Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 351–358

PERTURBATION OF NONHARMONIC FOURIER SERIES AND NONUNIFORM SAMPLING THEOREM

Hee Chul Pak and Chang Eon Shin

Abstract. For an entire function f whose Fourier transform has a compact support confined to $[-\pi, \pi]$ and restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$, we derive a nonuniform sampling theorem of Lagrange interpolation type with sampling points $\lambda_n \in \mathbb{R}, n \in \mathbb{Z}$, under the condition that

$$\limsup_{n\to\infty} |\lambda_n - n| < \frac{1}{4}.$$

1. Introduction

Any function $\hat{f}$ which has a compact support in $[-\pi, \pi]$ and belongs to $L^2[-\pi, \pi]$ can be expanded in terms of the orthonormal basis $\left\{\frac{e^{inx}}{\sqrt{2\pi}} : n \in \mathbb{Z}\right\}$ as

$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} \frac{e^{inx}}{\sqrt{2\pi}}, \tag{1.1}$$

where $\langle g, h\rangle_{L^2[-\pi,\pi]} = \int_{-\pi}^{\pi} g(x)\overline{h(x)}dx$. Taking the inverse Fourier transform in (1.1), we obtain the classical Whittaker-Shannon-Kotel'nikov [WSK] sampling theorem,

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}(t-n), \tag{1.2}$$

where $\mathrm{sinc}\, t = \frac{\sin \pi t}{\pi t}$ is the cardinal sinc function and f is the inverse Fourier transform of the function $\hat{f}$. This classical theorem expresses the possibility of recovering a certain kind of signals from a sequence of *regularly spaced* samples. From many practical points of view it is necessary to develop sampling theorems for a sequence of samples taken with a *nonuniform* distribution along the real line. The first answer for this direction was given by Paley and Wiener ([7, p.108]), and later an advanced result was presented by Levinson ([6]). The result is, in fact, related with the perturbation of a Hilbert basis $\{e^{inx}\}_{n\in\mathbb{Z}}$

Received September 27, 2006.
2000 *Mathematics Subject Classification.* 94A20.
Key words and phrases. Riesz basis, frame, nonuniform sampling, nonharmonic Fourier series.

for the function space $L^2[-\pi,\pi]$ in such a way that the perturbed sequence $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is also a Riesz basis for the same space. The *maximum* perturbation of the system $\{e^{inx}\}_{n\in\mathbb{Z}}$ (in the sense of the following theorem) is found by Kadec ([5]) whose result is commonly referred to as the *Kadec's "One-quarter" theorem.* It states that:

Theorem 1.1 (Kadec). *Let $\{\mu_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ satisfying*

$$|\mu_n - n| \le D < \frac{1}{4} \quad \text{for any } n\in\mathbb{Z}. \tag{1.3}$$

Then the set $\{e^{i\mu_n x} : n\in\mathbb{Z}\}$ is a Riesz basis for $L^2[-\pi,\pi]$.

The number 1/4 is the best possible constant for the set $\{e^{i\mu_n x} : n\in\mathbb{Z}\}$ to be a Riesz basis ([8, p. 44]). Despite the fact that 1/4 is the best possible constant, the Kadec's One-quarter theorem does not argue that a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ beyond the Kadec's condition (1.1) can not be a Riesz basis for $L^2[-\pi,\pi]$. Depending on the above theorem, nonuniform sampling expansion of band-limited function with finite energy is possible ([4]).

In this paper, we prove that if $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of distinct real numbers satisfying

$$\limsup_{n\to\infty} |\lambda_n - n| < \frac{1}{4}, \tag{1.4}$$

then the set $\{e^{i\lambda_n x} : n\in\mathbb{Z}\}$ becomes a Riesz basis. Here we note that finitely many sampling points may not satisfy (1.3) and be arbitrarily distributed. Based on the above result, we generalize the Paley-Wiener-Levinson irregular sampling theorem ([3], [6]) to say that any $[-\pi,\pi]$-bandlimited function f whose restriction on $\mathbb{R}$ belongs to $L^2(\mathbb{R})$ can be recovered from its sample values $\{f(\lambda_n)\}_{n\in\mathbb{Z}}$ based on irregular sampling points $\{\lambda_n\}_{n\in\mathbb{Z}}$ along the real line with the condition (1.4): To put it explicitly,

$$f(t) = \sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{L(t)}{L'(\lambda_n)(t-\lambda_n)},$$

where the series converges uniformly on any bounded horizontal strips of $\mathbb{C}$ and

$$L(t) = (t-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{\lambda_n}\right)\left(1-\frac{t}{\lambda_{-n}}\right).$$

