Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 331–336

NESTING POLYNOMIALS IN INFINITE RADICALS

PETER J. HUMPHRIES

ABSTRACT. We consider infinite nested radicals in which the arguments are positive polynomial sequences. It is shown that the evaluation of such a nesting is always finite, and we prove necessary and sufficient conditions for the evaluation to be a finite polynomial.

1. Introduction

A famous problem posed by Ramanujan asks for the evaluation of the infinite nested radical

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}.$$

If we instead try to evaluate a more general expression, where we replace the increasing sequence by an arithmetic progression in x, namely

$$L(x)=\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\cdots}}}}$$

then it can be seen that $L(x)$ satisfies the functional equation

$$L(x)^2=1+xL(x+1).$$

The solution to this is $L(x)=x+1$, giving the evaluation of Ramanujan's example correctly as 3. In fact, this numerical example is merely a special case of a more complicated identity in three variables (see the end of Section 2).

Several identities concerning infinite nested radicals may be found in [1], [2] and [3]. In [1], nested radicals involving arithmetic sequences in n-th roots are considered. The purpose of the current paper is to study the case, where the radicals have two polynomials as their arguments.

Throughout this paper, we denote the natural numbers (without zero) and the real numbers by $\mathbb{N}$ and $\mathbb{R}$ respectively. The ring of polynomials in x with real coefficients will be specified by $\mathbb{R}[x]$, and we note further that any use of square roots automatically implies a positive square root. A sequence a_n of

Received August 28, 2006.
2000 *Mathematics Subject Classification.* 11A99.
Key words and phrases. infinite radical, nested radical.

positive real numbers is called a *positive polynomial sequence* if there exists a polynomial $a(x) \in \mathbb{R}[x]$ such that $a_i = a(i)$ for all $i \in \mathbb{N}$.

To remove the possibility of any ambiguity, we formalize the concept of evaluating an infinite nested radical

$$\sqrt{a_1 + b_1\sqrt{a_2 + b_2\sqrt{a_3 + \cdots}}}$$

to be the limit

$$\lim_{n\to\infty} \sqrt{a_1 + b_1\sqrt{a_2 + \cdots + b_{n-1}\sqrt{a_n + b_n}}}, \tag{1}$$

where a_n, b_n are sequences of real numbers.

In Section 2, we characterize when an infinite nested radical involving polynomials from $\mathbb{R}[x]$ has a simple closed form as another polynomial in $\mathbb{R}[x]$. Section 3 is devoted to proving that, for all positive polynomial sequences a_n, b_n, the limit in (1) exists and is finite.

2. Identities involving nested radicals

The following lemma does not require proof, being a consequence of viewing the infinite nested radical as being a limit of an infinite sequence.

Lemma 2.1. *Let $L(x), p(x), q(x)$ be polynomials in $\mathbb{R}[x]$. Then*

$$L(x) = \sqrt{p(x) + q(x)\sqrt{p(x+d) + q(x+d)\sqrt{p(x+2d) + \cdots}}}$$

if and only if

$$L(x) = \sqrt{p(x) + q(x)L(x+d)}.$$

An analogous statement can be made for higher-order roots, and we may further replace the ring $\mathbb{R}[x]$ by any class of function in one or more variables. However, for the purposes of this paper we are primarily interested in polynomials in one variable.

From the above lemma, we get any number of results. More importantly, though, given a nested radical to evaluate, we can now concentrate on solving the non-linear functional equation

$$L(x)^2 = p(x) + q(x)L(x+d) \tag{2}$$

rather than on the radical itself, where $L(x)$ is assumed to take positive values on the domain of interest.

Given $L(x)$, $q(x)$ and d, we can always find a $p(x)$ that satisfies equation (2). That is, $p(x) = L(x)^2 - q(x)L(x+d)$. A more interesting problem is, given $p(x)$, $q(x)$ and d, to find the function $L(x)$. In particular, we want to find some $L(x)$ that is a polynomial of finite degree.

