Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 309–313

CUBIC OPERATOR NORM ON X_λ SPACE

Soon-Mo Jung

Abstract. By applying ideas from [M. S. Moslehian, et al., Norms of operators in X_λ spaces, Appl. Math. Lett. (2007), doi:10.1016/j.aml.2006.11.009], we investigate the norm of the cubic operator on the function space X_λ.

1. Introduction

Throughout this paper, let X and Y be real (or complex) normed spaces. For a fixed nonnegative number λ, we denote by X_λ the set of all functions $f : X \to Y$ for which there exists a constant $M_f \geq 0$ with

$$\|f(x)\| \leq M_f e^{\lambda\|x\|}$$

for all $x \in X$. It is not difficult to show that the space X_λ is a normed space if it is equipped with the norm

$$\|f\| = \sup_{x \in X} e^{-\lambda\|x\|}\|f(x)\|.$$

By $[X^2]_\lambda$ we denote the set of all functions $\phi : X \times X \to Y$ for which there exists a constant $M_\phi \geq 0$ such that

$$\|\phi(x, y)\| \leq M_\phi e^{\lambda(\|x\|+\|y\|)}$$

for all $x, y \in X$. We easily see that $[X^2]_\lambda$ is a normed space if it is equipped with the norm

$$\|\phi\| = \sup_{x,y \in X} e^{-\lambda(\|x\|+\|y\|)}\|\phi(x, y)\|.$$

On the other hand, if $m \geq 2$ is an integer, we denote by X_λ^m the normed space

$$X_\lambda^m = \{(f_1, f_2, \ldots, f_m) : f_i \in X_\lambda \text{ for all } i\}$$

with the norm

$$\|(f_1, f_2, \ldots, f_m)\| = \max\{\|f_1\|, \|f_2\|, \ldots, \|f_m\|\}.$$

Received June 6, 2006.

2000 *Mathematics Subject Classification.* Primary 47A30, 47B38; Secondary 39B52.

Key words and phrases. cubic functional equation, cubic operator, operator norm, X_λ space.

This work was supported by 2006 Hongik University Research Fund.

In [1, 2], Czerwik and Dlutek investigated the norms of the Pexiderized Cauchy, quadratic and Jensen operators. These results have been extended in the paper [5]. In fact, Moslehian *et al.* investigated the Pexiderized generalized Jensen and Pexiderized generalized quadratic operators on the function spaces X_λ and gave more general results regarding their norms.

A function $f : X \to Y$ is called a cubic function if and only if f is a solution function of the cubic functional equation

$$\begin{aligned} (1) \qquad & f(x+y) + f(x-y) \\ = \; & 2f\left(\frac{1}{2}x + y\right) + 2f\left(\frac{1}{2}x - y\right) + 12f\left(\frac{1}{2}x\right). \end{aligned}$$

It is known that when X and Y are real normed spaces, a function $f : X \to Y$ is a cubic function if and only if there exists a function $B : X \times X \times X \to Y$ such that $f(x) = B(x, x, x)$ for all $x \in X$, and B is symmetric for each fixed one variable and is additive for fixed two variables (see [3]). Moreover, the author and Kim [4] applied a fixed point method to the proof of the Hyers-Ulam-Rassias stability of the cubic functional equation (1).

We will introduce a linear operator which is constructed from the Pexiderization of the cubic equation (1). We define the cubic operator $C_P : X_\lambda^5 \to [X^2]_\lambda$ by

$$\begin{aligned} & C_P(f, g, h, k, \ell) \\ := \; & f(x+y) + g(x-y) - 2h\left(\frac{1}{2}x + y\right) \\ & - 2k\left(\frac{1}{2}x - y\right) - 12\ell\left(\frac{1}{2}x\right). \end{aligned}$$

In this paper, we will give the exact norm of the cubic operator defined on the function space X_λ.

2. Main results

In the following theorem, we prove that the operator norm of C_P is 18. While it seems easy to prove the validity of the inequality $\|C_P\| \leq 18$, it requires much skill for proving the inverse inequality.

Theorem 2.1. *The cubic operator C_P is a bounded linear operator with*

$$\|C_P\| = 18.$$

Proof. First, we show that $\|C_P\| \leq 18$. Since it holds that

$$\max\left\{ \|x+y\|, \|x-y\|, \left\|\frac{1}{2}x + y\right\|, \left\|\frac{1}{2}x - y\right\|, \left\|\frac{1}{2}x\right\| \right\} \leq \|x\| + \|y\|$$

for all $x, y \in X$, we get

$$
\begin{aligned}
&\|C_P(f,g,h,k,\ell)\| \\
&= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \left\| f(x+y) + g(x-y) - 2h\left(\frac{1}{2}x+y\right) \right. \\
&\qquad \left. - 2k\left(\frac{1}{2}x-y\right) - 12\ell\left(\frac{1}{2}x\right) \right\| \\
&\le \sup_{x,y\in X} e^{-\lambda\|x+y\|}\|f(x+y)\| + \sup_{x,y\in X} e^{-\lambda\|x-y\|}\|g(x-y)\| \\
&\quad + 2\sup_{x,y\in X} e^{-\lambda\|(1/2)x+y\|}\left\|h\left(\frac{1}{2}x+y\right)\right\| \\
&\quad + 2\sup_{x,y\in X} e^{-\lambda\|(1/2)x-y\|}\left\|k\left(\frac{1}{2}x-y\right)\right\| \\
&\quad + 12\sup_{x\in X} e^{-\lambda\|(1/2)x\|}\left\|\ell\left(\frac{1}{2}x\right)\right\| \\
&= \|f\| + \|g\| + 2\|h\| + 2\|k\| + 12\|\ell\| \\
&\le 18\max\{\|f\|,\ \|g\|,\ \|h\|,\ \|k\|,\ \|\ell\|\} \\
&= 18\|(f,g,h,k,\ell)\|
\end{aligned}
$$