2. Preliminaries

We introduce some basic terminologies related to Riesz bases and frames on a Hilbert space. Let H be a separable Hilbert space. Two bases $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ for H are said to be *equivalent* if there exists a bounded linear bijection $T : H \to H$ such that $Ty_n = x_n$ for all $n\in\mathbb{Z}$. A basis $\{x_n : n\in\mathbb{Z}\}$ for H is called a *Riesz basis* if it is equivalent to an orthonormal basis $\{e_n : n\in\mathbb{Z}\}$ for

H. A sequence of distinct vectors $\{x_n\}_{n\in\mathbb{Z}}$ in H is said to be a *frame* if there exist positive constants A and B such that

$$A\|f\|_H^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, x_n\rangle|^2 \leq B\|f\|_H^2 \tag{2.1}$$

for any $f \in H$. We note that every Riesz basis for H is a frame, but not conversely. A frame that ceases to be a frame when any one of its elements is removed is said to be an *exact frame*. It is well-known that a subset of a Hilbert space is an exact frame if and only if it is a Riesz basis. The proofs and complete discussions of these matters can be found in [8].

The following auxiliary Lemma concerning the nonharmonic Fourier series $\{e^{i\mu_n x}\}_{n\in\mathbb{Z}}$ presents a criterion on Riesz bases for the Hilbert space $L^2[-\pi,\pi]$, which is useful for the proof of Theorem 3.2.

Lemma 2.1 ([1, p. 353]). *If $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a frame in $L^2[-a,a]$ but fails to be a frame in $L^2[-a,a]$ by the removal of some function of the set, then it fails to be a frame in $L^2[-a,a]$ by the removal of any function of the set.*

We next define a space of functions whose expansions as sampling series will be discussed.

The function $\hat{f}$ represents the Fourier transform of f defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x)dx$$

when the integral exists in some sense. We denote by B_π^2 the space of all entire functions f such that the support of $\hat{f}$ is contained in $[-\pi,\pi]$ and the restriction of f to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. For an entire function f it follows from the Paley-Wiener Theorem in [2] that $f \in B_\pi^2$ if and only if $f|_{\mathbb{R}} \in L^2(\mathbb{R})$ and there exists $A > 0$ such that

$$|f(z)| \leq Ae^{\pi|y|} \quad \text{for any } z = x+iy \text{ in } \mathbb{C}. \tag{2.2}$$

An entire function f satisfying (2.2) is called a $[-\pi,\pi]$-bandlimited function. If $f \in B_\pi^2$, then by the Hölder inequality we can easily see that

$$|f(z)| \leq \|\hat{f}\|_{L^2[-\pi,\pi]} e^{\pi|y|} \qquad \text{for any } z = x+iy. \tag{2.3}$$

An entire function f is said to be a function of finite order if there exist a positive constant a and $r_0 > 0$ such that $|f(z)| < e^{|z|^a}$ for $|z| > r_0$, and the number $\rho = \inf\{a > 0 : |f(z)| < e^{|z|^a}$ for $|z|$ sufficiently large$\}$ is called the *order* of f.

Theorem 2.2 (Hadamard). *Let $f(z)$ be an entire function of order ρ,*

$$\{\lambda_n\}_{n=-\infty}^{\infty}$$

be its zeros, and $\lambda_n \neq 0$ for $n \neq 0$. Let p be the smallest integer for which the series

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|\lambda_n|^{p+1}}$$

converges, where the prime indicates the summation over nonzero values of n. *Then*

$$f(z) = z^m e^{P(z)} \prod_{n\in\mathbb{Z}\setminus\{0\}} E\left(\frac{z}{\lambda_n}; p\right),$$

where m *is the multiplicity of the zero at the origin,* $P(z)$ *is a polynomial of degree not exceeding* ρ *and* $E(u;p) = (1-u)e^{u+u^2/2+\cdots+u^p/p}$ *if* p *is a positive integer, and* $E(u;0) = 1-u$.

3. Stability of nonharmonic Fourier series

We establish the stability of a Riesz basis consisting of nonharmonic Fourier series, that is, if $\{e^{i\mu_n x} : n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[-\pi,\pi]$ and a sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ satisfies the condition (1.4), then the set $\{e^{i\lambda_n x} : n \in \mathbb{Z}\}$ is also a Riesz basis.

