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PERSISTENCE OF PERIODIC TRAJECTORIES OF PLANAR SYSTEMS UNDER TWO PARAMETRIC PERTURBATIONS

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Abstract. We consider a two parametric family of the planar systems with the form

$$\begin{aligned} \dot{x} &= P(x,y) + \epsilon_1 p_1(x,y) + \epsilon_2 p_2(x,y), \\ \dot{y} &= Q(x,y) + \epsilon_1 q_1(x,y) + \epsilon_2 q_2(x,y), \end{aligned}$$

where the unperturbed equation ($\epsilon_1 = \epsilon_2 = 0$) is assumed to have at least one periodic solution or limit cycle. Our aim here is to study the behavior of the system under two parametric perturbations; in fact, using the Poincare - Andronov technique, we impose conditions on the system which guarantee persistence of the periodic trajectories. At the end, we apply the result on the Van der Pol equation ; where, we consider the effect of nonlinear damping on the equation. Also the Hopf bifurcation for the Van der Pol equation will be investigated.

1. Introduction

Planar systems with periodic trajectories have many applications in physics and engineering; such systems could be a one-freedom Hamiltonian system and two dimensional oscillator system see [4]. There are many literature concerning the one parametric perturbations of the linear and nonlinear oscillator systems, such as ([1], [2], [15]). Recently, Chicone, in his papers [5], [6], [7], [8], [9] considered the system

$$\dot{x} = f(x) + \epsilon g(x,t), \qquad \epsilon \in R, \quad x \in R^n, \tag{1}$$

and supposed that the phase space of the unperturbed system has a annual region of the periodic solutions or it has a limit cycle Γ. The method, he implemented, gives a new formula for the Melnikov integral and conditions for persistence of the periodic solutions near resonance, that is, there exists a one-parameter family Γ_ϵ of periodic solutions emerging from Γ. In this paper, we extends his results for two parametric perturbations. Also, we apply the method for the Van der Pol equation. The second section is devoted to the

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definitions and theorems which are going to be used in the other sections. The two parametric perturbations are considered in section three, where we give conditions for the persistence of the periodic orbits. In the last section we apply the method on the Van der Pol equation, we show that how the nonlinear damping effects on the equation.

2. Preliminary

Consider a parametric family of planar vector fields $(x, y) \mapsto V(x, y, \epsilon) = V_\epsilon$, where ϵ is small parameter(s). The phase space of the unperturbed system $V_0(x, y)$ contains a annual periodic orbits or a limit cycle Γ. The Poincare-Andronov theorem state that a hyperbolic fixed point of the Poincare map is corresponding to a stable or unstable periodic orbit of the vector field (see [12], [14], [16]). Therefore the fixed points of the Poincare map corresponding to the system $V_\epsilon(x, y)$ provide us with a family of the periodic orbits Γ_ϵ with $\Gamma_0 = \Gamma$.

Since Diliberto's theorem contains the important information about the solutions of the linear variation equation along the trajectories of the plane vector field V, here we will give the theorem, for the proof and more information about the theorem see [5, 10].

Diliberto's theorem; Consider a smooth planar vector field $V = (V_1, V_2)$, we define

$$
\begin{aligned}
V^\perp &= (-Q, P), \\
K &= ||V||^{-3}(P\dot{Q} - Q\dot{P}), \\
\text{Curl } V &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \\
\operatorname{div} V &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \\
U_{V^\perp} &= \frac{1}{||V||^2} V^\perp,
\end{aligned}
$$

where $\dot{V} = \frac{dV}{dt}$. If $V(p) \neq 0$, then the fundamental matrix of the first linear variational equation along the integral curve $t \mapsto \phi_p(t)$, i.e., $\dot{X} = DV(\phi_p(t))X$, has the form $\Phi(t) = [V(\phi_p(t)), \bar{X}(t)]$, where

$$
\begin{aligned}
&\bar{X}(t) = \alpha(t)V(\phi_p(t)) + \beta(t)U_{V^\perp}(\phi_p(t)), \\
(2) \qquad &\alpha(t) = \textstyle\int_0^t \{\frac{1}{||V||}(2K||V|| - \text{Curl } V\}(\phi_s(p))\beta(s)ds, \\
&\beta(t) = \exp(\textstyle\int_0^t \operatorname{div} V(\phi_s(p))ds).
\end{aligned}
$$

The theorem was proven by Diliberto [10], but Chicone modified the proof of it in [5].

This theorem illustrate the important result about the *geometric* meaning of the two functions α, β that are defined in the theorem which we state in the following theorem. Before we state the theorem, we need to introduce some definitions. Let p and q be points in R^2 on the trajectory $t \longmapsto \phi_t(p)$ of V with $q = \phi_\tau(p)$. We consider two sections for the flow of V, Σ at p and Δ at q, given by plane curves $s \longmapsto \sigma(s)$ and $s \longmapsto \delta(s)$ with $\sigma(x_0) = p$. Moreover, we let Y denote the tangent vector field of σ and Z the tangent vector field of δ. We use $\hat{h} : \Sigma \longmapsto \Delta$ to denote the *transition map*. It assigns to a point $r \in \Sigma$ the point where the trajectory of V starting from r first meets Δ. We use $\hat{T} : \Sigma \longmapsto R$ to denote the *transition time function* that assigns to $r \in \Sigma$ the minimum positive time required for the transition. There will be an open interval $I \subset R$ such that for $s \in I$ both $\hat{h}$ and $\hat{T}$ are define on $\sigma(I)$. We define $J := \delta^{-1} \circ \hat{h} \circ \sigma(I)$. Then, we can express the transition map and the transition time function in the local coordinates on the two sections defined by the functions σ and δ. In fact we define the *scalar transition map* $h : I \longmapsto J$ by the formula

