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BEREZIN TRANSFORMS AND TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACES OF THE HALF-PLANE

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Abstract. We deal with Bergman functions defined on the half-plane. We study integral operators, and some relationship between Berezin transforms and Toeplitz operators, and also show that some Berezin transforms are Möbius invariant.

1. Introduction

Let $H = \{x + iy : y > 0\}$ be the half-plane in the complex plane $\mathbb{C}$. For $1 \leq p < \infty$ and $r > -\frac{1}{2}$, the weighted Bergman space $B^{p,r}$ of the half-plane is the space of analytic functions in $L^p(H, dA_r)$, where dA is the area measure on H, $K(z,w) = -\frac{1}{\pi(z-\overline{w})^2}$, and $dA_r(z) = (2r+1)K(z,z)^{-r}dA$. In fact, $K(z,\cdot)$ is the reproducing kernel for $B^{2,0}$(see [1]) and $K(z,\cdot)^{1+r}$ is the reproducing kernel for $B^{2,r}$(see [4]).

Section 2 of this paper contains the linearity and continuity of point evaluations, that is, they belong to $(B^{p,r})^*$ and the boundedness of each element of $B^{p,r}$ implies that $B^{p,r}$ is a closed subspace of $L^p(H, dA_r)$.

In Section 3, we define the Berezin transform. To do so, we need the normalized reproducing kernel. We show that the normalized reproducing kernel converges weakly to 0 in $B^{2,r}$ as $\operatorname{Im} z \to 0$ and we deal with some kernel operators. Moreover, we introduce Toeplitz operators with symbol which is an element of $B^{2,r} \cup L^\infty$.

We study integral operators, and some relationships between Berezin transforms and Toeplitz operators. We also show that some Berezin transforms are Möbius invariant and we get some quantity of Berezin transforms. Throughout this paper, we use the symbol $A \lesssim B$ ($A \approx B$, resp.) for nonnegative constants A and B to indicate that A is dominated by B times some positive constant ($A \lesssim B$ and $B \lesssim A$, resp.).

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2. Reproducing kernels

In this section, we deal with the point evaluation map and the closedness of $B^{p,r}$. To do so, we study the boundedness of each element of $B^{p,r}$.

Proposition 2.1. *Suppose $r > -\frac{1}{2}$ and $z = x + iy \in H$. Then*

$$|f(z)| \lesssim \left(\frac{4}{2r+1}\right)^{\frac{1}{p}} \frac{1}{(\pi y^2)^{\frac{1+r}{p}}} \| f \|_{p,r}$$

for every $f \in B^{p,r}$.

Proof. The mean-value property and Jensen's inequality imply that

$$|f(z)|^p \leq \frac{1}{A(B(z, \frac{y}{2}))} \int_{B(z,\frac{y}{2})} |f|^p dA.$$

If $w \in B(z, \frac{y}{2})$ then $\frac{3}{2}y > \operatorname{Im} w > \frac{y}{2}$ and hence $K(w,w) = \dfrac{1}{4\pi(\operatorname{Im} w)^2} \lesssim \dfrac{1}{\pi y^2}$.
Hence

$$\begin{aligned} |f(z)|^p &\lesssim \frac{1}{\pi \frac{y^2}{4}} \int_H |f(w)|^p K(w,w)^{-r} \frac{1}{(\pi y^2)^r} dA(w) \\ &= \frac{4}{(2r+1)(\pi y^2)^{1+r}} \| f \|_{p,r}^p, \end{aligned}$$

that is, $|f(z)| \lesssim \left(\dfrac{4}{(2r+1)(\pi y^2)^{1+r}}\right)^{\frac{1}{p}} \| f \|_{p,r}$ for all $f \in B^{p,r}$. □

Theorem 2.2. *For fixed $z \in H$, the point evaluation Λ_z defined by $\Lambda_z(f) = f(z)$ for all $f \in B^{p,r}$ is continuous on $B^{p,r}$.*

Proof. Let $z = x + iy$. By Proposition 2.1, for any $f \in B^{p,r}$,

$$|f(z)| \lesssim \left(\frac{4}{2r+1} \frac{1}{(\pi y^2)^{1+r}}\right)^{\frac{1}{p}} \| f \|_{p,r}$$

and hence $\| \Lambda_z \| \lesssim \left(\dfrac{4}{(2r+1)(\pi y^2)^{1+r}}\right)^{\frac{1}{p}}$, that is, Λ_z is an element of $(B^{p,r})^*$. □

