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A REFINEMENT OF THE CLASSICAL CLIFFORD INEQUALITY

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Abstract. We offer a refinement of the classical Clifford inequality about special linear series on smooth irreducible complex curves. Namely, we prove about curves of genus g and odd gonality at least 5 that for any linear series g^r_d with $d \leq g+1$, the inequality $3r \leq d$ holds, except in a few sporadic cases. Further, we show that the dimension of the set of curves in the moduli space for which there exists a linear series g^r_d with $d < 3r$ for $d \leq g+l$, $0 \leq l \leq \frac{g}{2}-3$, is bounded by $2g-1+\frac{1}{3}(g+2l+1)$.

1. Introduction and preliminary results

In the article we deal with complex algebraic curves and linear series on them. The classical Clifford theorem says that if D is a divisor on a smooth curve of genus g, then $2\dim|D| \leq \deg D$, provided that $0 \leq \deg D \leq 2g$. It is known that if we reduce the range for $\deg D$ then the inequality can be strengthen. So, the paper is a contribution to the following question:

Question. *What is the maximal $d_0 = d_0(k)$ such that for any linear series g^r_d on a smooth curve of genus g and gonality $k \geq k_0$ such that $0 \leq d \leq d_0$, the inequality*

$$3r \leq d \tag{1}$$

holds?

Results related to this questions have been obtained by Coppens, Kato, Keem, Martens among others. The first main result in this paper is contained in the next theorem:

Theorem 1. *Let C be an odd-gonal smooth curve of genus $g \geq 10$. Then for any linear series g^r_d on C with $d \leq g+1$, the inequality (1) holds unless*

- *C is 3-gonal;*
- *C is 5-gonal and :*
  (1) *C has a plane model of degree 7 and*

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(i) $g = 14$ *and there exists a very ample* $g^4_{12} = g^2_7 + g^1_5$, *or*
(ii) $g = 13$ *and there exists a simple* g^4_{12} *having exactly one double point given by the unique* g^2_7 *as* $g^2_7 + g^1_5$, *or*

(2) *for* $r = 4, \ldots, 7$, $g = 3r - 2$, *in which case* $g^r_d \equiv g^r_{3r-1}$ *is very ample, or*

(3) $g = 22$, $r = 8$, *in which case* $g^r_d \equiv g^8_{23}$ *is very ample, its dual* g^6_{19} *is also very ample and decomposes into* $g^6_{19} = g^1_5 + g^4_{14}$, *where* g^4_{14} *is very ample too and embeds* $C \hookrightarrow \mathbb{P}^4$ *as an extremal curve.*

In fact, it refines the following result obtained by Martens.

Proposition 1.1. [19, p.81] *For an odd-gonal curve* C *of genus* g *and a linear series* g^r_d *on it with* $0 < d \leq g - 1$, *the inequality* (1) *holds. Further, it becomes equality if and only if*

- *C is 3-gonal and*
  (i) $g^r_d = rg^1_3$ (g^1_3 *is unique for* $g > 4$), *or*
  (ii) $g \equiv 1 \mod 3$ *and* $|K - g^r_d| = \frac{g-1}{3} g^1_3$, (*for* $g > 4$ *this is a multiple of* g^1_3, *i.e. a theta characteristic*)
- *or we are in one of the sporadic cases* (*with* $10 \leq g \leq 16$)*:*
  (1) *C is a smooth plane sextic,* $g^r_d \equiv g^2_6$, *or*
  (2) *C has a plane model of degree 7 and*
    (i) $g = 14$, $g^r_d \equiv g^4_{12} = g^2_7 + g^1_5$ *very ample* (g^2_7 *and* g^1_5 *are unique*), *or*
    (ii) $g = 13$, $g^r_d \equiv g^4_{12}$ *is simple having exactly one double point given by the unique* g^2_7 *as* $g^2_7 + g^1_5$ (g^1_5 *is one of the at most 2 different pencils of degree 5*), *or*
  (3) *C is an extremal space curve of degree 10* ($\Rightarrow g = 16$), $g^r_d \equiv g^5_{15} = g^3_{10} + g^1_5$ *is very ample* (g^3_{10} *is unique and very ample,* g^1_5 *is one of the at most 2 different pencils of degree 5*).

