assume that there are $((n-2)k - k^2 + 1)!_k$ number of recursive k-trees with $n-1$ vertices. Then by Lemma 1, the number of $(k-1)$-cells(a complete subgraph of order k) in the k-tree with $(n-1)$ vertices is

$$(n-1)k - k^2 + 1.$$

This means there are $(n-1)k - k^2 + 1$ ways of selecting $(k-1)$-cells, when we add n^{th} vertex to the recursive k-tree. That is, for each recursive k-tree T_{n-1}^k, we produce $(n-1)k - k^2 + 1$ number of recursive k-tree T_n^k. Hence, by the definition of k^{th} factorial, the number of recursive k-tree with n vertices is

$$\begin{aligned} &\{((n-2)k - k^2 + 1)!_k\} \cdot \{(n-1)k - k^2 + 1\} \\ &= ((n-1)k - k^2 + 1)!_k. \end{aligned}$$

□

3. Main Theorem

Now before stating our main result of this paper we need the *generalized ordering recursive k-tree.* For a usual recursive k-tree, we fixed the order of positive integer $[n] := \{1, 2, \ldots, n\}$ for the labeling. Now let's consider the all possible ordering of $[n]$. Obviously there are $n!$ ways to order the set $[n]$. Since generalized ordering recursive k-trees are rooted on the $(k-1)$-cell, it produces exactly same generalized ordering recursive k-trees with this root. And it gives us $k!$. Hence the total number of generalized ordering recursive k-tree is

$$\frac{n!}{k!}(nk - k^2 - k + 1)!_k. \tag{3.1}$$

Theorem 1. *The number of ways to shell k-trees T of order n is* 1 *when* $n = k$, *and*

$$\frac{n!}{(k+1)!}(nk - k^2 - k + 1)!_k, \tag{3.2}$$

when $n > k$.

Proof. Let a labeled k-tree of order n be given. We shell this k-tree starting from any k-cell F_1 containing $k+1$ vertices labeled by $i_1, i_2, \ldots, i_{k+1}$ and choose another k-cell F_2 from the remaining cells of T such that $F_2 \bigcap F_1$ is $(k-1)$-cell, and so on. During this shelling we can get a sequence of vertices, say $i_1, i_2, \ldots, i_n$ where i_j $(1 \leq j \leq n)$ are in $[n]$. Now with this order of vertices if we make generalized ordering recursive k-trees rooted on the k vertices with $i_1, i_2, \ldots, i_{k+1}$ vertices, then there are only $\binom{k+1}{k} = k+1$ ways of selecting this root. Now by joining the $(k+2)^{th}$ vertex to each of k vertices of the same recursive tree, $(k+1)$ generalized ordering recursive k-trees with $k+2$ vertices are obtained. And the next $(k+3)^{th}$ vertex is added, and so on. We keep this process along the order of $i_1, i_2, \ldots, i_n$. Then we can make $(k+1)$ generalized ordering recursive k-trees.

On the other hand, consider the labeling the k-cell of generalized ordering recursive k-tree T rooted on the $(k-1)$-cell with the given order of vertices, say $i_1, i_2, \ldots, i_n$. Note that T may be rooted on the different $(k-1)$-cell that contains vertices $i_1, \ldots, \widehat{i_j}, \ldots, i_{k+1}$, for all $j = 1, \ldots, k+1$, where $\widehat{i_j}$ means this vertex is omitted. It produces the same shelling of k-tree with this order of vertices. Hence if we identify the $(k+1)$ generalized ordering recursive k-trees rooted on the $(k-1)$-cells, we can define isomorphism from the set of all shellings of k-tree to the set of all generalized ordering recursive k-trees with the identification of $(k+1)$ generalized ordering recursive k-trees rooted on each cells of k-cell with vertices labeled by $i_1, i_2, \ldots, i_{k+1}$.

Therefore this isomorphism gives us the formula

$$\frac{n!}{k!}(nk - k^2 - k + 1)!_k \frac{1}{k+1} = \frac{n!}{(k+1)!}(nk - k^2 - k + 1)!_k.$$

□

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