Proposition 2.3. *For $1 \leq p < \infty$ and $r > -\frac{1}{2}$, $B^{p,r}$ is a closed subspace of $L^{p,r}$.*

Proof. Take any Cauchy sequence $\{f_n\}$ in $B^{p,r}$. Since $L^{p,r}$ is complete, $\{f_n\}$ converges to f for some $f \in L^{p,r}$. By Proposition 2.1, $\{f_n\}$ converges uniformly on each compact subset of H and hence $\{f_n\}$ converges uniformly on compact subsets of H to a function g that is analytic on H. Since $\{f_n\}$ converges to f in $L^{p,r}$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f pointwise almost everywhere on H. This implies that $f = g$ almost everywhere on H. Hence $f \in B^{p,r}$. □

Since $B^{p,r}$ is a closed subspace of $L^{p,r}$, $B^{2,r}$ is a Hilbert space with inner product $\langle f,g\rangle_r = \int_H f(z)\overline{g(z)}dA_r(z)$. Theorem 2.2 states that for each $z \in H$, the point evaluation Λ_z is a bounded linear functional on $B^{2,r}$ and hence the Riesz representation theorem implies that there exists a unique function h in $B^{2,r}$ such that $\Lambda_z(f) = \langle f,h\rangle_r$ for all $f \in B^{2,r}$. In fact, $h(w) = K(z,w)^{1+r} = \Big(-\dfrac{1}{\pi(z-\overline{w})^2}\Big)^{1+r}$ (see [4]).

3. Integral operators, Berezin transforms, and Toeplitz operators

Section 3 deals with some kernel operators and the Berezin transform. To do so, for $-\frac{1}{2} < r$, let

$$I_\beta(z) = \int_H \frac{1}{|z-\overline{w}|^{2+2r+\beta}} dA_r(w).$$

Theorem 3.1. *$I_\beta(z) \approx \dfrac{1}{(\operatorname{Im} z)^\beta}$ whenever $\beta > 0$.*

Proof. Let $z = x+iy$ and $w = s+it$ be in H. Since $K(w,w)^{-r} = (\pi(2\operatorname{Im} w)^2)^r$,

$$\begin{aligned}
I_\beta(z) &= \int_H \frac{1}{|z-\overline{w}|^{2+2r+\beta}} dA_r(w) \\
&= \int_H \frac{(4\pi)^r(\operatorname{Im} w)^{2r}}{|z-\overline{w}|^{2+2r+\beta}} dA(w) \\
&\lesssim \int_H \frac{dA(w)}{|z-\overline{w}|^{2+\beta}} \\
&= \int_0^\infty\int_{-\infty}^\infty \frac{1}{\{(x-s)^2+(y+t)^2\}^{1+\frac{\beta}{2}}} dsdt \\
&\le \int_0^\infty\int_{-\infty}^\infty \frac{y+t}{(x-s)^2+(y+t)^2}\frac{1}{(y+t)^{\beta+1}} dsdt \\
&\approx \int_0^\infty \frac{1}{(y+t)^{1+\beta}} dt \\
&= -\frac{1}{\beta}\frac{1}{(y+t)^\beta}\Big]_0^\infty \\
&\approx \frac{1}{y^\beta} = \frac{1}{(\operatorname{Im} z)^\beta}.
\end{aligned}$$

Since $z = x+iy$ and $\frac{y}{2} > 0$, the Euclidean ball $B(z,\frac{y}{2})$ is contained in H and hence

$$\begin{aligned}
I_\beta(z) &= \int_H \frac{1}{|z-\overline{w}|^{2+2r+\beta}} dA_r(w) \\
&\gtrsim \int_{B(z,\frac{y}{2})} \frac{(\frac{y}{2})^{2r}}{y^{2+2r+\beta}} dA(w)
\end{aligned}$$

$$\begin{aligned}
&\approx \int_{B(z,\frac{y}{2})} y^{-2-\beta} dA(w) \\
&= \frac{1}{y^{2+\beta}} \pi (\frac{y}{2})^2 \\
&\approx \frac{1}{y^{\beta}} = \frac{1}{(\operatorname{Im} z)^{\beta}}.
\end{aligned}$$

Hence $I_\beta(z) \approx \frac{1}{(\operatorname{Im} z)^{\beta}}$. □

Theorem 3.2. *For any real numbers a and b, we define integral operators T and S by*

$$Tf(z) = (\operatorname{Im} z)^a \int_H \frac{(\operatorname{Im} w)^b}{(z - \overline{w})^{2+a+b}} f(w) dA(w)$$

and

$$Sf(z) = (\operatorname{Im} z)^a \int_H \frac{(\operatorname{Im} w)^b}{|z - \overline{w}|^{2+a+b}} f(w) dA(w).$$

Suppose $1 \le p < \infty$ and $-pa < 2r < pb$. Then T is bounded on $L^{p,r}$.