$$h := \delta^{-1} \circ \hat{h} \circ \sigma,$$

and the *scalar transition time function* $T : I \longmapsto R$ by

$$T := \hat{T} \circ \sigma.$$

If p is a periodic point of V we can take $\Sigma = \Delta$. In this case, the transition map is called the *return map* or the *Poincare map* on the *Poincare section* Σ.

Now we state the Variation lemma which gives us an explicit formula for the solution of the inhomogeneous linear variation equation along a trajectory of V; this formula will need for the perturbed vector field, which we will give it in the next section.

Variation Lemma; Let $V = (P(x,y), Q(x,y))$ be a smooth planar vector field and ϕ_t be the flow of V. If $p \in R^2$, and $V(p) \neq 0$, then the solution $t \mapsto W(t)$ of the initial value problem

$$\dot{W} = DV(\phi_p(t))W + B(\phi_p(t)), \qquad W(0) = 0\ , \quad B = (B_1, B_2) \tag{3}$$

is given by

$$W(t) = (N(t) + \alpha(t)M(t))\, V(\phi_p(t)) + \beta(t)M(t)U_{V^\perp}(\phi_p(t)), \tag{4}$$

where $\langle .,. \rangle$ is the inner product of vectors and

$$N(t) = \int_0^t \{\tfrac{1}{||V||}\langle V, B\rangle - \tfrac{\alpha(s)}{\beta(s)} V \wedge B\}(\phi_s(p))ds,$$

$$M(t) = \int_0^t \{\tfrac{1}{\beta(s)} V \wedge B\}(\phi_s(p))ds, \tag{5}$$

$$V \wedge B = PB_2 - QB_1.$$

Here the functions $\alpha(t)$ and $\beta(t)$ are defined in the statement of Diliberto's theorem. In addition, if m is a positive number, p is a point in the periodic annual of V with local section Σ given at p by the integral curve $s \mapsto \sigma(s)$ of vector field $V^\perp$ with $\sigma(x_0) = p$ and T denotes the scalar period function defined on Σ, then $\alpha(mT(x)) = -\frac{m}{||V(p)||^2}T^{'}(x)$ and $\beta(mT(x)) = 1$; furthermore,

$$
\begin{aligned}
(6) \qquad W(mT(x)) &= \left(N(mT(x)) - \frac{m}{||V(p)||^2}T^{'}(x)\right) M(mT(x))V(p) \\
&\quad + M(mT(x))U_{V^{\perp}}(x).
\end{aligned}
$$

For proof see [5].

Let us consider the planar system V_ϵ, where $\epsilon \in R^2, \epsilon = (\epsilon_1, \epsilon_2)$. Also let ϕ_t^ϵ denote the flow of V_ϵ. We can always arrange the coordinates so that a certain horizontal line segment $\Sigma : y = y_0$ is transverse to the flow of V_0 in a annual region A. Then there is some $\epsilon_0 > 0$ such that $(x, y) \longmapsto V(x, y, \epsilon)$ is transverse to Σ for all ϵ satisfying $|\epsilon| < \epsilon_0$. We suppose that Γ is one of the periodic trajectories in A transverse to Σ and let x denote the usual distance coordinate along Σ. Then, both scalar transition time function $(x, \epsilon) \longmapsto T(x, \epsilon)$ and the scalar Poincare return map$(x, \varepsilon) \longmapsto h(x, \epsilon)$ are defined on Σ. Assume that $V_0^{\perp}$ to be the vector field perpendicular to the vector field V_0, so $V_0^{\perp} = (-Q, P)$. We define $H(x, \epsilon)$ to be the vector $(d(x, \epsilon), 0)$, where the function d is the *displacement function* define by $d = d(x, \varepsilon) = h(x, \varepsilon) - x$, it is obvious that the zeroes of the function d are the fixed points of the Poincare map corresponding to the system V_ϵ. Now we define the *normalized displacement function* F by

$$F = F(x, \epsilon) = V_0^{\perp}(x, y_0) \cdot H(x, \epsilon) = -Q(x, y_0)d(x, \epsilon).$$

Remark. Note that we always can chose y_0 in away that $Q(x, y_0) \neq 0$ in a neighborhood of point $x_0 \in \Sigma$. Therefore $F(x, \epsilon) = 0$ if and only if the trajectory $(x, y) \longmapsto V(x, y, \epsilon)$ through (x, ϵ) is periodic and $F(x, 0) \equiv 0$. Hence for finding a family of the periodic orbits for the system V_ϵ, we look for the zeroes of the function F which are corresponding to the zeroes of the function d.

