ENDPOINT ESTIMATES FOR MULTILINEAR INTEGRAL OPERATORS

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ABSTRACT. In this paper, the endpoint estimates for some multilinear operators related to certain integral operators are obtained. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

1. Introduction and Notations

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-6]). In [10], the boundedness properties of the commutators for the extreme values of p are obtained. The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type integral operators. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

First, let us introduce some notations (see [7-9], [14-16]). Throughout this paper, Q will denote a cube of \( R^n \) with sides parallel to the axes. For a cube Q and a locally integrable function \( f \), let \( f_Q = |Q|^{-1} \int_Q f(x) dx \) and \( f^#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| dy \). Moreover, \( f \) is said to belong to \( BMO(R^n) \) if \( f^# \in L^\infty \) and define \( ||f||_{BMO} = ||f^#||_{L^\infty} \); We also define the central \( BMO \) space by \( CMO(R^n) \), which is the space of those functions \( f \in L_{loc}(R^n) \) such that

\[ ||f||_{CMO} = \sup_{r > 1} |Q(0, r)|^{-1} \int_Q |f(y) - f_Q| dy < \infty. \]

It is well-known that (see [8], [9])

\[ ||f||_{CMO} \approx \sup_{r > 1} \inf_{c \in C} |Q(0, r)|^{-1} \int_Q |f(x) - c| dx. \]

Also, we give the concepts of the atom and \( H^1 \) space. A function \( a \) is called as \( H^1 \) atom if there exists a cube Q such that \( a \) is supported on Q, \( ||a||_{L^\infty} \leq |Q|^{-1} \).

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and \( \int a(x)dx = 0 \). It is well known that the Hardy space \( H^1(R^n) \) has the atomic decomposition characterization (see [9]).

For \( k \in Z \), define \( B_k = \{ x \in R^n : |x| \leq 2^k \} \) and \( C_k = B_k \setminus B_{k-1} \). Denote by \( \chi_k \) the characteristic function of \( C_k \) and \( \bar{\chi}_k \) the characteristic function of \( C_k \) for \( k \geq 1 \) and \( \bar{\chi}_0 \) the characteristic function of \( B_0 \).

**Definition 1.** Let \( 0 < p < \infty \) and \( \alpha \in R \).

(1) The homogeneous Herz space \( \dot{K}_p^\alpha(R^n) \) is defined by
\[
\dot{K}_p^\alpha(R^n) = \{ f \in L^p_{loc}(R^n \setminus \{0\}) : \| f \|_{\dot{K}_p^\alpha} < \infty \},
\]
where
\[
\| f \|_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha} \| f \chi_k \|_{L^p};
\]

(2) The nonhomogeneous Herz space \( K_p^\alpha(R^n) \) is defined by
\[
K_p^\alpha(R^n) = \{ f \in L^p_{loc}(R^n) : \| f \|_{K_p^\alpha} < \infty \},
\]
where
\[
\| f \|_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} \| f \bar{\chi}_k \|_{L^p}.
\]
If \( \alpha = n(1 - 1/p) \), we denote that \( \dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n) \), \( K_p^\alpha(R^n) = K_p(R^n) \).

**Definition 2.** Let \( 0 < \delta < n \) and \( 1 < p < n/\delta \). We shall call \( B_p^\delta(R^n) \) the space of those functions \( f \) on \( R^n \) such that
\[
\| f \|_{B_p^\delta} = \sup_{r > 1} r^{-n(1/p-\delta/n)} \| f \chi_{Q(0,r)} \|_{L^p} < \infty.
\]

**Definition 3.** Let \( 1 < p < \infty \).

(1) The homogeneous Herz type Hardy space \( H\dot{K}_p(R^n) \) is defined by
\[
H\dot{K}_p(R^n) = \{ f \in S'(R^n) : G(f) \in \dot{K}_p(R^n) \},
\]
where
\[
\| f \|_{H\dot{K}_p} = \| G(f) \|_{\dot{K}_p};
\]

(2) The nonhomogeneous Herz type Hardy space \( HK_p(R^n) \) is defined by
\[
HK_p(R^n) = \{ f \in S'(R^n) : G(f) \in K_p(R^n) \},
\]
where
\[
\| f \|_{HK_p} = \| G(f) \|_{K_p};
\]
where \( G(f) \) is the grand maximal function of \( f \).

The Herz type Hardy spaces have the atomic decomposition characterization.
Definition 4. Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(n(1 - 1/p), p)$-atom (or a central $(n(1 - 1/p), p)$-atom of restrict type), if
1) $\text{Supp}a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$);
2) $||a||_{L^p} \leq |B(0, r)|^{1/p-1}$,
3) $\int_{\mathbb{R}^n} a(x)dx = 0$.

Lemma 1. (see [8], [15]). Let $1 < p < \infty$. A temperate distribution $f$ belongs to $\dot{H}^k_p(\mathbb{R}^n)$ (or $\dot{H}^k_p(\mathbb{R}^n)$) if and only if there exist central $(n(1 - 1/p), p)$-atoms (or central $(n(1 - 1/p), p)$-atoms of restrict type) $a_j$ supported on $B_j = B(0, 2^{j+1})$ and constants $\lambda_j$, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, and

$$||f||_{\dot{H}^k_p} \sim \sum_j |\lambda_j|.$$ 

2. Theorems

In this paper, we will study a class of multilinear operators related to some non-convolution type integral operators, whose definition are following.