Proof. Since the boundedness of S implies that of T, it is enough to show that S is bounded on $L^{p,r}$. Define $h(z) = (\operatorname{Im} z)^s$. Then h is a positive measurable function on H and

$$\begin{aligned}
&\int_H \frac{(\operatorname{Im} z)^a (\operatorname{Im} w)^b}{|z - \overline{w}|^{2+a+b+2r}} h(w)^q dA_r(w) \\
= \ &(\operatorname{Im} z)^a \int_H \frac{(\operatorname{Im} w)^{sq+b}}{|z - \overline{w}|^{2+a+b+2r}} dA_r(w).
\end{aligned}$$

Since $a + b = sq + b + \beta$ implies $\beta = a - sq$,

$$\int_H \frac{(\operatorname{Im} z)^a (\operatorname{Im} w)^b}{|z - \overline{w}|^{2+a+b+2r}} h(w)^q dA_r(w) \approx (\operatorname{Im} z)^{sq} = h(z)^q.$$

On the other hand,

$$\begin{aligned}
&\int_H \frac{(\operatorname{Im} z)^a (\operatorname{Im} w)^b}{|z - \overline{w}|^{2+a+b}} h(z)^p dA_r(z) \\
\approx \ &(\operatorname{Im} w)^b \int_H \frac{(\operatorname{Im} z)^a}{|z - \overline{w}|^{2+a+b+2r}} (\operatorname{Im} z)^{sp} dA_r(z) \\
\approx \ &(\operatorname{Im} w)^b \int_H \frac{(\operatorname{Im} z)^{a+sp+2r}}{|z - \overline{w}|^{2+a+b+2r}} dA_{r+\frac{a+sp}{2}}(z) \\
\approx \ &(\operatorname{Im} w)^b \frac{1}{(\operatorname{Im} w)^{b-sp}} = (h(w))^p.
\end{aligned}$$

According to Theorem 3.1, these estimates are correct if $sq + b \ge 0$, $a - sq > 0$, and $a + sp + 2r \ge 0$, $b - sp - 2r > 0$ and hence $-\frac{b}{q} \le s < \frac{a}{q}$ and

$\frac{-a-2r}{p} \le s < \frac{b-2r}{p}$. We note that $-pa < 2r$ if and only if $-pa+a < 2r+a$ if and only if $\frac{a(-p+1)}{p} < \frac{2r+a}{p}$ if and only if $\frac{a}{q} > -\frac{2r+a}{p}$. If $2r < pb = b\frac{q}{q-1}$ then $2r(q-1) < bq$ if and only if $-2r < (b-2r)q$. Since $-pb < -2r$, $-pb < (b-2r)q$ and hence $-\frac{b}{q} < \frac{b-2r}{p}$. Thus $\Big[-\frac{b}{q}, \frac{a}{q}\Big) \cap \Big[-\frac{2r+a}{p}, \frac{b-2r}{p}\Big) \neq \emptyset$. By Schur's Theorem, S is bounded on $L^{p,r}$. This completes the proof. □

Proposition 3.3. *For each $w \in H$, $\int_H \Big(\frac{(2\operatorname{Im} w)^2}{\pi|z-\overline{w}|^4}\Big)^{1+r} dA_r(z) = 1$.*

Proof. Take any $w \in H$. Then

$$\begin{aligned}
&\int_H \Big(\frac{(2\operatorname{Im} w)^2}{\pi|z-\overline{w}|^4}\Big)^{1+r} dA_r(z)\\
=\;& (2\operatorname{Im} w)^{2(1+r)} \int_H \frac{1}{(z-\overline{w})^{2(1+r)}} (-1)^{1+r} \overline{\Big(-\frac{1}{\pi(z-\overline{w})^2}\Big)^{1+r}} dA_r(z)\\
=\;& (-1)^{1+r}(2\operatorname{Im} w)^{2(1+r)} \int_H \frac{1}{(z-\overline{w})^{2(1+r)}} K(w,z)^{1+r} dA_r(z)\\
=\;& 1.
\end{aligned}$$

