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SOME POPULAR WAVELET DISTRIBUTION

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ABSTRACT. The modern approach for wavelets imposes a Bayesian prior model on the wavelet coefficients to capture the sparseness of the wavelet expansion. The idea is to build flexible probability models for the marginal posterior densities of the wavelet coefficients. In this note, we derive exact expressions for a popular model for the marginal posterior density.

1. Introduction

Recently, various Bayesian approaches have been proposed for nonlinear wavelet thresholding and nonlinear wavelet shrinkage estimators. These approaches impose a prior distribution on the variability of the observed wavelet coefficients. The prior model is designed to capture the sparseness of wavelet expansions. Then, the image is estimated by applying a suitable Bayesian rule to the resulting posterior distribution of the wavelet coefficients. Different choices of loss function lead to different Bayesian rules and hence to different nonlinear wavelet shrinkage and wavelet thresholding rules, see Walnut [4].

The aim of this note is to derive a popular model for the posterior distribution of the observed wavelet coefficients. As stated above, we assume a prior on the variability of the observed wavelets. It is also assumed in the prior model that the wavelet coefficients of the true image are mutually independent random variables and independent of the noise process. Let X denote the random variable representing the observed wavelet coefficients. Arguably, the most popular model for X is the white Gaussian noise model. Thus, assume that X has the normal distribution with mean μ and standard deviation λ. We need a prior for λ. For the past 40 to 50 years, the Student's t distribution has been the most popular prior distribution because elicitation of prior information in various physical, engineering, and financial phenomena is closely associated with that distribution, see Kotz and Nadarajah [2]. So, if we assume that

$$p(\lambda) \quad \propto \quad \left(1 + \frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2}$$

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then the joint posterior will be

$$p(\mu, \lambda \mid x) \quad \propto \quad \frac{1}{\lambda} \exp\left\{-\frac{(x-\mu)^2}{2\lambda^2}\right\} \left(1+\frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2}$$

Hence, the marginal posterior of μ will be

$$(1) \qquad p(\mu \mid x) \quad \propto \quad \int_0^\infty \frac{1}{\lambda} \exp\left\{-\frac{(x-\mu)^2}{2\lambda^2}\right\} \left(1+\frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2} d\lambda.$$

The density in (1) is the same as that of the product XY when X and Y are normal and Student's t random variables distributed independently of each other. Hence, calculating the marginal posterior of μ amounts to deriving the exact distribution of XY.

In this note, we derive the marginal posterior distribution given by (1), which amounts to deriving the distribution of $\mid XY \mid$ when X and Y are independent random variables with the pdfs

$$(2) \qquad f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$

and

$$(3) \qquad f_Y(y) \quad = \quad \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \left(1+\frac{y^2}{\nu}\right)^{-(1+\nu)/2},$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\sigma > 0$ and $\nu > 0$. The explicit expressions for the pdf and the cdf of $\mid XY \mid$ are given in Section 2. The calculations involve several special functions, including the complementary error function defined by

$$\operatorname{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt,$$

the error function of an imaginary argument defined by

$$\operatorname{erfi}(x) \quad = \quad \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(t^2\right) dt,$$

the modified Bessel function of the first kind defined by

$$I_m(x) \quad = \quad \frac{x^m}{\sqrt{\pi} 2^m \Gamma(m+1/2)} \int_{-1}^1 \left(1-t^2\right)^{m-1/2} \exp(\pm xt)\, dt,$$

the Kummer function defined by

$$\mathbf{K}(a, b; x) \quad = \quad \frac{1}{\Gamma(a)} \int_0^\infty \exp(-xt)\, t^{a-1} (1+t)^{b-a-1} dt$$

and the hypergeometric function defined by

$${}_2F_2(a, b; c, d; x) \quad = \quad \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* [3] and Gradshteyn and Ryzhik [1].

2. Exact distribution of the product

Theorems 1 and 2 derive explicit expressions for the pdf and the cdf of $\mid XY \mid$ in terms of the Kummer and the hypergeometric functions.