This is not always possible, as the following example shows. If we take $p(x) = 1$, $q(x) = x$ and $d = 2$, then we wish to find some $L(x)$ that satisfies

$$L(x)^2 = 1 + xL(x+2).$$

It can be seen that the degree of $L(x)$ must be one, and moreover the linear term will be x. However, if we try to evaluate a constant term a, we run into problems:

$$\begin{aligned}(x+a)^2 &= 1 + x(x+2+a)\\ ax + a^2 &= 2x + 1.\end{aligned}$$

Comparing the linear coefficients gives $a = 2$, but the constant terms give the solution $a = \pm 1$.

Our aim is to characterize when an infinite nested radical with polynomial arguments has a polynomial solution. That is, for what combinations of $p(x), q(x)$ and d can we find some $L(x) \in \mathbb{R}[x]$ satisfying equation (2). It is known ([3]) that if both $p(x)$ and $q(x)$ are constants, p and q say, then $L(x)$ is also constant, and solves the quadratic equation $L^2 - qL - p = 0$.

Let $\deg(f)$ denote the degree of a polynomial $f(x)$, and $[x^i]f(x)$ denote the coefficient of x^i in the function $f(x)$. Then we have the following two lemmas, which both follow from equation (2):

Lemma 2.2. *If $L(x) \in \mathbb{R}[x]$ solves equation (2) for some $p(x), q(x) \in \mathbb{R}[x]$, then $\deg(L) = \max\{\frac{\deg(p)}{2}, \deg(q)\}$.*

Lemma 2.3. *Let $p(x), q(x)$ be polynomials in $\mathbb{R}[x]$, and let*

$$F(x) = L(x)^2 - p(x) - q(x)L(x+d),$$

where $L(x) = a_k x^k + \cdots + a_0$. Then there exist $a_0, \ldots, a_k \in \mathbb{R}$ such that $L(x)$ solves equation (2) if and only if there exist $a_0, \ldots, a_k \in \mathbb{R}$ such that $[x^i]F(x) = 0$ for all $i \geq 0$.

This now allows us to find a solution to equation (2) by comparing coefficients of $F(x)$. While in the last lemma it is stated that $[x^i]F(x)$ must be zero for all $i \geq 0$, it suffices by Lemma 2.2 for this to hold only for values of i not exceeding the maximum of $\deg(p)$ and $2\deg(q)$. While, for $L(x)$ of degree k, this could potentially involve solving up to $2k+1$ simultaneous polynomials in the $k+1$ coefficients of $L(x)$, we can use the next lemma to find the solution systematically by solving only one quadratic equation (taking the positive root) and at most k linear equations.

Lemma 2.4. *Let $p(x), q(x)$ be polynomials in $\mathbb{R}[x]$, and let*

$$F(x) = L(x)^2 - p(x) - q(x)L(x+d),$$

where $L(x) = a_k x^k + \cdots + a_0$, and $k = \max\{\frac{\deg(p)}{2}, \deg(q)\}$. Then

(i) *$[x^{2k}]F(x)$ is quadratic in a_k;*
(ii) *$[x^j]F(x)$ is linear in a_{j-k} for all $k \leq j < 2k$; and*

(iii) $[x^j]F(x)$ *is independent of* a_i *for all* $i < j - k$, *where* $k \leq j \leq 2k$.