Lemma 3.1. *Let* $\{e^{i\mu_n x} : n \in \mathbb{Z}\}$ *be a Riesz basis of* $L^2[-\pi,\pi]$, $S = \{n_k : k = 1,\dots,\ell\}$ *be a finite subset of* $\mathbb{Z}$ *and let* $\{\tilde{\mu}_{n_k}\}_{k=1}^{\ell}$ *be a finite sequence of distinct complex numbers such that* $\tilde{\mu}_{n_k} \neq \mu_n$ *for any* $n \in \mathbb{Z}$ *and* $k = 1,\dots,\ell$. *Then the set*

$$\{e^{i\mu_n x} : n \in \mathbb{Z}\setminus S\} \cup \{e^{i\tilde{\mu}_{n_k} x} : k = 1,\dots,\ell\}$$

is also a Riesz basis.

Proof. Since the system $\{e^{i\mu_n x} : n \in \mathbb{Z}\}$ is a frame, so is $\{e^{i\mu_n x} : n \in \mathbb{Z}\} \cup \{e^{i\tilde{\mu}_{n_1} x}\}$. Then by virtue of Lemma 2.1, the set $\mathcal{A} = \{e^{i\mu_n x} : n \in \mathbb{Z}\setminus\{n_1\}\} \cup \{e^{i\tilde{\mu}_{n_1} x}\}$ is a frame since $\{e^{i\mu_n x} : n \in \mathbb{Z}\}$ is a frame. The fact that $\mathcal{A}\setminus\{e^{i\tilde{\mu}_1 x}\}$ is not a frame illustrates that $\mathcal{A}$ should be, in fact, an exact frame and hence a Riesz basis. The result follows by applying the same argument repeatedly. □

As an application of Lemma 3.1, we present a nonuniform sampling theorem with sampling points beyond the Kadec's $\frac{1}{4}$-condition. We consider a sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ of distinct real numbers satisfying the condition (1.4) which is equivalent to the condition that there exists $N \in \mathbb{N}$ such that

$$D_N = \sup_{|n|\geq N} |\lambda_n - n| < \frac{1}{4}. \tag{3.1}$$

By the Kadec's One-quarter theorem (Theorem 1.1) and Lemma 3.1, we have:

Theorem 3.2 (Perturbation of nonharmonic Fourier series). *Let* $\{\lambda_n\}_{n\in\mathbb{Z}}$ *be a sequence of distinct real numbers satisfying* (3.1) *for some* $N \in \mathbb{N}$. *Then the set* $\{e^{i\lambda_n x} : n \in \mathbb{Z}\}$ *is a Riesz basis for* $L^2[-\pi,\pi]$.

Proof. Define a sequence $\{\mu_n\}$ by

$$\mu_n = \begin{cases} n & \text{if } |n| < N, \\ \lambda_n & \text{if } |n| \geq N. \end{cases}$$

By Theorem 1.1, the set $\{e^{i\mu_n x} : n \in \mathbb{Z}\}$ is a Riesz basis of $L^2[-\pi,\pi]$. Let $S = \{n : |n| < N\}$. It follows from Lemma 3.1 that the set $\{e^{i\lambda_n x} : n \in \mathbb{Z}\}$ is a Riesz basis. □

For a sequence $\{\lambda_n\}$ satisfying (3.1) we define an infinite product $L(z)$ by

$$L(z) = (z - \lambda_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\lambda_{-n}}\right).$$

Here we assume that each λ_n, $n \neq 0$, is non-zero. Then L is a well-defined entire function, and satisfies the following Lemma.

Lemma 3.3. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of distinct real numbers satisfying (3.1) for some $N \in \mathbb{N}$ and $\lambda_n \neq 0$ for $n \neq 0$. There exists $C > 0$ such that for any $z = x + iy$ $(x, y \in \mathbb{R})$ with $|y| \geq 1$,*

$$|L(z)| \geq C \frac{e^{\pi|y|}}{(1+|z|)^{4D_N+1}}.$$

Proof. Define a rational function

$$Q(z) = \frac{z}{z - \lambda_0} \prod_{n=1}^{N-1} \frac{\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)}{\left(1 - \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\lambda_{-n}}\right)}.$$

From Lemma 16.1 in [6], there exists a constant $C_0 > 0$ such that

$$|Q(z)L(z)| \geq C_0 \frac{|y| e^{\pi|y|}}{(1+|z|)^{4D_N+1}}, \quad z = x + iy.$$