Theorem 1. *Suppose X and Y are distributed according to (2) and (3), respectively. Then, the pdf of $Z = \mid XY \mid$ can be expressed as*

$$f_Z(z) = \frac{\sqrt{2}\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi\nu}\sigma B\left(\nu/2, 1/2\right)} \mathbf{K}\left(\frac{1+\nu}{2}, 1; \frac{z^2}{2\sigma^2\nu}\right), \tag{4}$$

for $z > 0$.

Proof. The general formula for the pdf of $\mid XY \mid$ is

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\mid y \mid} \left\{ f_X\left(\frac{z}{\mid y \mid}\right) + f_X\left(-\frac{z}{\mid y \mid}\right) \right\} f_Y(y) dy.$$

Since the given forms for $f_X(\cdot)$ and $f_Y(\cdot)$ are both symmetric around zero, the above can be expressed as

$$\begin{aligned} & f_Z(z) \\ &= 4\int_0^{\infty} \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y) dy \\ &= \frac{4}{\sqrt{2\pi\nu}\sigma B\left(\nu/2, 1/2\right)} \int_0^{\infty} \frac{1}{y} \exp\left(-\frac{z^2}{2\sigma^2 y^2}\right) \left(1 + \frac{y^2}{\nu}\right)^{-(1+\nu)/2} dy \\ &= \frac{\sqrt{2}}{\sqrt{\pi\nu}\sigma B\left(\nu/2, 1/2\right)} \int_0^{\infty} w^{(\nu-1)/2} \exp\left(-\frac{z^2 w}{2\sigma^2}\right) \left(w + \frac{1}{\nu}\right)^{-(1+\nu)/2} dw, \end{aligned} \tag{5}$$

where the last step follows by substituting $w = 1/y^2$. The result of the theorem follows by applying equation (2.3.6.9) in Prudnikov *et al.* ([3], volume 1) to calculate the integral in (5). □

Theorem 2. *Suppose X and Y are distributed according to (2) and (3), respectively. Then, the cdf of $Z = \mid XY \mid$ can be expressed as*

$$F_Z(z) = \frac{2z\left\{2C + \Psi\left((1+\nu)/2\right)\right\}}{\sqrt{2\pi\nu}\sigma B\left(\nu/2, 1/2\right)} {}_2F_2\left(\frac{1+\nu}{2}, \frac{1}{2}; \frac{3}{2}, 1; \frac{z^2}{2\sigma^2\nu}\right) \tag{6}$$

for $z > 0$, where C denotes Euler's constant and $\Psi(x) = d\log\Gamma(x)/dx$ is the digamma function.

Proof. The general formula for the cdf of $\mid XY \mid$ is

$$F_Z(z) = \int_{-\infty}^{\infty} \left\{ F_X\left(\frac{z}{\mid y \mid}\right) - F_X\left(-\frac{z}{\mid y \mid}\right) \right\} f_Y(y) dy. \tag{7}$$

Considering

$$F_X(x) = 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sigma\sqrt{2}}\right), \tag{8}$$

(7) can be expressed as

$$\begin{aligned} F_Z(z) & \\ &= 1 - \frac{1}{\sqrt{\nu}B(\nu/2,1/2)}\int_{-\infty}^{\infty}\mathrm{erfc}\left(\frac{z}{\sqrt{2}\sigma \mid y \mid}\right)\left(1+\frac{y^2}{\nu}\right)^{-(1+\nu)/2}dy \\ &= 1 - \frac{2}{\sqrt{\nu}B(\nu/2,1/2)}\int_{0}^{\infty}\mathrm{erfc}\left(\frac{z}{\sqrt{2}\sigma y}\right)\left(1+\frac{y^2}{\nu}\right)^{-(1+\nu)/2}dy \\ &= 1 - \frac{2}{\sqrt{\nu}B(\nu/2,1/2)}\int_{0}^{\infty} w^{\nu-1}\left(w^2+1/\nu\right)^{-(1+\nu)/2}\mathrm{erfc}\left(\frac{wz}{\sqrt{2}\sigma}\right)dw, \end{aligned} \tag{9}$$

where the last step follows by substituting $w = 1/y$. The result of the theorem follows by using equation (2.8.3.5) in Prudnikov *et al.* ([3], volume 2) to calculate the integral in (9). □

Using special properties of the hypergeometric function, one can derive simpler forms for (6) when ν takes integer values. This is illustrated in the corollary below.