Just as it is the case here, many results about linear series on algebraic curves of genus g are often formulated in terms of gonality, which of its turn provides a stratification of the moduli space as $\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \cdots \subset \mathcal{M}^1_{g,[\frac{g+3}{2}]} \equiv \mathcal{M}_g$, where $\mathcal{M}^1_{g,k}$ is the subset of curves of gonality k, and as it is well known, $\dim \mathcal{M}^1_{g,k} = 2g + 2k - 5$, $k = 2, 3, \ldots, [\frac{g+3}{2}]$. This suggests that it is also relevant to ask for the codimension of the complement to the subset in the moduli space for which the desired properties hold. Thus the second main result in the paper presents an upper bound for the dimension of the subset of curves in $\mathcal{M}_g$ on which there exists g^r_d such that inequality (1) fails.

Theorem 2. *Denote by* $\mathcal{M}^{(3)} := \{C \in \mathcal{M}_g | \exists g^r_d \text{ on } C, d \leq g + l, d < 3r\}$ *where it is assumed that* g *and* l *are integers such that* $g \geq 7$ *and* $0 \leq l \leq \frac{g}{2} - 3$. *Then*

$$\dim \mathcal{M}^{(3)} \leq 2g - 1 + \frac{1}{3}(g + 2l + 1).$$

Our interest in this topic was motivated by a simple observation derived from the following statement proven by Coppens and Martens:

Proposition 1.2. [6, Theorem B] *On a k-gonal curve C, $k \geq 3$, of genus g, any linear series g^r_d of degree $k-3 \leq d \leq 2g-2-(k-3)$ satisfies*

$$2r \leq d-(k-3).$$

From here it is easy to see that on a general curve (in the sense of moduli) inequality (1) will hold for all values of d up to a number close to the maximal one allowed by the Riemann-Roch theorem.

Corollary 1.3. *Let C be a nonsingular curve of genus $g \geq 3$ and gonality $k = [\frac{g+3}{2}]$. Then inequality (1) holds for any linear series g^r_d on C such that $0 \leq d \leq 2g-2-k$, unless C is a plane quintic and $g^r_d \equiv g^2_5$ is very ample.*

Proof. If $g = 2g'+1$ is odd, we get $k = [\frac{g+3}{2}] = g'+2$, and since $r \leq g-k = 2g'+1-g'-2 = g'-1$, we obtain $d \geq 2r+g'-1 \geq 3r$. If $g = 2g'+2$ is even, we have $k = g'+2$, $r \leq g-k = g'$ and then $d \geq 2r+g'-1 \geq 3r-1$. Therefore, (1) could possibly fail if and only if

$$r = g' \text{ and } d = 3g'-1.$$

But if this is the case, we easily check that the residual linear series $g^{r'}_{d'} \equiv |K_C(-g^r_d)|$ is very ample and $r' = 2$, where as usual K_C is the canonical sheaf and $r' = g-d+r-1$, $d' = 2g-2-d$. To see this, notice that $d' = g'+3 = k+1$ and by Riemann-Roch theorem $r' = g'-3g'+1+2g'+2-1 = 2$. Also, for any two points $P, Q \in C$ we have $\deg|g^{r'}_{d'}(-P-Q)| = d'-2 = k-1$, therefore $\dim|g^2_{g'+3}(-P-Q)| \leq 0$, and thus $g^{r'}_{d'} = g^2_{g'+3}$ must be very ample. Therefore C has a smooth plane model of degree $g'+3$ and by the genus formula $\frac{(g'+2)(g'+1)}{2} = 2g'+2$. Hence, $g' = 2$, i.e., $g = 6$, and $g^r_d \equiv g^{r'}_{d'} \equiv g^2_5$ gives the embedding in $\mathbb{P}^2$. □

This suggests that it is natural to expect that with increasing the gonality k of the curve C the range $0 \leq d \leq d_0(k)$ where the inequality (1) holds for any g^r_d on C will also increase.