for any $(f,g,h,k,\ell) \in X_\lambda^5$. This implies that

$$\|C_P\| \le 18. \tag{2}$$

For a fixed $v \in Y$ with $\|v\| = 1$ and a sequence $\{a_n\}$ of positive numbers decreasing to 0 as $n \to \infty$, we define

$$
f_n(x) = \begin{cases} e^{2\lambda a_n} v & \text{for } \|x\| = 0 \text{ or } 2a_n, \\ -e^{2\lambda a_n} v & \text{for } \|x\| = (1/2)a_n \text{ or } (3/2)a_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}
$$

Then, we have

$$
e^{-\lambda\|x\|}\|f_n(x)\| = \begin{cases} e^{2\lambda a_n} & \text{for } \|x\| = 0, \\ e^{(3/2)\lambda a_n} & \text{for } \|x\| = (1/2)a_n, \\ e^{(1/2)\lambda a_n} & \text{for } \|x\| = (3/2)a_n, \\ 1 & \text{for } \|x\| = 2a_n, \\ 0 & \text{otherwise} \end{cases}
$$

for any $n \in \mathbb{N}$. Thus, we see that $\|f_n\| = e^{2\lambda a_n}$, and hence we conclude that $f_n \in X_\lambda$ for all $n \in \mathbb{N}$.

Now, let $u \in X$ be given with $\|u\| = 1$. If we take $x_n = y_n = a_n u$, then it follows from (3) that

$$\begin{aligned} & \|C_P(f_n, f_n, f_n, f_n, f_n)\| \\ = & \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \left\| f_n(x+y) + f_n(x-y) - 2f_n\left(\frac{1}{2}x + y\right) \right. \\ & \left. - 2f_n\left(\frac{1}{2}x - y\right) - 12f_n\left(\frac{1}{2}x\right) \right\| \\ \geq & e^{-2\lambda a_n} \|e^{2\lambda a_n}v + e^{2\lambda a_n}v + 2e^{2\lambda a_n}v + 2e^{2\lambda a_n}v + 12e^{2\lambda a_n}v\| \\ = & 18 \end{aligned} \tag{4}$$

for each $n \in \mathbb{N}$.

If we assume that $\|C_P\| < 18$, then we can choose a positive constant ε with

$$\|C_P(f_n, f_n, f_n, f_n, f_n)\| \leq (18 - \varepsilon)\|(f_n, f_n, f_n, f_n, f_n)\| \tag{5}$$

for any $n \in \mathbb{N}$. Since $\|f_n\| = e^{2\lambda a_n}$ for all $n \in \mathbb{N}$, it follows from (4) and (5) that

$$\begin{aligned} 18 & \leq \|C_P(f_n, f_n, f_n, f_n, f_n)\| \\ & \leq (18 - \varepsilon)\|(f_n, f_n, f_n, f_n, f_n)\| \\ & = (18 - \varepsilon)\|f_n\| \\ & = (18 - \varepsilon)e^{2\lambda a_n} \\ & \to (18 - \varepsilon) \text{ as } n \to \infty, \end{aligned}$$

which leads to a contradiction. Hence, it yields that $\|C_P\| \geq 18$.

Finally, considering (2), we can conclude that $\|C_P\| = 18$. □

As a variation of C_P, we may consider an operator $C_P^v : X_\lambda^5 \to [X^2]_\lambda$ defined by

$$\begin{aligned} C_P^v(f, g, h, k, \ell) := & f(x+y) + g(x-y) + h\left(\frac{1}{2}x + y\right) \\ & + k\left(\frac{1}{2}x - y\right) + \ell\left(\frac{1}{2}x\right) \end{aligned}$$

and we can apply a similar argument to prove that the operator norm of C_P^v is 5.

Corollary 2.2. *The operator C_P^v is a bounded linear operator with $\|C_P^v\| = 5$.*

Proof. We only need to replace the definition (3) by

$$f_n(x) = \begin{cases} e^{2\lambda a_n} v & \text{for } \|x\| = 0,\ (1/2)a_n,\ (3/2)a_n \text{ or } 2a_n, \\ 0 & \text{otherwise} \end{cases}$$

and follow the exact steps in the proof of Theorem 2.1. □

References

[1] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[2] S. Czerwik and K. Dlutek, *Cauchy and Pexider operators in some function spaces*, Functional equations, inequalities and applications, 11–19, Kluwer Acad. Publ., Dordrecht, 2003.

[3] K.-W. Jun and H.-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 867–878.

[4] S.-M. Jung and T.-S. Kim, *A fixed point approach to the stability of cubic functional equation*, Boletin de la Sociedad Matematica Mexicana **12** (2006), 51–57.

[5] M. S. Moslehian, T. Riedel, and A. Saadatpour, *Norms of operators in X_λ spaces*, Appl. Math. Lett. (2007), doi:10.1016/j.aml.2006.11.009.

Mathematics Section
College of Science and Technology
Hong-Ik University
Chochiwon 339-701, Korea
E-mail address: smjung@hongik.ac.kr