Let $m_j$ be the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $\mathbb{R}^n$ $(j = 1, \ldots, l)$. Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\beta| \leq m_j} \frac{1}{\beta!} D^\beta A_j(y)(x - y)^\beta$$

and

$$Q_{m_j+1}(A_j; x, y) = R_{m_j}(A_j; x, y) - \sum_{|\beta| = m_j} \frac{1}{\beta!} D^\beta A_j(x)(x - y)^\beta.$$ 

Fixed $0 \leq \delta < n$. Let $F_l(x, y)$ define on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$. Set

$$F_l^l(f)(x) = \int_{\mathbb{R}^n} F_l(x, y)f(y)dy$$

and

$$F_{l+1}^A(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{l+1} \frac{R_{m_j+1}(A_j; x, y)}{|x - y|^{m_j}} F_l(x, y)f(y)dy$$

for every bounded and compactly supported function $f$. Let $H$ be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_l(f)(x)$ and $F_{l+1}^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to $H$. Then, the multilinear operator related to $F_l$ is defined by

$$T_{\delta}^l(f)(x) = ||F_{l+1}^A(f)(x)||$$

where $F_l$ satisfies: for fixed $\varepsilon > 0$,

$$||F_l(x, y)|| \leq C|x - y|^{-n+\delta}$$

and

$$||F_l(y, x) - F_l(z, x)|| \leq C|y - z|^\varepsilon|x - z|^{-n-\varepsilon+\delta}$$

and

$$||F_l(y, x) - F_l(z, x)|| \leq C|y - z|^\varepsilon|x - z|^{-n-\varepsilon+\delta}$$
if $2|y - z| \leq |x - z|$. We define that $T_\delta(f)(x) = ||F_\delta(f)(x)||$. We also consider
the variant of $T_\delta^A$, which is defined by

$$
\tilde{T}_\delta^A(f)(x) = ||\tilde{F}_\delta^A(f)(x)||,
$$

where

$$
\tilde{F}_\delta^A(f)(x) = \int_{R^n} \prod_{j=1}^l \frac{Q_{m_{j+1}}(A_j; x, y)}{|x - y|^n} F_i(x, y) f(y) dy.
$$

Note that when $m = 0$, $T_\delta^A$ is just the multilinear commutators of $T_\delta$ and $A$ (see [1], [11-13], [18]). It is well-known that multilinear operator, as a nontrivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3-6]). In [2], the weak ($H^1$, $L^1$)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the endpoint continuity properties of the multilinear operators $T_\delta^A$ and $\tilde{T}_\delta^A$. In Section 4, we will give some applications of Theorems in this paper.

Now we state our results as following.

**Theorem 1.** Let $0 \leq \delta < n$ and $D^\beta A_j \in BMO(R^n)$ for all $\beta$ with $|\beta| = m_j$ and $j = 1, \ldots, l$. Suppose that $T_\delta$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r, s \in (1, +\infty)$ with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$. Then $T_\delta^A$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.

**Theorem 2.** Let $0 \leq \delta < n$ and $D^\beta A_j \in BMO(R^n)$ for all $\beta$ with $|\beta| = m_j$ and $j = 1, \ldots, l$. Suppose that $\tilde{T}_\delta^A$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r, s \in (1, +\infty)$ with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$. Then $\tilde{T}_\delta^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.

**Theorem 3.** Let $0 \leq \delta < n$, $1 < p < n/\delta$ and $D^\beta A_j \in BMO(R^n)$ for all $\beta$ with $|\beta| = m_j$ and $j = 1, \ldots, l$. Suppose that $T_\delta$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r, s \in (1, +\infty)$ with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$. Then $T_\delta^A$ is bounded from $B^p(R^n)$ to $CMO(R^n)$.

**Theorem 4.** Let $0 \leq \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $D^\beta A_j \in BMO(R^n)$ for all $\beta$ with $|\beta| = m_j$ and $j = 1, \ldots, l$. Suppose that $\tilde{T}_\delta^A$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r, s \in (1, +\infty)$ with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$. Then $\tilde{T}_\delta^A$ is bounded from $HK^p(R^n)$ to $K^\alpha_q(R^n)$ with $\alpha = n(1 - 1/p)$.

**Remark.** Theorem 4 is also hold for nonhomogeneous Herz and Herz type Hardy space.

### 3. Proofs of Theorems

To prove the theorems, we need the following lemma.
Lemma 2. (see [6]) Let $A$ be a function on $\mathbb{R}^n$ and $D^\beta A \in L^q(\mathbb{R}^n)$ for $|\beta| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\beta| = m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$. 

Proof of Theorem 1. It is only to prove that there exists a constant $C_Q$ such that

$$\frac{1}{|Q|} \int_Q |T_\delta^A(f)(x) - C_Q| dx \leq C||f||_{L^{n/\delta}}$$

holds for any cube $Q$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\beta| = m} \frac{1}{|\beta|}(D^\beta A_j)\tilde{Q}x^\beta$.

then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)\tilde{Q}$ for $|\beta| = m_j$.

We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$F_t^A(f)(x)$$

$$= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 \sum_{|\beta_j| = m_j} \frac{1}{\beta_j!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\beta_2}}{|x - y|^m} \times D^{\beta_2} \tilde{A}_2(y) F_t(x, y) f_1(y) dy + \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\beta_2}}{|x - y|^m} \times D^{\beta_2} \tilde{A}_2(y) F_t(x, y) f_1(y) dy }$$

Then

$$\left| T_\delta^A(f)(x) - T_\delta^\tilde{A}(f_2)(x_0) \right|$$

$$= \left| ||F_t^A(f)(x)|| - ||F_t^\tilde{A}(f_2)(x_0)|| \right|$$
\[
\begin{align*}
&\leq \left\| F_t^A(f)(x) - F_t^\tilde{A}(f_2)(x_0) \right\| \\
&\leq \left| \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_m j (\tilde{A}_j; x, y)}{|x - y|^m} F_t(x, y) f_1(y) dy \right| \\
&+ \left| \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{\mathbb{R}^n} \frac{R_m j (\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} D^{\beta_2} \tilde{A}_1(y) F_t(x, y) f_1(y) dy \right| \\
&+ \left| \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{\mathbb{R}^n} \frac{R_m j (\tilde{A}_1; x, y)(x - y)^{\beta_2}}{|x - y|^m} D^{\beta_2} \tilde{A}_2(y) F_t(x, y) f_1(y) dy \right| \\
&+ \left| \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{\beta_1! \beta_2!} \right| \\
&\times \left| \int_{\mathbb{R}^n} (x - y)^{\beta_1 + \beta_2} D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) F_t(x, y) f_1(y) dy \right| \\
&+ \left| T_{\delta}^\tilde{A}(f_2)(x) - T_{\delta}^\tilde{A}(f_2)(x_0) \right| \\
&:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\end{align*}
\]