In the rest of this section, we briefly explain the Hopf bifurcation theorem which we are to use in the last section; the complimentary literature can be found in [13], [16]. Let $\dot{x} = A(\mu)x + f(x, \mu)$ be a c^r, $r \geq 3$, planar vector field depending on a scalar parameter μ, such that $A(\mu)$ is a 2×2 matrix, $f(0, \mu) = 0$ and $D_x f(0, \mu) = 0$, for all sufficiently small $|\mu|$. Assume that, $A(\mu)$ has the complex eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, suppose that, the eigenvalues cross the imaginary axis with nonzero speed, that is, $\alpha^{'}(0) \neq 0$. Then, there exists $\mu_0 > 0$ such that for any neighborhood U containing the origin, there exists μ with $0 < |\mu| < \mu_0$ such that the system has a nontrivial periodic solution in U. The stability of the periodic solution can be also investigated (see [11]).

3. Two parametric perturbations

Consider the parametric system $\dot{X} = V_\epsilon(X)$, where $X \in R^2$ and $\epsilon = (\epsilon_2, \epsilon_2)$ with the following form

$$(7)\qquad \begin{cases} \dot{x} = P(x,y) + \epsilon_1 p_1(x,y) + \epsilon_2 p_2(x,y) \\ \\ \dot{y} = Q(x,y) + \epsilon_1 q_1(x,y) + \epsilon_2 q_2(x,y), \end{cases}$$

and suppose that for $\epsilon = (\epsilon_1, \epsilon_2) = (0,0) = \bar{0}$, $V_{\bar{0}}$ has a annual region A of periodic orbits or has at least one limit cycle Γ_0. We would like to see if any of the periodic orbits can persists under small perturbation, i.e., there exists a continuous family of periodic orbits Γ_ϵ.

As we mentioned before, we choose Σ as the Poincare cross section and $F = F(x, \epsilon)$ as the normalized displacement function. Let x be the usual distance coordinates along Σ. Then $(x, \epsilon) \mapsto T(x, \epsilon)$, the first return time, and the return map $(x, \epsilon) \mapsto P(x, \epsilon)$ define on Σ, if $F(x, \epsilon) = 0$ then (x, y_0) lies on a periodic orbit. We will show that if $\frac{\partial F}{\partial x}(x_0, \bar{0}) \neq 0$, then by the implicit function theorem, there exists an open neighborhood U containing $\bar{0}$ in (ϵ_1, ϵ_2) Plane, and a C^1 map $x(\epsilon_1, \epsilon_2)$ with $x(0,0) = x_0$ such that $F(x, \epsilon_1, \epsilon_2) = 0$. The next theorem present conditions for which the persistence of a periodic orbit in the annual region A.

Theorem 3.1. *Suppose that we have the system V_ϵ with corresponding flow ϕ_t^ϵ. Let V_0 has a periodic annual A with Poincare cross section Σ. Then, if there is a point $x_0 \in \Sigma$ such that the first partial derivatives of the normalized function F with respect to parameters $\epsilon = (\epsilon_1, \epsilon_2)$ at $(x_0, \bar{0})$, $\bar{0} = (0,0)$, are zeroes, but the second partial derivatives with respect to x and ϵ are not zeroes. Then there is a continuous family Γ_ϵ of periodic trajectories for the system V_ϵ emerging from $\Gamma = \Gamma_0$.*

For proving this theorem we need to use the Implicit function Theorem, so first we prove the following lemma.

Parameter Lemma. *Consider the parametric system V_ϵ and suppose that V_0 has a periodic annual region A.*

(I) *Let $p = (x_0, y_0)$ be an internal point of A and $\Sigma_{y_0} = J \times \{y_0\}$ be the Poincare cross section ; also, assume that $F(x, \epsilon_1, \epsilon_2) : J \times (-\delta\,,\, \delta) \times (-\delta\,,\, \delta) \to R$, $\delta > 0$ is the family of the normalized displacement functions based on Σ_{y_0}. If x_0 is such that $\frac{\partial F}{\partial \epsilon_1}(x_0, \bar{0}) = 0$ and $\frac{\partial^2 F}{\partial \epsilon_1 \partial x}(x_0, \bar{0}) \neq 0$, then, there exists $r > 0$ such that for each $0 < |\epsilon_1| \leq r$ and $|\epsilon_2| \leq \sqrt{\frac{\epsilon_1^2(r^2 - \epsilon_1^2)}{1+\epsilon_1^2}}$, there is a C^1 map $x(\epsilon_1, \epsilon_2)$ such that $F(x(\epsilon_1, \epsilon_2), \epsilon_1, \epsilon_2) = 0$. We denote this region for existence of zeroes in $\epsilon_1\epsilon_2$ plane by $A_r^{\epsilon_1}$.*

(II) *We suppose that $\epsilon_1 = f(\epsilon_2) = f$, where f is a polynomial of degree n and $f(0) = 0$, $\frac{\epsilon_1}{\epsilon_2}|_{\epsilon_2 \neq 0}$. Therefore we can writ $\epsilon_1 = \epsilon_2 f_1(\epsilon_2)$. Then there exists a function $\beta : \epsilon_2 \longmapsto \beta(\epsilon_2)$ such that $F(\beta(\epsilon_2), \epsilon_1, \epsilon_2) = 0$.*