In the first half of the paper, we will consider odd-gonal curves, studying those series g^r_d for which r is maximal for the given degree d. As we extend the scope of validity of inequality (1) from previously known results, we may assume that g^r_d is base point free and complete linear series. Remark that in [6] Coppens and Martens have proven that (1) holds for all g^r_d's on a smooth odd-gonal curve C of genus g, assuming $d \leq g-1$, Martens have identified in [19] the cases in which there is equality in (1), while the case of even-gonal C has been studied by Kato in [16]. Since for $d \leq g-1$ Theorem 1 is simply the result of Coppens and Martens, we need to focus on the cases $d = g$ and $d = g+1$ only. We will resolve them in the next section, where we will also show through examples the existence of curves (and linear series on them) like those in cases (2) and (3) of Theorem 1, i.e. the new ones in comparison to

Proposition 1.1. In the second half of the paper, we will offer some further extensions of the Clifford inequality mostly oriented toward dealing with linear series g^r_d of degree $d \geq g$. In Proposition 3.1 we will identify the possible exceptions essentially in terms of Clifford index of the curve, while in Theorem 2 we will do this in terms of moduli of curves.

Notations and some preliminary results

The notations in the paper are for the most part like those in [3]. In addition, we occasionally write $g^r_d \in W^r_d(C)$ or $E \in W^r_d(C)$ for a complete linear series g^r_d or a positive divisor of degree d meaning in fact the associated line bundle. Regarding the tensor product of line bundles, we write $L(-E)$, or $L(-g^r_d)$ meaning the tensor product over $\mathbb{C}$ of L with inverse of the line bundle associated to the divisor E or the linear series g^r_d, correspondingly. Also, we denote by K_C or simply K the canonical line bundle of the curve C. By $\mathrm{Cliff}(L)$ we denote the Clifford index of a line bundle L on C, i.e. $\mathrm{Cliff}(L) = \deg L - 2(h^0(C,L) - 1)$, and by $\mathrm{Cliff}(C)$ we will denote the minimal number $\mathrm{Cliff}(L)$ among those for which also $h^0(C,L) \geq 2$ and $h^1(C,L) \geq 2$. When working with a fixed curve, as in section 2, we denote its gonality with a single letter, but when we deal with families of curves, as in section 3, we opt for the notation $\mathrm{gon}(C)$ for an element C. A line bundle L on a smooth curve C will be called *strictly birationally very ample* if the associated morphism Φ_L is a birational, but not very ample. The last is equivalent to $h^0(C, L(-p-q)) = h^0(C,L) - 2$ except for finitely many (but at least one) pairs of points $p, q \in C$.

Before moving to the proofs, recall the following results which we will refer to several times.

Proposition 1.4. [11, p.525] *The minimal possible degree for a surface S in $\mathbb{P}^r$ is $r-1$. If $\deg S = r-1$ then it is either a Veronese surface in $\mathbb{P}^5$ or a rational normal surface scroll in $\mathbb{P}^r$.*

Proposition 1.5. [12, p.87] *Let $C \hookrightarrow \mathbb{P}^r$ be an integral curve properly contained in $\mathbb{P}^r$, $r \geq 3$. Then*

$$p_a(C) \leq \pi(d,r),$$

where $\pi(d,r) = \binom{m}{2}(r-1) + m\varepsilon$, with $m := [\frac{d-1}{r-1}]$ and $\varepsilon = d-1-m(r-1)$. Further, equality is possible only if C lies on a surface of minimal degree in $\mathbb{P}^r$.