□

Since $K(w,w) = \frac{1}{\pi(2\operatorname{Im} w)^2}$, for each $w \in H$, we define

$$k_w(z) = \Big(\frac{K(z,w)}{\sqrt{K(w,w)}}\Big)^{1+r}.$$

Then we have the following :

Corollary 3.4. *For each $w \in H$, k_w is a unit vector in $B^{2,r}$, that is, k_w is a normalized reproducing kernel.*

Proof. Since $K(z,w) = \frac{1}{-\pi(z-\overline{w})^2}$ and $\sqrt{K(w,w)} = \frac{1}{\sqrt{\pi}2\operatorname{Im} w}$,

$$\begin{aligned}
\langle k_w, k_w\rangle_r &= \int_H k_w(z)\overline{k_w(z)} dA_r(z)\\
&= \int_H \Big(\frac{|K(z,w)|^2}{K(w,w)}\Big)^{1+r} dA_r(z)\\
&= \int_H \Big(\frac{(2\operatorname{Im} w)^2}{\pi|w-\overline{z}|^4}\Big)^{1+r} dA_r(z) = 1.
\end{aligned}$$

□

Suppose T is a linear operator on $B^{2,r}$. We define $\tilde{T}(z) = \langle Tk_z, k_z\rangle_r$ for all $z \in H$, where k_z is the normalized reproducing kernel. Since $B^{2,r}$ is a closed subspace of a Hilbert space $L^{2,r}$, there is a unique orthogonal projection

$P : L^{2,r} \longrightarrow B^{2,r}$ such that $P(f)(w) = \int_H f(z)K(w,z)^{1+r}dA_r(z)$ for all $f \in L^{2,r}$. The reproducing kernel property of $K(w,z)^{1+r}$ implies that $P|_{B^{2,r}} = I$. Moreover, we can extend P to $L^{p,r}$ whenever $1 < p < \infty$ and $r > -\frac{1}{2}$ and $P : L^{p,r} \longrightarrow B^{p,r}$ is a bounded linear operator (see[4]). For $f \in L^\infty$, we define $T_f : B^{2,r} \longrightarrow B^{2,r}$ by $T_f(g) = P(fg)$ for all $g \in B^{2,r}$. Since $f \cdot g$ is in $L^{2,r}$, T_f is well-defined. Since $B^{2,r} \cap L^\infty$ is dense in $B^{2,r}$(see Lemma 3.7), for any $f \in B^{2,r}$ and any $w = s + it$ in H, there is a sequence $\{f_n\}$ in $B^{2,r} \cap L^\infty$ such that $\lim_{n\to\infty} f_n = f$ in $B^{2,r}$ and hence $\left| \int_H f(z)g(z)K(w,z)^{1+r}dA_r(z) - \int_H f_n(z)g(z)K(w,z)^{1+r}dA_r(z) \right| \leq \int_H |(f_n - f)(z)||g(z)||K(w,z)^{1+r}|dA_r(z) \longrightarrow 0$ as $n \to \infty$ because the integral is dominated by $\|f_n - f\|_{2,r}\|g\|_{2,r}$ and we can define $T_f(g)(w) = \int_H f(z)g(z)K(w,z)^{1+r}dA_r(z)$. Let $X = L^\infty \cup B^{2,r}$. Then for any $f \in X$, we can define $T_f : B^{2,r} \longrightarrow B^{2,r}$ by $T_f(g) = P(fg)$ for all $g \in B^{2,r}$. Moreover, T_f is a bounded linear operator and called the Toeplitz operator with symbol f. We consider the Berezin transform $\tilde{T}$ of Toeplitz operators T. In particular, for any Toeplitz operator with symbol f, we write $\tilde{T_f} = \tilde{f}$ which is called the Berezin transform of the function f.

Proposition 3.5. *For any* $f \in X$, $\tilde{T_f}(z) = \tilde{f}(z) = \int_H f(w)|k_z(w)|^2 dA_r(w)$.