Proof. For proof of part (I), Since $F(x,\bar{0}) = 0$, $x \in J$, so by the Taylor expansion of F at $\epsilon = \bar{0}$, we have

$$\begin{aligned} F(x,\epsilon_1,\epsilon_2) &= \epsilon_1 \frac{\partial F}{\partial \epsilon_1}(x,0,0) + \epsilon_2 \frac{\partial F}{\partial \epsilon_2}(x,0,0) \\ &+ \epsilon_1^2 \frac{\partial^2 F}{\partial \epsilon_1}(x,a\epsilon_1,b\epsilon_2) + 2\epsilon_1\epsilon_2 \frac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2) \\ &+ \epsilon_2^2 \frac{\partial^2 F}{\partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2), \end{aligned}$$

where $0 \leq a, b \leq 1$. Therefore, for $\epsilon_1 \neq 0$,

$$\begin{aligned} F(x,\epsilon_1,\epsilon_2) &= \epsilon_1 [\frac{\partial F}{\partial \epsilon_1}(x,0,0) + \left(\frac{\epsilon_2}{\epsilon_1}\right) \frac{\partial F}{\partial \epsilon_2}(x,0,0) \\ &+ \epsilon_1 \frac{\partial^2 F}{\partial \epsilon_1}(x,a\epsilon_1,b\epsilon_2) + 2\epsilon_2 \frac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2) \\ &+ \left(\frac{\epsilon_2^2}{\epsilon_1}\right) \frac{\partial^2 F}{\partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2)]. \end{aligned}$$

Let us define the map

$$\begin{aligned} \Theta \;&: J \times (-\delta\,,\,\delta) \times (-\delta\,,\,\delta) \times (-\delta\,,\,\delta) \to R \\ \Theta \;&: (x,\epsilon_1,\epsilon_2,\nu) \mapsto \frac{\partial F}{\partial \epsilon_1}(x,0,0) + \nu \frac{\partial F}{\partial \epsilon_2}(x,0,0) + \epsilon_1 \frac{\partial^2 F}{\partial \epsilon_1}(x,a\epsilon_1,b\epsilon_2) \\ &\qquad + 2\epsilon_2 \frac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2) + \epsilon_2 \nu \frac{\partial^2 F}{\partial \epsilon_2}(x,a\epsilon_1,b\epsilon_2). \end{aligned}$$

Then $\Theta(x_0,0,0,0) = \frac{\partial F}{\partial \epsilon}(x_0,0,0) = 0$ and $\frac{\partial \Theta}{\partial x}(x_0,0,0,0) = \frac{\partial^2 F}{\partial \epsilon_1 \partial x}(x_0,0,0) \neq 0$, therefore, by the implicit function theorem, there is $r > 0$ such that for each $\epsilon_1, \epsilon_2, \nu$ with $\epsilon_1^2 + \epsilon_2^2 + \nu^2 \leq r^2$, there exists C^1 map $\bar{x}(\epsilon_1,\epsilon_2,\nu)$ with $\bar{x}(0,0,0) = x_0$ such that $\Theta(\bar{x},\epsilon_1,\epsilon_2,\nu) = 0$. This shows that if $\epsilon_1 \neq 0$ and $\epsilon_1^2 + \epsilon_2^2 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \leq r^2$, then for $x(\epsilon_1,\epsilon_2) = \bar{x}(\epsilon_1,\epsilon_2,\frac{\epsilon_1}{\epsilon_2})$, we have $F(x(\epsilon_1,\epsilon_2),\epsilon_1,\epsilon_2) = 0$. The inequality $\epsilon_1^2 + \epsilon_2^2 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \leq r^2$ implies that $0 < |\epsilon_1| \leq r$ and $|\epsilon_2| \leq \sqrt{\frac{\epsilon_1^2(r^2-\epsilon_1^2)}{1+\epsilon_1^2}}$. This completes the proof of part one (see figure 1a).

Remark. In the proof of part (I) of the lemma, we may replace ϵ_1 with ϵ_2, then we find the region $A_r^{\epsilon_2}$ for existence of zeroes. Moreover if the conditions of the theorem hold for both parameters, i.e.,

$$\frac{\partial F}{\partial \epsilon_1}(x_0,0,0) = \frac{\partial F}{\partial \epsilon_2}(x_0,0,0) = 0,$$

$$\frac{\partial^2 F}{\partial \epsilon_1 \partial x}(x_0,0,0) . \frac{\partial^2 F}{\partial \epsilon_2 \partial x}(x_0,0,0) \neq 0,$$

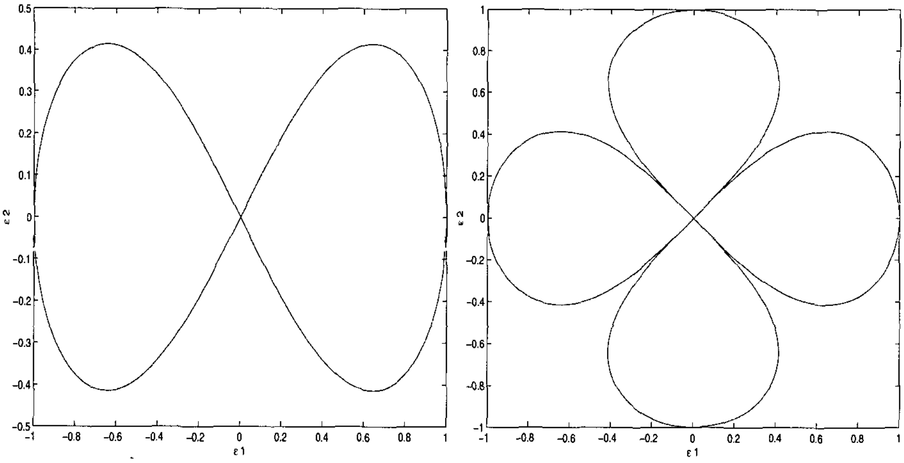

FIGURE 1. typical region $A_r^{\epsilon_2}$ and A_r for existence of zeroes. theorem 3.1, remark