Proof. The coefficient $[x^{2k}]F(x)$ is given by

$$\begin{aligned}[x^{2k}]F(x) &= ([x^k]L(x))^2 - [x^{2k}]p(x) - ([x^k]q(x))([x^k]L(x+d)) \\ &= a_k^2 - a_k[x^k]q(x) - [x^{2k}]p(x)\end{aligned}$$

proving part (i). Similarly, the coefficient $[x^j]F(x)$, where $k \leq j < 2k$ is

$$\begin{aligned}[x^j]F(x) &= \sum_{i=j-k}^{k} ([x^i]L(x))([x^{j-i}]L(x)) - [x^j]p(x) \\ &\quad - \sum_{i=j-k}^{k} ([x^i]q(x))([x^{j-i}]L(x+d)) \\ &= 2a_{j-k}a_k - a_{j-k}[x^k]q(x) - [x^j]p(x) + g(a_{j-k+1}, \dots, a_k),\end{aligned}$$

where $g(a_{j-k+1}, \dots, a_k)$ takes care of the extra terms in the summations. This proves (ii), and (iii) follows directly from the expansions above. □

We can now prove the main result of this section.

Theorem 2.5. *Let* $p(x), q(x)$ *be polynomials of degree* s, t *respectively in* $\mathbb{R}[x]$, *both with positive leading coefficients. Then there are* $\max\{\frac{s}{2}, t\}$ *equalities that must be satisfied by* d *and the coefficients of* $p(x), q(x)$ *in order for some* $L(x) \in \mathbb{R}[x]$ *that solves equation* (2) *to exist. Moreover, if these equalities are satisfied, then there is a general solution for* $L(x)$ *in terms of* d *and the coefficients of* $p(x), q(x)$.

Proof. We take $L(x), F(x)$ as in Lemma 2.4. Then, by the same lemma, we can find a positive $a_k \in \mathbb{R}$ that solves $[x^{2k}]F(x) = 0$. Further, given $a_i, \dots, a_k$, where $i > 0$, we can find a_{i-1} that solves $[x^{i+k-1}]F(x) = 0$. That is, we can find $a_0, \dots, a_k$ that simultaneously solve $[x^i]F(x) = 0$ for all $k \leq i \leq 2k$.

Now, by Lemma 2.3, for $L(x) \in \mathbb{R}[x]$ to exist, we need $[x^i]F(x) = 0$ for all $i \geq 0$. Since we have this equality for $k \leq i \leq 2k$, we need the remaining k equations to be satisfied. That is, $[x^i]F(x) = 0$ for $0 \leq i < k$. Hence there are k constraints on d and the coefficients of $p(x), q(x)$.

We complete the proof of the theorem by noting that, by Lemma 2.3 $L(x) = a_kx^k + \cdots + a_0$ solves equation (2) if and only if all of the k constraints are met with equality. □

We illustrate the theorem with a more concrete example. If $p(x)$ and $q(x)$ are both linear, then we wish to find $L(x) \in \mathbb{R}[x]$ such that

$$L(x)^2 = (p_1x + p_0) + (q_1x + q_0)L(x+d),$$

where we assume that both p_1 and q_1 are non-zero. In this case, it can be seen that $L(x)$ is of the form $a_1x + a_0$, and that in fact $a_1 = q_1$. So we have

$$(q_1x + a_0)^2 = (p_1x + p_0) + (q_1x + q_0)(q_1x + q_1d + a_0)$$
$$a_0q_1x + a_0^2 = (q_1q_0 + q_1^2d + p_1)x + (q_1q_0d + a_0q_0 + p_0).$$

By comparing the linear terms, we get $a_0 = q_0 + q_1d + \frac{p_1}{q_1}$, which on substitution into the constant terms gives

$$0 = (q_1^2d + p_1)^2 + q_1(p_1q_0 - p_0q_1). \tag{3}$$

That is, the solution $L(x) \in \mathbb{R}[x]$ exists if and only if equation 3 holds, in which case

$$L(x) = q_1x + \left(q_0 + q_1d + \frac{p_1}{q_1}\right).$$

The identity of Ramanujan's, which we alluded to in the introduction, is

$$x + n + a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + \cdots}},$$

where $p(x) = ax + (n+a)^2$, $q(x) = x$ and $d = n$. Applying the results we have just derived we find that the constraint in equation (3) is indeed satisfied, and the evaluation of the nested radical is $x + n + a$ as expected.