Corollary 1. *If* $\nu = 1, 2, \ldots, 10$ *then* (6) *reduces to*

$$F_Z(z) = (C/\pi)\mathrm{erfi}\left(\sqrt{u}\right),$$

$$F_Z(z) = (1/\sqrt{pi})\sqrt{u}\,(C+2-2\log 2)\exp(u/2)\, I_0(u/2),$$

$$F_Z(z) = \pi^{-3/2}(C+1)\left\{2\sqrt{u}\exp(u) + \sqrt{\pi}\mathrm{erfi}\left(\sqrt{u}\right)\right\},$$

$$\begin{aligned} F_Z(z) = {} & 1/\left(6\sqrt{\pi}\right)\sqrt{u}\,(3C+8-6\log 2)\exp(u/2)\left\{3I_0(u/2) + uI_0(u/2)\right. \\ & \left. + uI_1(u/2)\right\}, \end{aligned}$$

$$F_Z(z) = 1/\left(6\pi^{3/2}\right)(2C+3)\left\{4u^{3/2}\exp(u) + 10\sqrt{u}\exp(u) + 3\sqrt{\pi}\mathrm{erfi}\left(\sqrt{u}\right)\right\},$$

$$\begin{aligned} F_Z(z) = {} & 1/\left(120\sqrt{\pi}\right)\sqrt{u}\,(15C+46-30\log 2)\exp(u/2)\left\{2u^2 I_0(u/2)\right. \\ & \left. + 10uI_0(u/2) + 15I_0(u/2) + 2u^2 I_1(u/2) + 8uI_1(u/2)\right\}, \end{aligned}$$

$$\begin{aligned} F_Z(z) = {} & 1/\left(90\pi^{3/2}\right)(6C+11)\left\{66\sqrt{u}\exp(u) + 52u^{3/2}\exp(u) + 8u^{5/2}\exp(u)\right. \\ & \left. + 15\sqrt{\pi}\mathrm{erfi}\left(\sqrt{u}\right)\right\}, \end{aligned}$$

$$F_Z(z) = 1/\left(5040\sqrt{\pi}\right)\sqrt{u}\,(105C + 352 - 210\log 2)\exp(u/2)\Big\{105uI_0\left(u/2\right) + 40u^2 I_0\left(u/2\right) + 4u^3 I_0\left(u/2\right) + 105 I_0\left(u/2\right) + 36u^2 I_1\left(u/2\right) + 71uI_1\left(u/2\right) + 4u^3 I_1\left(u/2\right)\Big\},$$

$$F_Z(z) = 1/\left(1260\pi^{3/2}\right)(12C + 25)\left\{652u^{3/2}\exp(u) + 200u^{5/2}\exp(u) + 16u^{7/2}\exp(u) + 558\sqrt{u}\exp(u) + 105\sqrt{\pi}\,\mathrm{erfi}\left(\sqrt{u}\right)\right\},$$

and

$$F_Z(z) = 1/\left(120960\sqrt{\pi}\right)\sqrt{u}\,(315C + 1126 - 630\log 2)\exp(u/2)\Big\{1260uI_0\left(u/2\right) + 696u^2 I_0\left(u/2\right) + 136u^3 I_0\left(u/2\right) + 8u^4 I_0\left(u/2\right) + 945 I_0\left(u/2\right) + 572u^2 I_1\left(u/2\right) + 744uI_1\left(u/2\right) + 128u^3 I_1\left(u/2\right) + 8u^4 I_1\left(u/2\right)\Big\},$$