then the regions $A_r^{\epsilon_i}$, (i=1,2) exist for $r > 0$ small enough. It is easy to check that $A_r^{\epsilon_1} \cap A_r^{\epsilon_2} = \emptyset$, therefore, we can consider $A_r = A_r^{\epsilon_1} \cup A_r^{\epsilon_2}$ as the region of existence of zeroes and extend the C^1 map $x(\epsilon_1, \epsilon_2)$, on A (see figure 1b).

The condition $\frac{\partial F}{\partial \epsilon_1}(x_0, \bar{0}) = 0$ shows that the displacement function has no linear dependent on ϵ_1. Hence, if, for all $x \in J$, the Taylor expansion of $F(x, \epsilon)$ contains first order terms with respect to ϵ_1 and ϵ_2, then lemma is not valid more. In this case the Taylor expansion can be considered as an inner product, i.e.,

$$
\begin{aligned}
F(x, \epsilon_1, \epsilon_2) &= \epsilon_1 \tfrac{\partial F}{\partial \epsilon_1}(x, 0, 0) + \epsilon_2 \tfrac{\partial F}{\partial \epsilon_2}(x, 0, 0) \\
&\quad + \epsilon_1^2 \tfrac{\partial^2 F}{\partial \epsilon_1}(x, a\epsilon_1, b\epsilon_2) + 2\epsilon_1\epsilon_2 \tfrac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x, a\epsilon_1, b\epsilon_2) \\
&\quad + \epsilon_2^2 \tfrac{\partial^2 F}{\partial \epsilon_2}(x, a\epsilon_1, b\epsilon_2) \qquad (8) \\
&= \langle (\epsilon_1, \epsilon_2) \,,\, (\tfrac{\partial F}{\partial \epsilon_1}(x, 0, 0) + A \,,\, \tfrac{\partial F}{\partial \epsilon_2}(x, 0, 0) + B) \rangle \\
&= \langle \epsilon \,,\, \overrightarrow{f}(x, \epsilon_1, \epsilon_2) \rangle,
\end{aligned}
$$

where $\langle ., . \rangle$ is the inner product operator and

$$A = \epsilon_1 \tfrac{\partial^2 F}{\partial \epsilon_1}(x, a\epsilon_1, b\epsilon_2) + \epsilon_2 \tfrac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x, a\epsilon_1, b\epsilon_2)$$

$$B = \epsilon_2 \tfrac{\partial^2 F}{\partial \epsilon_2}(x, a\epsilon_1, b\epsilon_2) + \epsilon_1 \tfrac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x, a\epsilon_1, b\epsilon_2),$$

with $0 \leq a, b \leq 1$. This shows that for $\epsilon \neq 0$, $F(x,\epsilon) = 0$ if and only if $\overrightarrow{f}(x,\epsilon_1,\epsilon_2) = 0$ or $\epsilon \perp \overrightarrow{f}(x,\epsilon_1,\epsilon_2)$. It is easy to check that, if for $x_0 \in J$,

$$(9) \qquad \begin{aligned} &\tfrac{\partial F}{\partial \epsilon_1}(x,\bar{0}) = \tfrac{\partial F}{\partial \epsilon_1}(x,\bar{0}) = 0 \\ &\tfrac{\partial^2 F}{\partial x \partial \epsilon_1}(x_0,\bar{0})\tfrac{\partial^2 F}{\partial \epsilon_1 \partial \epsilon_2}(x_0,\bar{0}) - \tfrac{\partial^2 F}{\partial x \partial \epsilon_2}(x_0,\bar{0})\tfrac{\partial^2 F}{\partial \epsilon_1^2}(x_0,\bar{0}) \neq 0, \end{aligned}$$

then, by the implicit function theorem, there exists an open interval containing ϵ_2 and a C^1 map $(x(\epsilon_2), \epsilon_1(\epsilon_2))$ with $(x(0), \epsilon_1(0)) = (x_0, 0)$ such that $\overrightarrow{f}(x(\epsilon_2), \epsilon_1(\epsilon_2), \epsilon_2) = 0$. Also we can give the rule of ϵ_1 to ϵ_2 and find the similar results.