Proof. Since $k_z(w) = \left(-\frac{2\,\mathrm{Im}\,z}{\sqrt{\pi}(w-\overline{z})^2}\right)^{1+r}$,

$$
\begin{aligned}
&\tilde{T_f}(z) = \tilde{f}(z)\\
= \;& \langle T_f k_z, k_z\rangle_r = \int_H T_f k_z(w)\overline{k_z(w)}dA_r(w)\\
= \;& \int_H P(fk_z)(w)\overline{k_z(w)}dA_r(w)\\
= \;& \int_H\int_H f(t)k_z(t)K(w,t)^{1+r}dA_r(t)\overline{k_z(w)}dA_r(w)\\
= \;& \int_H\int_H f(t)\Big(\frac{K(t,z)}{\sqrt{K(z,z)}}\Big)^{1+r}K(w,t)^{1+r}\overline{\Big(\frac{K(w,z)}{\sqrt{K(z,z)}}\Big)^{1+r}}dA_r(t)dA_r(w)\\
= \;& \frac{1}{K(z,z)^{1+r}}\int_H\int_H f(t)K(t,z)^{1+r}K(w,t)^{1+r}\overline{K(w,z)^{1+r}}dA_r(t)dA_r(w)\\
= \;& \frac{1}{K(z,z)^{1+r}}\int_H f(t)K(t,z)^{1+r}\int_H K(w,t)^{1+r}\overline{K(w,z)^{1+r}}dA_r(w)dA_r(t)\\
= \;& \frac{1}{K(z,z)^{1+r}}\int_H f(t)K(t,z)^{1+r}K(z,t)^{1+r}dA_r(t)\\
= \;& \frac{1}{K(z,z)^{1+r}}\int_H f(t)K(t,z)^{1+r}\overline{K(t,z)^{1+r}}dA_r(t)
\end{aligned}
$$

$$= \int_H f(w)|k_z(w)|^2 dA_r(w).$$

Thus one has the result. □

Corollary 3.6. *If $f \in B^{2,r}$ then $\tilde{T}_f(z) = \tilde{f}(z) = f(z)$.*

Proof. It is immediately from the fact that

$$\int_H f(w)K(w,z)^{1+r}K(z,w)^{1+r}dA_r(w) = f(z)K(z,z)^{1+r}.$$

□

Lemma 3.7. *$B^{2,r} \cap L^\infty$ is dense in $B^{2,r}$.*

Proof. Take any $\epsilon > 0$ and any $f \in B^{2,r}$ For each $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$ for all $z \in H$. Take any w in H. By the mean-value property,

$$f_\delta(w) = f(w + \delta i) = \frac{1}{|B(w + \delta i, \delta)|} \int_{B(w+\delta i,\delta)} f dA.$$

Jensen's inequality implies that

$$|f_\delta(w)|^2 \leq \frac{1}{|B(w,\delta)|} \parallel f \parallel_{2,r}^2 = \frac{1}{\pi\delta^2} \parallel f \parallel_{2,r}^2.$$

Hence f_δ is bounded on H. Since $f_\delta \in B^{2,r}$ and $C_c(H)$ is dense in $L^{2,r}$, there is $g \in C_c(H)$ such that $\parallel g - f \parallel_{2,r} < \epsilon$. Since g is uniformly continuous on H, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, $\parallel g_\delta - g \parallel_{2,r} < \epsilon$. Suppose $0 < \delta < \delta_0$. Since $\parallel f_\delta - g_\delta \parallel_{2,r} \leq \parallel f - g \parallel_{2,r} < \epsilon$, $\parallel f_\delta - f \parallel_{2,r} \leq \parallel f_\delta - g_\delta \parallel_{2,r} + \parallel g_\delta - g \parallel_{2,r} + \parallel g - f \parallel_{2,r} < 3\epsilon$ and hence $\lim_{\delta\to 0} \parallel f_\delta - f \parallel_{2,r} = 0$. Thus $B^{2,r} \cap L^\infty$ is dense in $B^{2,r}$. □

Proposition 3.8. *Suppose $z \in H$. Then k_z converges weakly to 0 in $B^{2,r}$ as $\operatorname{Im} z \to 0$.*

Proof. Take any f in $B^{2,r} \cap L^\infty$. Then

$$\begin{aligned}
\langle f, k_z \rangle_r &= \int_H f(w)\overline{k_z(w)} dA_r(w) \\
&= \frac{1}{\sqrt{K(z,z)^{1+r}}} \int_H f(w)\overline{K(w,z)^{1+r}} dA_r(w) \\
&= \frac{1}{\sqrt{K(z,z)^{1+r}}} \int_H f(w)K(z,w)^{1+r} dA_r(w) \\
&= (\sqrt{\pi} 2 \operatorname{Im} z)^{1+r} f(z).
\end{aligned}$$