thus,

\[
\frac{1}{|Q|} \int_{Q} \left| T_{\delta}^A(f)(x) - T_{\delta}^\tilde{A}(f_2)(x_0) \right| dx \\
\leq \frac{1}{|Q|} \int_{Q} I_1(x) dx + \frac{1}{|Q|} \int_{Q} I_2(x) dx + \frac{1}{|Q|} \int_{Q} I_3(x) dx \\
+ \frac{1}{|Q|} \int_{Q} I_4(x) dx + \frac{1}{|Q|} \int_{Q} I_5(x) dx \\
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Now, let us estimate \(I_1, I_2, I_3, I_4\) and \(I_5\), respectively. For \(I_1\), by Lemma 2, we get, for \(x \in Q\) and \(y \in Q\),

\[
R_m j (\tilde{A}_j; x, y) \leq C |x - y|^{m_j} \sum_{|\beta_j|=m_j} ||D^{\beta_j} \tilde{A}_j||_{BMO},
\]

thus, by \((L^{n/\delta}, L^{\infty})\)-boundedness of \(T_{\delta}\), we get

\[
I_1 \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} \tilde{A}_j||_{BMO} \right) \frac{1}{|Q|} \int_{Q} |T_{\delta}(f_1)(x)| dx \\
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} \tilde{A}_j||_{BMO} \right) ||T_{\delta}(f_1)||_{L^{\infty}}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}};
\]

For \( I_2 \), by \((L^p, L^q)\)-boundedness of \( T_\delta \) for \( 1/q = 1/p - \delta/n \), \( n/\delta > p > 1 \) and Hölder’s inequality, we get

\[
I_2 \leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \sum_{|\beta_1|=m_1} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 f_1)(x)| \, dx
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\beta_1} \tilde{A}_1 f_1)(x)|^q \, dx \right)^{1/q}
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \times \sum_{|\beta_1|=m_1} |Q|^{-1/q} \left( \int_{R^n} |D^{\beta_1} \tilde{A}_1 f_1(x)|^p \, dx \right)^{1/p}
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_{\tilde{Q}} |D^{\beta_1} A_1(x) - (D^{\beta_1} A_1)(\tilde{x})|^q \, dx \right)^{1/q} \|f\|_{L^{n/\delta}}
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}};
\]

For \( I_3 \), similar to the proof of \( I_2 \), we get

\[
I_3 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}};
\]

Similarly, for \( I_4 \), choose \( 1 < p < n/\delta \) and \( q, r_1, r_2 > 1 \) such that \( 1/q = 1/p - \delta/n \) and \( 1/r_1 + 1/r_2 + p\delta/n = 1 \), we obtain, by Hölder’s inequality,

\[
I_4 \leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)| \, dx
\leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \left( \frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)|^q \, dx \right)^{1/q}
\]
\[
\begin{align*}
\leq C & \sum_{|\beta_1|=m_1, |\beta_2|=m_2} |Q|^{-1/q} \left( \int_{\mathbb{R}^n} |D^{\beta_1} \tilde{A}_1(x) D^{\beta_2} \tilde{A}_2(x) f_1(x)|^p dx \right)^{1/p} \\
\leq C & \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_1} \tilde{A}_1(x)|^{pr_1} dx \right)^{1/pr_1} \\
& \times \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_2} \tilde{A}_2(x)|^{pr_2} dx \right)^{1/pr_2} ||f||_{L^{n/s}} \\
\leq C & \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/s}};
\end{align*}
\]

For \( I_5 \), we write
\[
F_t^\vA(f_2)(x) - F_t^\vA(f_2)(x_0) = \int_{\mathbb{R}^n} \left( F_t(x,y) - \frac{F_t(x_0,y)}{|x-y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)f_2(y)dy
\]
\[
+ \int_{\mathbb{R}^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x-y|^m} F_t(x_0,y)f_2(y)dy
\]
\[
+ \int_{\mathbb{R}^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x-y|^m} F_t(x_0,y)f_2(y)dy
\]
\[
- \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\beta_1}}{|x-y|^m} F_t(x,y) \\
- \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\beta_1}}{|x_0-y|^m} F_t(x_0,y) \right] D^{\beta_1} \tilde{A}_1(y)f_2(y)dy
\]
\[
- \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\beta_2}}{|x-y|^m} F_t(x,y) \\
- \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\beta_2}}{|x_0-y|^m} F_t(x_0,y) \right] D^{\beta_2} \tilde{A}_2(y)f_2(y)dy
\]
\[
+ \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{\beta_1! \beta_2!} \int_{\mathbb{R}^n} \left[ \frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m} F_t(x,y) \\
- \frac{(x_0-y)^{\beta_1+\beta_2}}{|x_0-y|^m} F_t(x_0,y) \right] D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y)f_2(y)dy
\]
\[
= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\]

By Lemma 1 and the following inequality (see [16])
\[
|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|)||b||_{BMO} \quad \text{for} \quad Q_1 \subset Q_2,
\]
we know that, for \( x \in Q \) and \( y \in 2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q} \),

\[
|R_{m,j}(\tilde{A}_j; x, y)| \leq C|x - y|^{m_j} \sum_{|\beta| = m_j} (||D^\beta A_j||_{BMO} + |(D^\beta A_j)\tilde{Q}(x, y) - (D^\beta A_j)\tilde{Q}|)
\leq Ck|x - y|^{m_j} \sum_{|\beta| = m_j} ||D^\beta A_j||_{BMO}.
\]

Note that \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in \mathbb{R}^n \setminus \tilde{Q} \), we obtain, by the condition of \( F_1 \),

\[
||I_5^{(1)}|| \leq C \int_{\mathbb{R}^n} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon - \delta}} \right)
\times \prod_{j=1}^2 |R_{m,j}(\tilde{A}_j; x, y)||f_2(y)|dy
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} ||D^{\beta_j} A_j||_{BMO} \right)
\times \prod_{j=1}^2 \left( \sum_{k=0}^\infty \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon - \delta}} \right) |f(y)|dy \right)
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} ||D^{\beta_j} A_j||_{BMO} \right) \sum_{k=1}^\infty k^2(2^{-k} + 2^{-k})||f||_{L^{n/\delta}}
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}.
\]