For proof of part (II), in this case again we writ the Taylor expansion of the map F at $\epsilon = \bar{0}$, Since $F(x,\bar{0}) = 0$ for $x \in A$, then :

$$\begin{aligned} F(x,\epsilon_1,\epsilon_2) &= \frac{\partial F}{\partial \epsilon_1}\epsilon_1 + \frac{\partial F}{\partial \epsilon_2}\epsilon_2 + O(|\epsilon|^2) \\ &= \frac{\partial F}{\partial \epsilon_2}\frac{\partial \epsilon_2}{\partial \epsilon_1}\epsilon_2 f_1(\epsilon_2) + \frac{\partial F}{\partial \epsilon_2}\epsilon_2 + O(|\epsilon|^2) \\ &= \epsilon_2 G(x,\epsilon_2) + O(|\epsilon|^2). \end{aligned}$$

Therefore the zeroes of $G(x,\epsilon_2)$ correspond to the zeroes of F. Now by hypotheses of the lemma, we can apply the Implicit function Theorem for the function G; then there exists a function β which satisfies the statement of the lemma. This will end proof of the lemma. □

Note that the hypotheses of the second part of the lemma help us to have the origin in the region of the zeroes for the function F.

Proof of the theorem 3.1. Since $F(x,\bar{0}) = 0$ for $x \in A$, then the Taylor expansion of F at $\epsilon = \bar{0}$ is

$$\begin{aligned} F(x,\epsilon) &= (\frac{\partial F}{\partial \epsilon_1}\epsilon_1 + \frac{\partial F}{\partial \epsilon_2}\epsilon_2) + O(|\epsilon|^2)|_{(x,\bar{0})} \\ &= \langle \epsilon, G(x,\epsilon_1,\epsilon_2)\rangle + O(|\epsilon|^2). \end{aligned}$$

Therefore hypotheses for the normalized function F implies that $G(x,\bar{0}) = 0$ and

$$D_x G(x,\bar{0}) \neq 0.$$

So by the parameter lemma, we can apply the Implicit function Theorem and have region in the parameter space $\epsilon = (\epsilon_1,\epsilon_2)$ which implies that the displacement function F in that region is equal to zeroes.

Now we give formula for computing F_ϵ: Let $T(x,\epsilon)$ denote the time of the first return of the solution $\phi_t(x,y_0,\epsilon) = (X(t,x,\epsilon), Y(t,x,\epsilon))$ to Σ, that is

$$X(T(x,\epsilon), x, y_0, \epsilon) = d(x,\epsilon),$$

$$Y(T(x,\epsilon), x, y_0, \epsilon) = y_0.$$

we define

$$T = T(x, \bar{0}),$$
$$T_\epsilon = (\frac{\partial T}{\partial \epsilon_1}, \frac{\partial T}{\partial \epsilon_2})|_{(x,\bar{0})},$$
$$d_\epsilon = (\frac{\partial d}{\partial \epsilon_1}, \frac{\partial d}{\partial \epsilon_2})|_{(x,\bar{0})}.$$

Also, we have

$$X_\epsilon = (\frac{\partial X}{\partial \epsilon_1}, \frac{\partial X}{\partial \epsilon_2})|_{(T,x,\bar{0})},$$
$$Y_\epsilon = (\frac{\partial Y}{\partial \epsilon_1}, \frac{\partial Y}{\partial \epsilon_2})|_{(T,x,\bar{0})}.$$

By derivative of these equations at $\epsilon = \bar{0}$, we have

$$\begin{pmatrix} d_\epsilon \\ \bar{0} \end{pmatrix} = \begin{pmatrix} \dot{X}(T, x, \bar{0})T_\epsilon \\ \dot{Y}(T, x, \bar{0})T_\epsilon \end{pmatrix} + \begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix}. \tag{10}$$

Now we define $W(t) = (W_1(t), W_2(t))$, where $W(t)$ is the matrix

$$W = \begin{pmatrix} \frac{\partial X}{\partial \epsilon_1} & \frac{\partial X}{\partial \epsilon_2} \\ \frac{\partial Y}{\partial \epsilon_1} & \frac{\partial Y}{\partial \epsilon_2} \end{pmatrix},$$

at $(t, x, \bar{0})$. Using the fact that $(\dot{X}, \dot{Y})|_{(T,x,\bar{0})} = V_0(\Gamma(t))$, then the equation (10) has the form

$$(d_\epsilon, \bar{0})^T = V_0 T_\epsilon + W(T).$$

Therefore by definition of the normalized displacement function, we have

$$F_\epsilon(x, \bar{0}) = V_0^\perp \cdot (d_\epsilon, \bar{0}) = V_0^\perp \cdot V_0 T_\epsilon + V_0^\perp \cdot W,$$

where $V_0^\perp \cdot V_0 = 0$

Let $X(t, x, \epsilon), Y(t, x, \epsilon) = (\bar{x}, \bar{y})$ be a solution of system V_ϵ, then

$$\dot{X} = P(\bar{x}, \bar{y}) + \epsilon_1 p_1(\bar{x}, \bar{y}) + \epsilon_2 p_2(\bar{x}, \bar{y}),$$
$$\dot{Y} = Q(\bar{x}, \bar{y}) + \epsilon_1 q_1(\bar{x}, \bar{y}) + \epsilon_2 q_2(\bar{x}, \bar{y})$$

Derivative of this system with respect to ϵ at $(0, 0)$ and the definition of $W_i(t) = (X_{\epsilon_i}, Y_{\epsilon_i})$ implies that $W_i(t)$ is the solution of the equation

$$\dot{W_i}(t) = DV_0(\Gamma(t)) \cdot W_i(t) + B_i(t).$$

Moreover from above formula for the function $F_\epsilon(x, \bar{0})$, we see that

$$\frac{\partial F}{\partial \epsilon_1}(x, \bar{0}) = -Q(x, y_0)\frac{\partial X}{\partial \epsilon_1}(x, \bar{0}) + P(x, y_0)\frac{\partial Y}{\partial \epsilon_1}(x, \bar{0}),$$
$$\frac{\partial F}{\partial \epsilon_2}(x, \bar{0}) = -Q(x, y_0)\frac{\partial X}{\partial \epsilon_2}(x, \bar{0}) + P(x, y_0)\frac{\partial Y}{\partial \epsilon_2}(x, \bar{0}).$$