Since the function $\frac{|y|}{|Q(x+iy)|}$ is bounded below by some positive constant $C_1 > 0$ on the set $\{x + iy : |y| \geq 1\}$, we get

$$|L(z)| \geq C_0 \frac{|y|}{|Q(z)|} \frac{e^{\pi|y|}}{(1+|z|)^{4D_N+1}} \geq C \frac{e^{\pi|y|}}{(1+|z|)^{4D_N+1}}, \qquad |y| \geq 1,$$

where $C = C_0 C_1$. □

Now we have the following nonuniform sampling theorem of the Lagrange type interpolation series with finitely many free sampling points.

Theorem 3.4. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of distinct real numbers satisfying (3.1) for some $N \in \mathbb{N}$ and $\lambda_n \neq 0$ for $n \neq 0$. For any $f \in B_\pi^2$, f can be recovered from its sample values $\{f(\lambda_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange type interpolation series*

$$f(z) = \sum_{n=-\infty}^{\infty} f(\lambda_n) \frac{L(z)}{L'(\lambda_n)(z - \lambda_n)},$$

where the series converges uniformly in any bounded horizontal strips of $\mathbb{C}$.

Proof. By Theorem 3.2, the set $\{e^{-i\lambda_n x} : n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[-\pi,\pi]$. It admits a biorthogonal basis $\{h_n(x) : n \in \mathbb{Z}\}$, that is, for every $m, n \in \mathbb{Z}$,

$$\langle h_n, e^{-i\lambda_m x}\rangle_{L^2[-\pi,\pi]} = \delta_{nm}. \tag{3.2}$$

Thus, every $\hat{f} \in L^2[-\pi,\pi]$ can be written in terms of h_n as

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \langle \hat{f}, e^{-i\lambda_n \xi}\rangle_{L^2[-\pi,\pi]} h_n(\xi) \qquad \text{in } L^2[-\pi,\pi], \tag{3.3}$$

that is,

$$\lim_{n\to\infty} \hat{H}_m(\xi) = 0 \qquad \text{in } L^2[-\pi,\pi], \tag{3.4}$$

where $\hat{H}_m(\xi) = \hat{f}(\xi) - \sum_{|n|\le m} \langle \hat{f}, e^{-i\lambda_n \xi}\rangle_{L^2[-\pi,\pi]} h_n(\xi)$. Let

$$g_n(z) = \mathcal{F}^{-1}(\sqrt{2\pi} h_n \chi_\pi)(z) = \int_{-\pi}^{\pi} h_n(\xi) e^{i\xi z} d\xi,$$

where χ_π is the characteristic function of the interval $[-\pi,\pi]$. Taking the inverse Fourier transform in (3.3), we have

$$f(z) = \sum_{n=-\infty}^{\infty} f(\lambda_n) g_n(z), \tag{3.5}$$

where the series converges in $L^2(\mathbb{R})$. Note that $h_n\chi_\pi \in L^2[-\pi,\pi]$ and $g_n \in B^2_\pi$. By (3.2), each λ_m is a zero of the function g_n for $m \neq n$. Suppose that there exists a zero μ of g_n other than λ_m, $m \neq n$. By Lemma 3.1 the set $\{e^{-i\lambda_m x} : m \in \mathbb{Z} \setminus \{n\}\} \cup \{e^{-i\mu x}\}$ is a frame over $[-\pi,\pi]$. Hence there exists $A > 0$ such that

$$\begin{aligned} A\|h_n\|^2_{L^2[-\pi,\pi]} &\le \sum_{m\in\mathbb{Z}\setminus\{n\}} |\langle h_n, e^{-i<\lambda_m,\,\cdot>}\rangle_{L^2[-\pi,\pi]}|^2 + |\langle h_n, e^{-i<\mu,\,\cdot>}\rangle_{L^2[-\pi,\pi]}|^2 \\ &= \sum_{m\in\mathbb{Z}\setminus\{n\}} |g_n(\lambda_m)|^2 + |g_n(\mu)|^2 = 0, \end{aligned}$$

which says $h_n \equiv 0$. This contradiction implies that λ_m, $m \neq n$, are the only zeros of g_n.