3. Convergence of nested radicals

At this point, we introduce a more compact notation for nested radicals. For two sequences a_n, b_n of positive real numbers, we define the operator $\mathbf{R}$ by

$$\mathbf{R}_{i=1}^n(a_i, b_i) = \sqrt{a_1 + b_1\sqrt{a_2 + \cdots + b_{n-1}\sqrt{a_n + b_n}}}.$$

It was proved by Herschfeld ([2]) that $\mathbf{R}_{i=1}^n(a_i, 1)$ converges if $a_n^{2^{-n}}$ has a finite upper limit as n tends to infinity.

Let p_n, q_n be positive polynomial sequences, and let the sequence r_n be given by

$$r_n = \mathbf{R}_{i=1}^n(p_i, q_i).$$

Then we wish to find whether or not r_n converges. The next lemma will be of use.

Lemma 3.1. *Let u_n, v_n, y_n, z_n be sequences of positive real numbers such that $u_i \leq y_i, v_i \leq z_i$ for all $i \in \mathbb{N}$. Then for all $n \in \mathbb{N}$*

$$\mathbf{R}_{i=1}^n(u_i, v_i) \leq \mathbf{R}_{i=1}^n(y_i, z_i).$$

Proof. The result is a straight-forward consequence of the sequences being strictly positive. □

Theorem 3.2. *Let p_n, q_n be positive polynomial sequences. Then the sequence $r_n = \mathbf{R}_{i=1}^n(p_i, q_i)$ converges.*

Proof. Let $p(x), q(x) \in \mathbb{R}[x]$ be polynomials such that $p(i) = p_i, q(i) = q_i$ for all $i \in \mathbb{N}$, and let $m \in \mathbb{N}$ be such that $2m > \deg(p), m > \deg(q) + 1$. Then let $L(x) = x^m$ and $v(x) = x^{m-1}$, and define $u(x)$ by

$$\begin{aligned} u(x) &= L(x)^2 - v(x)L(x+1) \\ &= x^{2m} + O(x^{2m-1}). \end{aligned}$$

We further define the sequences u_n, v_n by $u_i = u(i), v_i = v(i)$. Then there is some $k \in \mathbb{N}$ such that $p_j \leq u_j, q_j \leq v_j$ for all $j \geq k$. Hence, by Lemmas 2.1 and 3.1 , we have

$$\begin{aligned} \lim_{n\to\infty} \mathbf{R}_{i=k}^{n}(p_i, q_i) &\leq \lim_{n\to\infty} \mathbf{R}_{i=k}^{n}(u_i, v_i) \\ &= L(k) \\ &= k^m. \end{aligned}$$

This provides a finite upper bound on r_n by applying Lemma 3.1 again with the finite sequences $\langle p_1, \ldots, p_{k-1} \rangle$ and $\langle q_1, \ldots, q_{k-2}, k^m q_{k-1} \rangle$.

Now, there is also some $k \in \mathbb{N}$ such that $p_j + q_j > 1$ for all $j \geq k$. That is

$$\mathbf{R}_{i=1}^{j}(p_i, q_i) \leq \mathbf{R}_{i=1}^{j+1}(p_i, q_i)$$

for all $j \geq k$, and hence r_n converges to some finite limit. □

References

[1] J. M. Borwein and G. de Barra, *Nested radicals*, Amer. Math. Monthly **98** (1991), no. 8, 735–739.

[2] A. Herschfeld, *On Infinite Radicals*, Amer. Math. Monthly **42** (1935), no. 7, 419–429.

[3] M. McGuffin and B. Wong, *The Museum of Infinite Nested Radicals*, `http://www.dgp.toronto.edu/~mjmcguff/math/nestedRadicals.html`

Department of Mathematics and Statistics
University of Canterbury
Christchurch, New Zealand
E-mail address: `pjh96@student.canterbury.ac.nz`