Hence

$$F_{\epsilon_i} = V_0^\perp \cdot W_i(T), \quad i = 1, 2.$$

An application of the second part of the Variation Lemma for $m=1$ for each $W_i(t)$ implies that

$$F_i(x,\bar{0}) = V_0^{\perp} \cdot W_i(T) = M(T, V, B_i, \Gamma(0)),$$
$$B_i(t) = (p_i(\Gamma(t), q_i(\Gamma(t))^T, \quad i=1,2$$

In order to compute $F_\epsilon(x,\bar{0})$, we compute the functions $W_i(t)$ from Variation Lemma for each i, so we get

$$F_{\epsilon_1} = \frac{\partial F}{\partial \epsilon_1}(x,0) = \int_0^T [-Q(\Gamma)p_1(\Gamma) + P(\Gamma)q_1(\Gamma)] \exp(\int_0^t -\operatorname{div} V_0(\Gamma(t))dt)\, dt$$

and

$$F_{\epsilon_2} = \frac{\partial F}{\partial \epsilon_2}(x,0) = \int_0^T [-Q(\Gamma)p_2(\Gamma) + P(\Gamma)q_2(\Gamma)] \exp(\int_0^t -\operatorname{div} V_0(\Gamma(t))dt)\, dt.$$

If There is a simple zero x_0 for $F_{\epsilon_i}, i=1,2$ for which we have $F_x\epsilon_i \neq 0, i=1,2$, then by Implicit function theorem we have a family of the periodic trajectories Γ_ϵ which emerges from the trajectory Γ, this complete the proof of the theorem 3.1. □

4. Effect of nonlinear damping on Van der Pol

In this section, we apply the result on the Van der Pol equation. As we know this equation has many application in engineering and physical problems. Effect of linear damping on this equation has been considered. But here we will consider the effect of nonlinear damping on the equation. For this purpose we apply our result on the equation.

Let us consider the equation

$$\ddot{x} + \omega^2 x + \epsilon_1(x^2-1)\dot{x} + \epsilon_2(a\dot{x}^2 + b\dot{x}x) = 0$$

or

$$(11) \qquad \begin{cases} \dot{x} = \omega y - \frac{1}{3}\epsilon_1 x^3 - \frac{1}{2}b\epsilon_2 x^2 \\ \dot{y} = -\omega x + \frac{\epsilon_1}{\omega} y - \frac{\epsilon_2}{\omega} a y^2. \end{cases}$$

For $\epsilon_1 = \epsilon_2 = 0$, the system (11) has periodic solutions $(r\cos\omega t, r\sin\omega t)$. Note that the system (11) for $\epsilon_2 = 0$ is the Van der Pol equation. By numerical method one can find the periodic orbit $r=2$ for small ϵ_1. Here we will show that this periodic orbit will be persist under small ϵ_2 nonlinear damped perturbation. For this propose we apply the result of the theorem 3.1, i.e., the formula of the theorem. Here

$$P(x,y) = -\omega y, \quad p_1(x,y) = -\frac{1}{3}x^3, \quad p_2(x,y) = -\frac{1}{2}bx^2$$
$$Q(x,y) = -\omega x, \quad q_1(x,y) = \frac{y}{\omega}, \quad q_2(x,y) = -\frac{a}{\omega}y^2$$

Since we have $\operatorname{div} V_0 = 0$, then $\exp \int_0^t -\operatorname{div} V_0 ds = 1$, and therefore:

$$F_{\epsilon_1}(x,0) = a(1-\frac{1}{4}r^2)\pi r^2$$

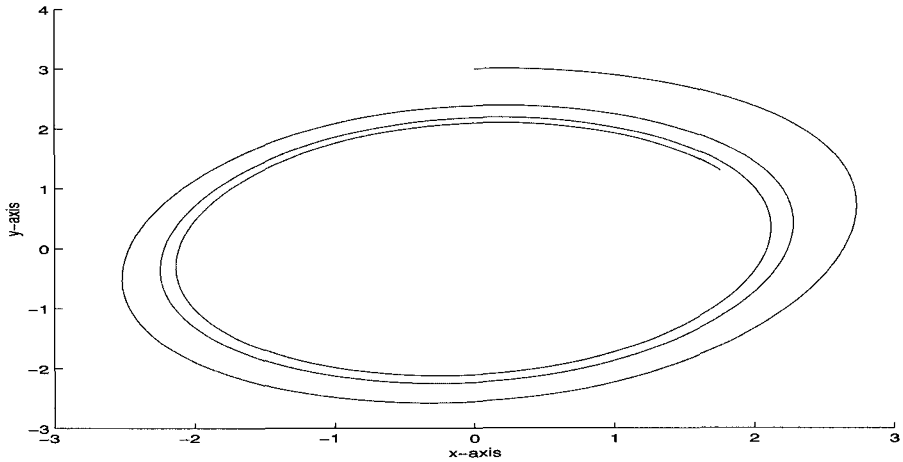

FIGURE 2

$$F_{\epsilon_2}(x,0) = 0.$$

Note that we calculated the F_{ϵ_1} and F_{ϵ_2} at $(r\cos\omega t, r\sin\omega t)$. From above results, we see that $F_{\epsilon_1} = 0$ for $r = 2$ and $F_{\epsilon_2} = 0$ for all values. Therefore from the circle $r = 2$, a family of the periodic trajectories Γ_ϵ can emerged for small $\epsilon = (\epsilon_1, \epsilon_2)$. Hence we have the following theorem for the Van der Pol equation:

Theorem 4.1. (i) *The closed trajectory $r = 2$ of the Van der Pol equation can persist under effect of small nonlinear damping. Also a family of the periodic trajectories will be emerge from $r = 2$, See the following figures. Note that these figures shows that the trajectory $r = 2$ is asymptotically stable.*

(ii) *For small parameter ϵ_1 we will have Hopf bifurcation for the Van der Pol equation.*

Proof. The proof for first part lemma that is (i) comes from the above result. For the second part that is (ii), we see that the origin is the fixed point of the equation. By direct calculation, we see that at the fixed point $(0,0)$ the linearized part of the system (5) has eigenvalues:

$$\lambda = \frac{1}{2}(\epsilon_1 \pm \sqrt{(\epsilon_1^2 - 4\omega^2)}),$$

where at $\epsilon_1 = 0$ the eigenvalues are pure imaginary. Therefore the Hopf bifurcation theorem, (see section 2), implies that the Van der Pol equation has Hopf bifurcation for small ϵ_1. □

Conclusion

1) In this paper we gave some conditions under which the persistence of a periodic trajectory guarantees. The persistence is under small autonomous

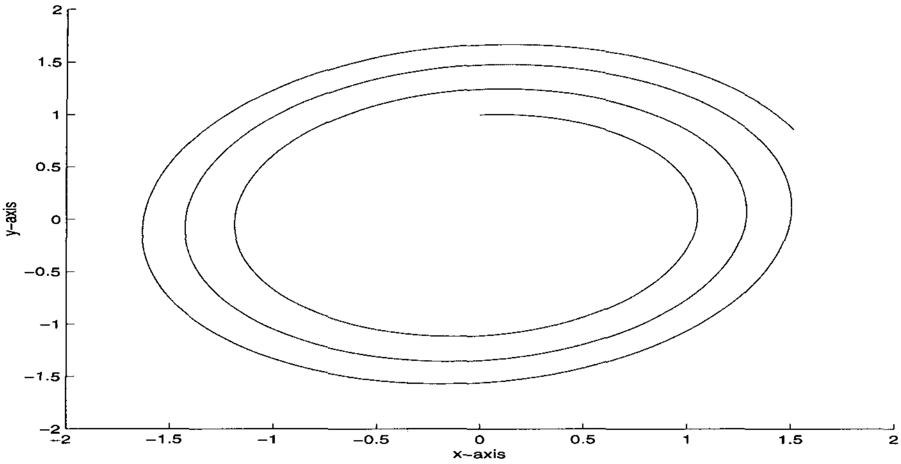

FIGURE 3. This figure draw for $\epsilon_1 = 0.1$, $\epsilon_2 = 0.01, x = -3 - 3, y = -3 - 3$.

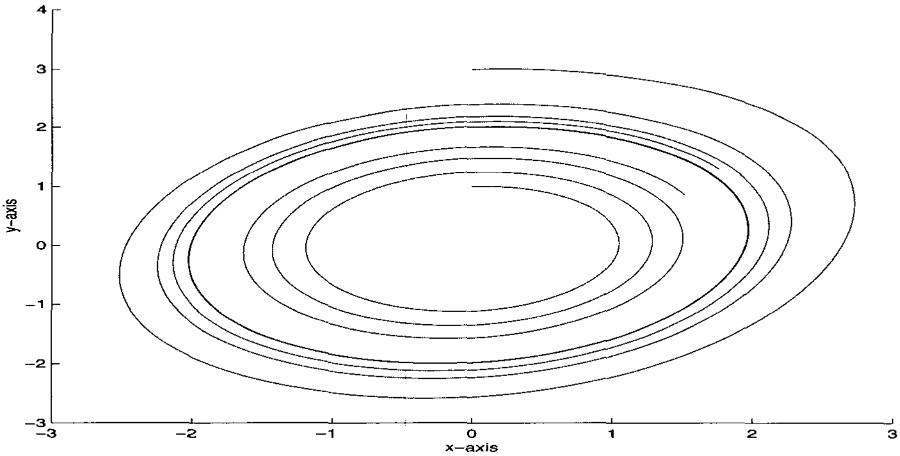

FIGURE 4. here $\epsilon_1 = 0.1$ and $\epsilon_2 = 0.01$

perturbation with two parameters. Therefore by this result one can see the effect of nonlinear damping on the physical and engineering problems.

2) Emerging a family of periodic trajectories from a periodic orbit implies the Hopf bifurcation.

3) We guess that the effect of two parametric non-autonomous perturbations will give similar results which we are going to consider it in a separate paper.

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