For \( I_5^{(2)} \), by the formula (see [6]):

\[
R_{m,j}(\tilde{A}_j; x, y) - R_{m,j}(\tilde{A}_j; x_0, y) = \sum_{|\gamma| < m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{A}_j; x, x_0)(x - y)^\gamma
\]

and Lemma 1, we have

\[
|R_{m,j}(\tilde{A}_j; x, y) - R_{m,j}(\tilde{A}_j; x_0, y)| \leq C \sum_{|\gamma| < m_j} \sum_{|\beta| = m_j} |x - x_0|^{m_j - |\gamma|}|x - y|^{\gamma}||D^\beta A_j||_{BMO},
\]
thus

\[ \|I_5^{(2)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^\beta_j A_j\|_{BMO} \right) \times \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} k^{\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}} |f(y)|dy \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^\beta_j A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}; \]

Similarly,

\[ \|I_5^{(3)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^\beta_j A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}; \]

For \( I_5^{(4)} \), taking \( r > 1 \) such that \( 1/r + \delta/n = 1 \), then

\[ \|I_5^{(4)}\| \leq C \sum_{|\beta_1|=m_1} \int_{R^n} \left\| (x-y)^{\beta_1} F_1(x,y) - (x_0-y)^{\beta_1} F_1(x_0,y) \right\| \times |R_{m_2}(\tilde{A}_2; x, y)||D^{\beta_1} \tilde{A}_1(y)||f_2(y)|dy \]

\[ + C \sum_{|\beta_1|=m_1} \int_{R^n} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)| \]

\[ \times \left| \frac{|(x_0-y)^{\beta_1} F_1(x_0,y)|}{|x_0-y|^m} \right| |D^{\beta_1} \tilde{A}_1(y)||f_2(y)|dy \]

\[ \leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \sum_{|\beta_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\epsilon k}) \]

\[ \times \left( \frac{1}{2^{k}Q} \int_{2^kQ} |D^{\beta_1} \tilde{A}_1(y)|^r dy \right)^{1/r} \|f\|_{L^{n/\delta}} \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^\beta_j A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}; \]

Similarly,

\[ \|I_5^{(5)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^\beta_j A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}; \]
For $I_5^{(6)}$, taking $r_1, r_2 > 1$ such that $\delta/n + 1/r_1 + 1/r_2 = 1$, then
\[
\|I_5^{(6)}\| \leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \int_{R_n} \frac{(x-y)^{\beta_1+\beta_2} F_t(x,y)}{|x-y|^m} \left| (x_0-y)^{\beta_1} \tilde{A}_1(y) \right| \left| (x_0-y)^{\beta_2} \tilde{A}_2(y) \right| |f_2(y)| \, dy \\
\leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \sum_{k=1}^{\infty} (2^{-k} + 2^{-\epsilon k}) \|f\|_{L^{n/\delta}} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_1} \tilde{A}_1(y)|^{r_1} \, dy \right)^{1/r_1} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_2} \tilde{A}_2(y)|^{r_2} \, dy \right)^{1/r_2} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}.
\]
Thus
\[
I_5 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{L^{n/\delta}}.
\]
This completes the proof of Theorem 1.

Proof of Theorem 2. It is only to show that there exists a constant $C > 0$ such that for every $H^1$-atom $a$ (that is that $a$ satisfies: supp $a \subset Q = Q(x_0,d)$, $\|a\|_{L^{\infty}} \leq |Q|^{-1}$ and $\int a(y) \, dy = 0$ (see [9])), the following holds:
\[
\|\tilde{T}_A^A(a)\|_{L^{n/(n-\delta)}} \leq C.
\]
Without loss of generality, we may assume $l = 2$. Write
\[
\int_{R^n} \left[ \tilde{T}_A^A(a)(x) \right]^{n/(n-\delta)} \, dx = \left[ \int_{|x-x_0| \leq 2d} + \int_{|x-x_0| > 2d} \right] \left[ \tilde{T}_A^A(a)(x) \right]^{n/(n-\delta)} \, dx := J_1 + J_2.
\]
For $J_1$, by the $(L^p, L^q)$-boundedness of $\tilde{T}_A^A$ for $1/q = 1/p - \delta/n$, $n/\delta > p > 1$, we get
\[
J_1 \leq C \|\tilde{T}_A^A(a)\|_{L^q}^{n/((n-\delta)q)} |2Q|^{1-n/((n-\delta)q)} \leq C \|a\|_{L^p}^{n/((n-\delta)q)} |Q|^{1-n/((n-\delta)q)} \leq C.
\]
To obtain the estimate of $J_2$, we denote that
\[
\tilde{A}_j(x) = A_j(x) - \sum_{|\beta_j|=m_j} \frac{1}{\beta_1!} (D^{\beta_j} A_j)_{2Q} x^\beta.
\]
Then \( Q_{m_j}(A_j; x, y) = Q_{m_j}(\tilde{A}_j; x, y) \). We write, by the vanishing moment of \( a \),
\[
\begin{align*}
& F_t^A(a)(x) \\
= & \int_{R^n} \left[ \frac{F_t(x, y)}{|x-y|^m} - \frac{F_t(x, x_0)}{|x-x_0|^m} \right] R_{m_1}(\tilde{A}_1; x, y)R_{m_2}(\tilde{A}_2; x, y)a(y)dy \\
+ & \int_{R^n} \frac{F_t(x, x_0)}{|x-x_0|^m} [R_{m_1}(\tilde{A}_1; x, y)R_{m_2}(\tilde{A}_2; x, y)]a(y)dy \\
& - \sum_{|\beta_2|=m_2} \int_{R^n} \left[ \frac{F_t(x, y)(x-y)^{\beta_2}}{|x-y|^m} - \frac{F_t(x, x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m} \right] R_{m_1}(\tilde{A}_1; x, y)R_{m_2}(\tilde{A}_2; x, y)a(y)dy \\
& \times R_{m_1}(\tilde{A}_1; x, y)D^{\beta_2} \tilde{A}_2(x)a(y)dy \\
& - \sum_{|\beta_1|=m_1} \int_{R^n} \left[ \frac{F_t(x, x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m} \right] [R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x, x_0)] \\
& \times D^{\beta_2} \tilde{A}_2(x)a(y)dy \\
& - \sum_{|\beta_1|=m_1} \int_{R^n} \left[ \frac{F_t(x, y)(x-y)^{\beta_1}}{|x-y|^m} - \frac{F_t(x, x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m} \right] R_{m_2}(\tilde{A}_2; x, y)R_{m_2}(\tilde{A}_2; x, x_0)a(y)dy \\
& \times D^{\beta_1} \tilde{A}_1(x)a(y)dy \\
& + \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \int_{R^n} \left[ \frac{F_t(x, y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m} - \frac{K(x, x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m} \right] \\
& \times D^{\beta_1} \tilde{A}_1(x)D^{\beta_2} \tilde{A}_2(x)a(y)dy,
\end{align*}
\]