Since $g_n \in B^2_\pi$, it follows from (2.2) there exists $C_n > 0$ such that

$$|g_n(z)| \le C_n e^{\pi|y|} \qquad z = x + iy. \tag{3.6}$$

Observing that

$$\sum_{n\neq 0} \frac{1}{|\lambda_n|} = \infty \quad \text{and} \quad \sum_{n\neq 0} \frac{1}{|\lambda_n|^2} < \infty,$$

by virtue of the Hadamard's Factorization theorem we have, for $n \neq 0$ and $z \neq \lambda_n$,

$$g_n(z) = e^{P_n(z)}(z-\lambda_0)\prod_{m\neq n}\left(1-\frac{z}{\lambda_m}\right)e^{\frac{z}{\lambda_m}}$$

$$(3.7) \qquad = e^{P_n(z)-\frac{z}{\lambda_n}}\frac{z-\lambda_0}{1-z/\lambda_n}\prod_{m\in\mathbb{N}}\left(1-\frac{z}{\lambda_m}\right)\left(1-\frac{z}{\lambda_{-m}}\right)e^{z\left(\frac{1}{\lambda_m}+\frac{1}{\lambda_{-m}}\right)}$$

$$(3.8) \qquad = e^{P_n(z)+\alpha z}\frac{z-\lambda_0}{1-z/\lambda_n}\prod_{m\in\mathbb{N}}\left(1-\frac{z}{\lambda_m}\right)\left(1-\frac{z}{\lambda_{-m}}\right)$$

$$(3.9) \qquad = e^{P_n(z)+\alpha z}\frac{L(z)}{1-z/\lambda_n},$$

where $P_n(z)$ is a linear or a constant function and $\alpha = \sum_{m\in\mathbb{N}}\left(\frac{1}{\lambda_m}+\frac{1}{\lambda_{-m}}\right)-\frac{1}{\lambda_n} \in \mathbb{R}$. For the case when $n = 0$, the factor $\frac{z-\lambda_0}{1-z/\lambda_n}$ in (3.7) $\sim$ (3.9) ought to be disappeared and $g_0(z) = e^{P_0(z)+\alpha z}\frac{L(z)}{z-\lambda_0}$. We observe that $P_n(z)+\alpha z$ is a constant. Indeed, by Lemma 3.3 and (3.6) there exists $D_n > 0$ such that

$$(3.10) \qquad C_n \geq D_n \frac{|e^{P_n(z)+\alpha z}|}{|z-\lambda_n|(1+|z|)^{4D_N+1}},$$

for large z. The function on the right side of (3.10) is unbounded unless $P_n(z)+\alpha z$ is a constant. Hence we get

$$g_n(z) = A_n\frac{L(z)}{z-\lambda_n}$$

for some constant A_n. Since $g_n(\lambda_n) = 1$, we have $A_n = 1/L'(\lambda_n)$ and in view of (3.5)

$$(3.11) \qquad f(z) = \sum_{n=-\infty}^{\infty} f(\lambda_n)\frac{L(z)}{L'(\lambda_n)(z-\lambda_n)},$$

where the series converges in $L^2(\mathbb{R})$. Observing by (3.4) that $\hat{H}_m$ converges to 0 as $m \to \infty$ in $L^2[-\pi,\pi]$, $H_m(z) = \mathcal{F}^{-1}(\hat{H}_m)(z) \in B^2_\pi$ and $H_m(z) = f(z) - \sum_{|n|\leq m} f(\lambda_n)\frac{L(z)}{L'(\lambda_n)(z-\lambda_n)}$, it follows from (2.2) that the series in (3.11) converges uniformly to $f(z)$ on any bounded horizontal strip of $\mathbb{C}$. □

References

[1] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[2] I. M. Gel'fand and G. E. Shilov, *Generalized functions. Vol. I: Properties and operations*, Translated by Eugene Saletan Academic Press, New York-London, 1964.

[3] J. R. Higgins, *Sampling Theoy in Fourier and Signal Analysis*, Oxford University Press Inc., New York, 1996.

[4] ———, *A sampling theorem for irregularly spaced sample points*, IEEE Trans. Information Theory **IT-22** (1976), no. 5, 621–622.

[5] M. I. Kadec, *The exact value of the Paley-Wiener constant*, Dokl. Akad. Nauk SSSR **155** (1964), 1253–1254.

[6] N. Levinson, *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940.

[7] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Reprint of the 1934 original. American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI, 1987.

[8] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, 93. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

Hee Chul Pak
Department of Applied Mathematics
Dankook University
Chungnam 330-714, Korea
E-mail address: hpak@dankook.ac.kr

Chang Eon Shin
Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: shinc@sogang.ac.kr