Since $\lim_{\operatorname{Im} z \to 0} (\sqrt{\pi} 2 \operatorname{Im} z)^{1+r} f(z) = 0$ and $(B^{2,r})^* = B^{2,r}$, k_z converges weakly to 0 in $B^{2,r}$ as $\operatorname{Im} z \to 0$. □

Proposition 3.9. *Suppose $f \in B^{2,r}$. If T_f is bounded (compact, respectively) then f is bounded ($\lim_{\operatorname{Im} z \to 0} f(z) = 0$, respectively).*

Proof. Corollary 3.6 implies that for each $z \in H$, $\tilde{T}_f(z) = f(z)$ and hence

$$\langle T_f k_z, k_z \rangle_r = f(z).$$

Since $|\langle T_f k_z, k_z \rangle_r| \leq \|T_f\|$, the bondedness of T_f implies the boundedness of f. Suppose T_f is compact. By Proposition 3.8, k_z converges weakly to 0 as $\operatorname{Im} z \to 0$ and hence $\langle T_f k_z, k_z \rangle_r \to 0$ as $\operatorname{Im} z \to 0$.Thus $\lim_{\operatorname{Im} z \to 0} f(z) = 0$. □

Theorem 3.10. *Suppose $f \in X$ and $\varphi \in Aut(H)$. Then $\widetilde{f \circ \varphi}(z) = \tilde{f} \circ \varphi(z)$ for all $z \in H$, that is, some Berezin transforms are Möbius invariant.*

Proof. Since $\varphi \in Aut(H)$, there are constants $a > 0$ and $b \in \mathbb{R}$ such that $\varphi(z) = az + b$ (see[2]) and hence $\varphi^{-1}(z) = \frac{1}{a}(z - b)$. We note that

$$k_z(\varphi^{-1}(w)) = \Big(\frac{K(\varphi^{-1}(w), z)}{\sqrt{K(z,z)}}\Big)^{1+r},$$

$$K(\varphi^{-1}(w), z) = -\frac{1}{\pi(\frac{1}{a}(w-b) - \overline{z})^2},$$

and

$$K(\varphi^{-1}(w), \varphi^{-1}(w)) = \frac{a^2}{\pi 4(\operatorname{Im} w)^2}.$$

Since

$$\begin{aligned}
& |k_z(\varphi^{-1}(w))|^2 \frac{1}{a^2} K(\varphi^{-1}(w), \varphi^{-1}(w))^{-r} \\
= & \left|\frac{4\pi(\operatorname{Im} z)^2}{\pi^2(\frac{1}{a}(w-b)-\overline{z})^4}\right|^{1+r} \Big(\frac{1}{a^2}\Big)^{1+r} (\pi 4(\operatorname{Im} w)^2)^r \\
= & \left|\frac{4\pi(\operatorname{Im} z)^2}{\pi^2 a^2(\frac{1}{a}(w-b)-\overline{z})^4}\right|^{1+r} (\pi 4(\operatorname{Im} w)^2)^r \\
= & \left|\frac{\pi a^2 4(\operatorname{Im} z)^2}{\pi^2 a^4(\frac{1}{a}(w-b)-\overline{z})^4}\right|^{1+r} (\pi 4(\operatorname{Im} w)^2)^r \\
= & |k_{\varphi(z)}(w)|^2 K(w,w)^{-r},
\end{aligned}$$

$$\begin{aligned}
& \widetilde{f \circ \varphi}(z) = \tilde{T}_{f\circ\varphi}(z) \\
= & \int_H f \circ \varphi(w) |k_z(w)|^2 dA_r(w) \\
= & \int_H f(\varphi(w)) |k_z(w)|^2 (2r+1) K(w,w)^{-r} dA(w) \\
= & \int_H f(w) |k_z(\varphi^{-1}(w))|^2 (2r+1) K(\varphi^{-1}(w), \varphi^{-1}(w))^{-r} |(\varphi^{-1})'(w)|^2 dA(w)
\end{aligned}$$

$$
\begin{aligned}
&= \int_H f(w)(2r+1)|k_z(\varphi^{-1}(w))|^2 \frac{1}{a^2} K(\varphi^{-1}(w),\varphi^{-1}(w))^{-r} dA(w) \\
&= \int_H f(w)(2r+1)|k_{\varphi(z)}(w)|^2 K(w,w)^{-r} dA(w) \\
&= \int_H f(w)|k_{\varphi(z)}(w)|^2 dA_r(w) \\
&= \tilde{T}_f(\varphi(z)) = \tilde{f}\circ\varphi(z).
\end{aligned}
$$