similar to the proof of Theorem 1, we obtain
\[
J_2 \leq C \left[ \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \right]^{n/(n-\delta)} \\
\times \sum_{k=1}^{\infty} k^2 \left[ 2^{-kn/(n-\delta)} + 2^{-kn\varepsilon/(n-\delta)} \right] \\
\leq C.
\]

This completes the proof of Theorem 2.

Proof of Theorem 3. It suffices to prove that there exists a constant \( C_Q \) such that
\[
\frac{1}{|Q|} \int_Q |T_{\delta}^A(f)(x) - C_Q|dx \leq C ||f||_{B^\delta_0}
\]
holds for any cube $Q = Q(0,d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\beta| = m} \frac{1}{\beta!} (D^\beta A_j)_Q x^\beta$, then $R_m (A_j; x, y) = R_m (\tilde{A}_j; x, y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)_Q$ for $|\beta| = m_j$. We write, for $f_1 = f \chi_Q$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x - y|^m} F_t(x, y) f(y) dy$$

$$= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} F_t(x, y) f_2(y) dy$$

$$+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} F_t(x, y) f_1(y) dy$$

$$- \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) F_t(x, y) f_1(y) dy$$

$$- \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\beta_2}}{|x - y|^m} D^{\beta_2} \tilde{A}_2(y) F_t(x, y) f_1(y) dy$$

$$+ \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{\beta_1! \beta_2!}$$

$$\times \int_{R^n} \frac{(x - y)^{\beta_1 + \beta_2} D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y)}{|x - y|^m} F_t(x, y) f_1(y) dy,$$

then

$$\frac{1}{|Q|} \int_Q \left| T_{\delta}^A(f)(x) - T_{\delta}^A(f_2)(0) \right| dx$$

$$\leq \frac{1}{|Q|} \int_Q \left\| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} F_t(x, y) f_1(y) dy \right\| dx$$

$$+ \frac{1}{|Q|} \int_Q \left\| \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!}$$

$$\times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) F_t(x, y) f_1(y) dy \right\| dx$$

$$+ \frac{1}{|Q|} \int_Q \left\| \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!}$$
\[
\begin{align*}
&\times \int_{\mathbb{R}^n} \frac{R_{m,1}(A_1;x,y)(x-y)\beta_2}{|x-y|^m} D^{\beta_2} A_2(y) F_1(x,y) f_1(y) dy dx \\
+ \frac{1}{|Q|} \int_Q \left| \sum_{|\beta_1|=m_1, \ |\beta_2|=m_2} \frac{1}{\beta_1! \beta_2!} \times \int_{\mathbb{R}^n} \frac{(x-y)^{\beta_1+\beta_2} D^{\beta_1} A_1(y) D^{\beta_2} A_2(y) F_t(x,y) f_1(y) dy}{|x-y|^m} \right| dx \\
+ \frac{1}{|Q|} \int_Q \left| T^\delta_2(f_2)(x) - T^\delta_2(f_2)(0) \right| dx \\
:= L_1 + L_2 + L_3 + L_4 + L_5.
\end{align*}
\]

Similar to the proof of Theorem 1, we get, for \(1/s = 1/r - \delta/n\), \(1 < r < p\), \(1 < t_1, t_2 < \infty\) and \(1/t_1 + 1/t_2 + \delta/p = 1\),

\[
L_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \frac{1}{|Q|} \int_Q |T^\delta(f_1)(x)| dx \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \left( \frac{1}{|Q|} \int_Q |T^\delta(f_1)(x)|^q dx \right)^{1/q} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) a^{-n(1/p-\delta/n)} \| f \chi_Q \|_{L^p} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \| f \|_{B^s_{p,q}} ;
\]

\[
L_2 \leq C \sum_{|\beta_2|=m_2} ||D^{\beta_2} A_2||_{BMO} \sum_{|\beta_1|=m_1} \frac{1}{|Q|} \int_Q |T^\delta(D^{\beta_1} A_1 f_1)(x)| dx \\
\leq C \sum_{|\beta_2|=m_2} ||D^{\beta_2} A_2||_{BMO} \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |T^\delta(D^{\beta_1} A_1 f_1)(x)|^s dx \right)^{1/s} \\
\leq C \sum_{|\beta_2|=m_2} ||D^{\beta_2} A_2||_{BMO} \sum_{|\beta_1|=m_1} |Q|^{-1/s} \|(D^{\beta_1} A_1 - (D^{\beta_1} A_1)_Q) f_1 \|_{L^p} \\
\leq C \sum_{|\beta_2|=m_2} ||D^{\beta_2} A_2||_{BMO} \\
\times \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_Q |D^{\beta_1} A_1(y)|^{pr/(p-r)} dy \right)^{(p-r)/pr} |Q|^{\delta/n-1/p} \| f_1 \|_{L^p} .
\]
\[
\begin{align*}
L_3 & \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B_p^s} \\
L_4 & \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{|Q|} \int_Q |T_5(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)| \, dx \\
& \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \left( \frac{1}{|Q|} \int_{R^n} |T_5(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)| \, dx \right)^{1/s} \\
& \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} |Q|^{-1/s} \left( \int_{R^n} |D^{\beta_1} \tilde{A}_1(x) D^{\beta_2} \tilde{A}_2(x) f_1(x)| \, dx \right)^{1/r} \\
& \leq C \sum_{|\beta_1| = m_1} \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_1} \tilde{A}_1(x)|^{r_t_1} \, dx \right)^{1/r_t_1} \\
& \quad \times \sum_{|\beta_2| = m_2} \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_2} \tilde{A}_2(x)|^{r_t_2} \, dx \right)^{1/r_t_2} |Q|^{\delta/n - 1/p} \|f_1\|_{L^p} \\
& \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B_p^s} \\
\end{align*}
\]