where $u = z^2/(2\sigma^2\nu)$ *and* C *denotes Euler's constant.*

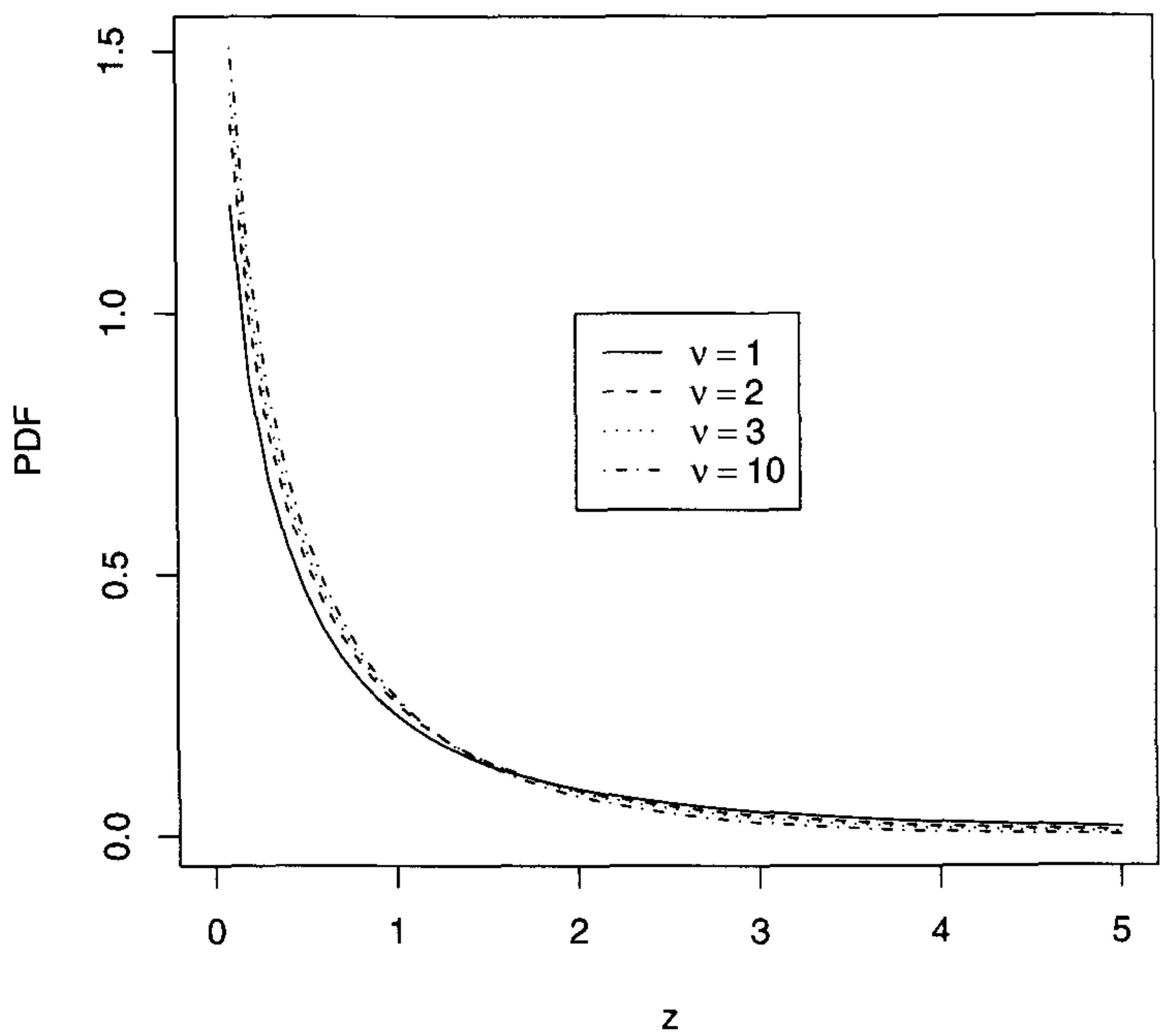

FIGURE 1. Plots of the pdf (4) for $\nu = 1, 2, 3, 10$ and $\sigma = 1$.

Figure 1 illustrates possible shapes of the pdf (4) for a range of values of ν. Note that the shapes are unimodal and that the value of ν largely dictates the behavior of the pdf near $z = 0$.

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