This completes the proof. □

Proposition 3.11. *For any $f \in X$, $T_f^* = T_{\overline{f}}$.*

Proof. Take any g, h in $B^{2,r}$. Fubini's Theorem implies that

$$
\begin{aligned}
&\langle T_f(g), h\rangle_r \\
&= \langle P(fg), h\rangle_r \\
&= \int_H P(fg)(w)\overline{h(w)}dA_r(w) \\
&= \int_H \int_H f(z)g(z)K(w,z)^{1+r}dA_r(z)\overline{h(w)}dA_r(w) \\
&= \int_H f(z)g(z)\overline{\int_H h(w)K(z,w)^{1+r}dA_r(w)}dA_r(z) \\
&= \int_H f(z)g(z)\overline{h(z)}dA_r(z) \\
&= \int_H f(z)\overline{h(z)}\int_H g(w)K(z,w)^{1+r}dA_r(w)dA_r(z) \\
&= \int_H g(w)\overline{\int_H \overline{f(z)}h(z)K(w,z)^{1+r}dA_r(z)}dA_r(w) \\
&= \int_H g(w)\overline{P(\overline{f}h)(w)}dA_r(w) \\
&= \langle g, T_{\overline{f}}(h)\rangle_r.
\end{aligned}
$$

Hence $T_f^* = T_{\overline{f}}$. □

Corollary 3.12. *For any $f \in X$, T_f is self-adjoint if and only if f is real-valued.*

Proposition 3.13. *Suppose $z \in H$ and f, $g \in X$. Then $\widetilde{f\cdot g}(z) - \tilde{f}(z)\cdot\tilde{g}(z) = \frac{1}{2}\int_H\int_H (f(u)-f(v))(g(u)-g(v))|k_z(u)|^2|k_z(v)|^2 dA_r(u)dA_r(v)$.*

Proof. Since k_z is a unit vector, Proposition 3.5 implies that

$$\frac{1}{2}\int_H\int_H (f(u)-f(v))(g(u)-g(v))|k_z(u)|^2|k_z(v)|^2 dA_r(u)dA_r(v)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_H\int_H [f(u)g(u) - f(v)g(u) - f(u)g(v) \\
&\quad + f(v)g(v)]|k_z(u)|^2|k_z(v)|^2 dA_r(u)dA_r(v) \\
&= \frac{1}{2}\Big[\int_H f(u)g(u)|k_z(u)|^2 dA_r(u) - \tilde{f}(z)\tilde{g}(z) - \tilde{f}(z)\cdot\tilde{g}(z) \\
&\quad + \int_H f(v)g(v)|k_z(u)|^2 dA_r(v)\Big] \\
&= \frac{1}{2}\Big[2\int_H f(u)g(u)|k_z(u)|^2 dA_r(u) - 2\tilde{f}(z)\tilde{g}(z)\Big] \\
&= \widetilde{f\cdot g}(z) - \tilde{f}(z)\tilde{g}(z).
\end{aligned}
$$

□

Corollary 3.14. *Suppose $z \in H$ and $f \in L^\infty$. Then* $|\tilde{f}|^2(z) - |\tilde{f}(z)|^2 = \int_H\int_H |f(u) - f(v)|^2|k_z(u)|^2|k_z(v)|^2 dA_r(u)dA_r(v)$.

Proof. In Proposition 3.13, put $g = \overline{f}$. Since $\overline{f}$ is also in L^∞, we have the result. □

References

[1] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, Graduate Texts in Mathematics, 137, Springer-Verlag, New York, 1992.

[2] S. G. Gindkin, *Analysis in Homogeneous Domains*, (Russian) Uspehi Mat. Nauk **19** (1964), no. 4 (118), 3–92.

[3] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, (English summary) Graduate Texts in Mathematics, 199. Springer-Verlag, New York, 2000.

[4] J. Y. Kim, *Weighted Analytic Bergman Spaces of the Half Plane and their Toeplitz Operators*, Ph. D. Thesis, Sookmyung Women's University, 2001.

[5] W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill, New York, 1987.

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