For $L_5$, we write, for $x \in Q$,

\[
\begin{align*}
F_t^\tilde{A}(f_2)(x) - F_t^\tilde{A}(f_2)(0) \\
= \int_{R^n} \left( \frac{F_t(x, y) - F_t(0, y)}{|x - y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) f_2(y) \, dy \\
+ \int_{R^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; 0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|y|^m} F_t(0, y) f_2(y) \, dy \\
+ \int_{R^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; 0, y) \right) \frac{R_{m_1}(\tilde{A}_1; 0, y)}{|y|^m} F_t(0, y) f_2(y) \, dy \\
- \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} F_t(x, y) \\
\end{align*}
\]
\[
- \frac{R_{m2}(\tilde{A}_2;0,y)(-y)^{\beta_1}}{|y|^m} F_t(0,y) \right] D^{\beta_1} \tilde{A}_1(y) f_2(y)dy \\
- \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{\mathbb{R}^n} \left[ \frac{R_{m1}(\tilde{A}_1;0,y)(x-y)^{\beta_2}}{|x-y|^m} F_t(x,y) \right] D^{\beta_2} \tilde{A}_2(y) f_2(y)dy \\
+ \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{\mathbb{R}^n} \left[ \frac{R_{m1}(\tilde{A}_1;0,y)(x-y)^{\beta_1}}{|x-y|^m} F_t(x,y) - \frac{(-y)^{\beta_1+\beta_2}}{|y|^m} F_t(0,y) \right] D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(0,y) f_2(y)dy \\
= L_5^{(1)} + L_5^{(2)} + L_5^{(3)} + L_5^{(4)} + L_5^{(5)} + L_5^{(6)}.
\]

Similar to the proof of Theorem 1, we get, for \( 1 < r_1, r_2 < \infty \) and \( 1/r_1 + 1/r_2 = 1/p = 1 \),

\[
\|L_5^{(1)}\| \leq C \int_{\mathbb{R}^n} \left( \frac{|x|}{|y|^{m+n+1-\delta}} + \frac{|x|^{e}}{|y|^{m+n+\epsilon-\delta}} \right) \prod_{j=1}^{2} \|R_{m_j}(\tilde{A}_j;x,y)\| f_2(y)dy \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \times \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} k^2 \left( \frac{|x|}{|y|^{n+1-\delta}} + \frac{|x|^{e}}{|y|^{n+\epsilon-\delta}} \right) |f(y)|dy \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-e-k})(2^k d)^{-n(1/p-\delta/n)} \|f \chi_{2^k Q}\|_{L^p} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-e-k}) \|f\|_{B^p} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^p}; \\
\|L_5^{(2)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} k \frac{|x|}{|y|^{n+1-\delta}} |f(y)|dy
\]
\[
\begin{align*}
&\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^s_p} \\
&\|L_5^{(3)}\| \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^s_p} \\
&\|L_5^{(4)}\| \leq C \sum_{|\beta_1|=m_1} \int_{\mathbb{R}^n} \left| \frac{(x-y)^{\beta_1} F_1(x,y) - (-y)^{\beta_1} F_1(0,y)}{|x-y|^m} \right| \\
&\quad \times |R_{m_2}(\tilde{A}_2; x, y)||D^{\beta_1} \tilde{A}_1(y)||f_2(y)| dy \\
&\quad + C \sum_{|\beta_1|=m_1} \int_{\mathbb{R}^n} \left| R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; 0, y) \right| \\
&\quad \times \left| \frac{(-y)^{\beta_1} F_1(0,y)}{|y|^m} \right||D^{\beta_1} \tilde{A}_1(y)||f_2(y)| dy \\
&\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \\
&\quad \times \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)} \|f \chi_{2^k Q}\|_{L^p} \\
&\quad \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_1} \tilde{A}_1(y)|^{p'} dy \right)^{1/p'} \\
&\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^s_p} \\
&\|L_5^{(5)}\| \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^s_p} \\
&\|L_5^{(6)}\| \leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \int_{\mathbb{R}^n} \left| \frac{(x-y)^{\beta_1+\beta_2} F_1(x,y) - (-y)^{\alpha_1+\alpha_2} F_1(0,y)}{|x-y|^m} \right| \\
&\quad \times |D^{\beta_1} \tilde{A}_1(y)||D^{\beta_2} \tilde{A}_2(y)||f_2(y)| dy \\
&\leq C \sum_{k=1}^{\infty} (2^{-k} + 2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)} \|f \chi_{2^k Q}\|_{L^p} \\
&\quad \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_1} \tilde{A}_1(y)|^{r_1} dy \right)^{1/r_1} \\
&\quad \times \sum_{|\beta_2|=m_2} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_2} \tilde{A}_2(y)|^{r_2} dy \right)^{1/r_2}
\end{align*}
\]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{B^p_q}; \]

Thus
\[ L_5 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{B^p_q}. \]

This finishes the proof of Theorem 3. \( \square \)

**Proof of Theorem 4.** Without loss of generality, we may assume \( l = 2 \). Let \( f \in H\hat{K}_p(R^n) \), by Lemma 1, \( f = \sum_{j=-\infty}^{\infty} \lambda_j a_j \), where \( a_j \)'s are the central \((n(1-1/p), p)\)-atom with \( \text{supp} a_j \subset B_j = B(0, 2^j) \) and \( ||f||_{H\hat{K}_p} \sim \sum_j |\lambda_j| \). We write
\[ ||\hat{T}_A^A(f)||_{L^q} = \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} ||\chi_k \hat{T}_A^A(f)||_{L^q} \]
\[ \leq \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=-\infty}^{k-1} ||\lambda_j|| ||\chi_k \hat{T}_A^A(a_j)||_{L^q} \]
\[ + \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} ||\lambda_j|| ||\chi_k \hat{T}_A^A(a_j)||_{L^q} = M_1 + M_2. \]

For \( M_2 \), by the \((L^p, L^q)\)-boundedness of \( \hat{T}_A^A \) for \( 1/q = 1/p - \delta/n \), we get
\[ M_2 \leq C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} ||\lambda_j|| ||a_j||_{L^p} \]
\[ \leq C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} ||\lambda_j|| 2^{jn(1/p-1)} \]
\[ \leq C \sum_{j=-\infty}^{\infty} ||\lambda_j|| \sum_{k=-\infty}^{j} 2^{(k-j)n(1-1/p)} \]
\[ \leq C \sum_{j=-\infty}^{\infty} ||\lambda_j|| \leq C ||f||_{H\hat{K}_p}. \]

To obtain the estimate of \( M_1 \), we denote that
\[ \hat{A}_j(x) = A_j(x) - \sum_{|\beta_j|=m_j} \frac{1}{\beta!} (D^{\beta} A_j)_{2Q} x^\beta. \]

Then \( Q_{m_j}(A_j; x, y) = Q_{m_j}(\hat{A}_j; x, y) \). We write, by the vanishing moment of \( a \),
\[ \hat{F}_t^A(a)(x) \]
\[ = \int_{R^n} \left[ \frac{F_t(x, y)}{|x-y|^m} - \frac{F_t(x, 0)}{|x|^m} \right] R_{m_1}(\hat{A}_1; x, y) R_{m_2}(\hat{A}_2; x, y) a(y) dy \]
\[ + \int_{\mathbb{R}^n} \frac{F_t(x,0)}{|x|^m} \left[ R_{m_1} (\tilde{A}_1; x, y) R_{m_2} (\tilde{A}_2; x, y) - R_{m_1} (\tilde{A}_1; x, 0) R_{m_2} (\tilde{A}_2; x, 0) \right] a(y) dy \\
- \sum_{|\beta_2| = m_2} \int_{\mathbb{R}^n} \left[ \frac{F_t(x,y)(x-y)^{\beta_2}}{|x-y|^m} - \frac{F_t(x,0)x^{\beta_2}}{|x|^m} \right] \\
\times R_{m_1} (\tilde{A}_1; x, y) D_{\beta_2} \tilde{A}_2(x) a(y) dy \\
- \sum_{|\beta_1| = m_1} \int_{\mathbb{R}^n} \left[ \frac{F_t(x,y)(x-y)^{\beta_1}}{|x-y|^m} - \frac{F_t(x,0)x^{\beta_1}}{|x|^m} \right] \\
\times R_{m_2} (\tilde{A}_2; x, y) D_{\beta_1} \tilde{A}_1(x) a(y) dy \\
- \sum_{|\beta_1| = m_1} \int_{\mathbb{R}^n} \left[ \frac{F_t(x,0)x^{\beta_1}}{|x|^m} \left[ R_{m_2} (\tilde{A}_2; x, y) - R_{m_2} (\tilde{A}_2; x, 0) \right] \\
\times D_{\beta_1} \tilde{A}_1(x) a(y) dy \\
+ \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \int_{\mathbb{R}^n} \left[ \frac{F_t(x,y)(x-y)^{\beta_1 + \beta_2}}{|x-y|^m} - \frac{F_t(x,0)x^{\beta_1 + \beta_2}}{|x|^m} \right] \\
\times D_{\beta_1} \tilde{A}_1(x) D_{\beta_2} \tilde{A}_2(x) a(y) dy. \]

Similar to the proof of Theorem 1 and Theorem 2, we get

\[ M_1 \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^\beta_j A_j\|_{BMO} \right) \\
\times \sum_{k=-\infty}^{\infty} 2^{kn(1-\delta/n)} \sum_{j=-\infty}^{k-1} |\lambda_j| \left[ \frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\epsilon}}{2^{k(n+\epsilon-\delta)}} \right] \\
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^\beta_j A_j\|_{BMO} \right) \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} [2^{j-k} + 2^{(j-k)\epsilon}] \\
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^\beta_j A_j\|_{BMO} \right) \sum_{j=-\infty}^{\infty} |\lambda_j| \\
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^\beta_j A_j\|_{BMO} \right) \|f\|_{H^K_{\epsilon}}. \]

This completes the proof of Theorem 4. \qed
4. Applications

Now we shall apply the theorems of the paper to some particular operators such as Littlewood-Paley operators and Marcinkiewicz operators.

**Application 1.** Littlewood-Paley operator.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$ and $\mu > (3n + 2 - 2\delta)/n$. Let $\psi$ be a fixed function which satisfies the following properties:

1. $\int_{\mathbb{R}^n} \psi(x)dx = 0$,
2. $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
3. $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon (1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g^A_\psi(f)(x) = \left( \int_0^\infty |F^A_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S^A_\psi(f)(x) = \left[ \int \int_{\Gamma(x)} |F^A_t(f)(x, y)|^2 \frac{dydt}{tn+1} \right]^{1/2}$$

and

$$g^A_\mu(f)(x) = \left[ \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{1 + |x - y|} \right)^{n\mu} |F^A_t(f)(x, y)|^2 \frac{dydt}{tn+1} \right]^{1/2},$$

where

$$F^A_t(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, y) \psi_t(x - y)f(y)dy,$$

$$F^A_t(f)(x, y) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, z) \frac{f(z)}{|x - z|^m} \psi_t(y - z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. The variants of $g^A_\psi$, $S^A_\psi$ and $g^A_\mu$ are defined by

$$\tilde{g}^A_\psi(f)(x) = \left( \int_0^\infty |\tilde{F}^A_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$\tilde{S}^A_\psi(f)(x) = \left[ \int \int_{\Gamma(x)} |\tilde{F}^A_t(f)(x, y)|^2 \frac{dydt}{tn+1} \right]^{1/2},$$

and

$$\tilde{g}^A_\mu(f)(x) = \left[ \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{1 + |x - y|} \right)^{n\mu} |\tilde{F}^A_t(f)(x, y)|^2 \frac{dydt}{tn+1} \right]^{1/2},$$

where

$$\tilde{F}^A_t(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l Q_{m_j+1}(A_j; x, y) \psi_t(x - y)f(y)dy.$$
and
\[
\tilde{F}^A_t(f)(x, y) = \int_{R^n} \prod_{j=1}^l Q_{m_j+1}(A_j; x, z) \frac{\psi_t(y - z) f(z) dz}{|x - z|^m}.
\]
Set \(F_t(f)(y) = f * \psi_t(y)\). We also define that
\[
g_{\psi}(f)(x) = \left( \int_{0}^{\infty} |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\]
\[
S_{\psi}(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}
\]
and
\[
g_{\mu}(f)(x) = \left( \int \int_{R^n+1} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
which are the Littlewood-Paley operators (see [17]). Let \(H\) be the space
\[
H = \left\{ h : ||h|| = \left( \int_{0}^{\infty} |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \right\}
\]
or
\[
H = \left\{ h : ||h|| = \left( \int \int_{R^n+1} |h(y, t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} < \infty \right\},
\]
then, for each fixed \(x \in R^n\), \(F^A_t(f)(x)\) and \(F^A_t(f)(x, y)\) may be viewed as the mapping from \([0, +\infty)\) to \(H\), and it is clear that
\[
g_{\psi}^A(f)(x) = ||F^A_t(f)(x)||, \quad g_{\psi}(f)(x) = ||F_t(f)(x)||,
\]
\[
S_{\psi}^A(f)(x) = ||\chi_{\Gamma(x)}F^A_t(f)(x, y)||, \quad S_{\psi}(f)(x) = ||\chi_{\Gamma(x)}F_t(f)(y)||
\]
and
\[
g_{\mu}^A(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F^A_t(f)(x, y) \right\|,
\]
\[
g_{\mu}(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|.
\]
It is easily to see that \(g_{\psi}, S_{\psi}\) and \(g_{\mu}\) satisfy the conditions of Theorem 1, 2, 3 and 4, thus the conclusions of Theorem 1, 2, 3 and 4 hold for \(g_{\psi}^A\) and \(\tilde{g}_{\psi}^A\), \(S_{\psi}^A\) and \(\tilde{S}_{\psi}^A\), \(g_{\mu}^A\) and \(\tilde{g}_{\mu}^A\).

**Application 2.** Marcinkiewicz operator.

Fixed \(0 \leq \delta < n\), Fix \(\lambda > \max(1, 2n/(n + 2 - 2\delta))\) and \(0 < \gamma \leq 1\). Let \(\Omega\) be homogeneous of degree zero on \(R^n\) with \(\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0\). Assume that \(\Omega \in Lip_{\gamma}(S^{n-1})\). The Marcinkiewicz multilinear operators are defined by (see [12], [18])
\[
\mu^A_{\Omega}(f)(x) = \left( \int_{0}^{\infty} |F^A_t(f)(x)|^2 \frac{dt}{t^{3}} \right)^{1/2},
\]
\[ \mu^A_S(f)(x) = \left[ \int \int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2} \]

and

\[ \mu^A_A(f)(x) = \left[ \int \int_{R^+_{n+1}} \left( \frac{t}{t + |x-y|} \right)^n |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}, \]

where

\[ F_t^A(f)(x) = \int_{|x-y| \leq t} \frac{\prod_{j=1}^{l} R_{m_{j+1}}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy \]

and

\[ F_t^A(f)(x, y) = \int_{|y-z| \leq t} \frac{\prod_{j=1}^{l} R_{m_{j+1}}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z) dz; \]

The variants of \( \mu^A_O, \mu^A_S \) and \( \mu^A_A \) are defined by

\[ \tilde{\mu}^A_O(f)(x) = \left( \int_{0}^{\infty} |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \]

\[ \tilde{\mu}^A_S(f)(x) = \left[ \int \int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2} \]

and

\[ \tilde{\mu}^A_A(f)(x) = \left[ \int \int_{R^+_{n+1}} \left( \frac{t}{t + |x-y|} \right)^n |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}, \]

where

\[ \tilde{F}_t^A(f)(x) = \int_{|x-y| \leq t} \frac{\prod_{j=1}^{l} Q_{m_{j+1}}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy \]

and

\[ \tilde{F}_t^A(f)(x, y) = \int_{|y-z| \leq t} \frac{\prod_{j=1}^{l} Q_{m_{j+1}}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z) dz. \]

Set

\[ F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy; \]

We also define that

\[ \mu^A_O(f)(x) = \left( \int_{0}^{\infty} |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \]

\[ \mu^A_S(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} \]
and
\[ \mu_\lambda(f)(x) = \left( \int \int_{R^{n+1}_+} \left( \frac{t}{t + |x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]

which are the Marcinkiewicz operators (see [18]). Let \( H \) be the space
\[ H = \left\{ h : ||h|| = \left( \int_0^\infty |h(t)|^2 dt / t^3 \right)^{1/2} < \infty \right\} \]
or
\[ H = \left\{ h : ||h|| = \left( \int \int_{R^{n+1}_+} |h(y,t)|^2 dydt / t^{n+3} \right)^{1/2} < \infty \right\}. \]
Then, it is clear that
\[ \mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||, \]
\[ \mu_S^A(f)(x) = ||\chi_{\Gamma(x)} F_t^A(f)(x,y)||, \quad \mu_S(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)|| \]
and
\[ \mu_\lambda^A(f)(x) = \left\| \left( \frac{t}{t + |x-y|} \right)^{n\lambda/2} F_t^A(f)(x,y) \right\|, \]
\[ \mu_\lambda(f)(x) = \left\| \left( \frac{t}{t + |x-y|} \right)^{n\lambda/2} F_t(f)(y) \right\|. \]
It is easily seen that \( \mu_\Omega, \mu_S \) and \( \mu_\lambda \) satisfy the conditions of Theorem 1, 2, 3 and 4, thus Theorem 1, 2, 3 and 4 hold for \( \mu_\Omega^A \) and \( \mu_\lambda^A \), \( \mu_S^A \), \( \mu_\lambda^A \) and \( \mu_\lambda \).

References


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