Proposition 1.6. [12, p.99] *Let $C \hookrightarrow \mathbb{P}^r$ be an integral curve properly contained in $\mathbb{P}^r$, $r \geq 3$. If we have for its arithmetic genus that*

$$p_a(C) > \pi_1(d,r),$$

where $\pi_1(d,r) = \binom{m_1}{2}r + m_1\varepsilon_1 + \mu_1$, with $m_1 := [\frac{d-1}{r}]$, $\varepsilon_1 = d-1-m_1 r$ and $\mu_1 = 0$ for $\varepsilon_1 \neq r-1$ and $\mu_1 = 1$ only if $\varepsilon_1 = r-1$, then C lies on a surface of minimal degree in $\mathbb{P}^r$. Further, if $p_a(C) = \pi_1(d,r)$ then C may lie on surface of degree r in $\mathbb{P}^r$ as well.

Proposition 1.7. [21] *Let S be a surface scroll (possibly singular) and $C \subset S$ an integral curve on S not lying in the singular locus of S. Let n be the degree of the mapping from C to a general hyperplane section H of S induced by the ruling on S. Then*

$$p_a(C) = (n-1)(\deg C - 1 - \frac{1}{2} n \deg S) + n g_0,$$

where g_0 is the sectional genus of S, i.e. $g_0 = p_g(H)$.

2. Proof of Theorem 1 and some related examples

2.1. Proof of Theorem 1

Before launching into the proof let's note that case (1) from Proposition 1.1 is included in case (3) of Theorem 1 (the one in which $r = 4$ and $g = 10$), while the absence of case (2) from the proposition among the exceptional cases in our theorem will become clear in the course of the proof.

Assume that the gonality of C is at least 5 and also that C is none of the curves described in sporadic cases (1) and (2) of Proposition 1.1. Assume further that g^r_d on C is a linear series for which inequality (1) is violated. Then it is base point free because if it were not base point free, we could obtain the claim of the theorem simply by induction once we have the case $d = g$ resolved. For it, let $g^r_d \equiv g^r_g$ be a linear series on C. Then its dual is $g^{r'}_{d'} \equiv g^{r-1}_{g-2}$. Applying Proposition 1.1, we get $g - 2 - 3(r-1) \geq 1$, or $0 \leq g - 3r = d - 3r$. Remark that here we could not possibly encounter case (3) from Proposition 1.1, since in such a case the only possible violation of (1) could be for $g^r_d \equiv g^r_g \equiv g^6_{16}$, but then we would have $g^{r'}_{d'} = g^5_{14}$ which is impossible under the assumptions in the theorem.

In this way, we may consider the natural morphism $\Phi_{g^r_d}$ associated with the given linear series and focus on the case $d = g + 1$.

Lemma 2.1. *Let C be a non-singular curve of genus $g \geq 6$ and gonality $k \geq 3$, such that if k is even then we assume in addition that C is not a double cover of a curve of genus no less than $\frac{k-3}{2}$. Let g^r_d with $g \leq d \leq 2g - k$ be a base point free linear series on C for which inequality (1) fails. Then the associated morphism $\Phi_{g^r_d}$ is simple.*

Proof. Let C' be the image of C under the morphism $\Phi_{g^r_d}$ associated to g^r_d, i.e. $C' := \Phi_{g^r_d}(C)$. Let $\pi : \widetilde{C}' \to C'$ be the normalization of C'. Assume that g^r_d is not simple, i.e. $n := deg(\Phi_{g^r_d}) \geq 2$. The degree of π is 1 (π is birational morphism), wherefrom there is a linear series $\widetilde{g}'^r_{\frac{d}{n}}$ on $\widetilde{C}'$, such that

$$g^r_d = \phi^*(\widetilde{g}'^r_{\frac{d}{n}}),$$

where $\phi : C \to \widetilde{C}'$ is the morphism induced by $\Phi_{g^r_d}$, i.e. $\pi \circ \phi \equiv \Phi_{g^r_d}$. It is easy to see that $\widetilde{g}'^r_{\frac{d}{n}}$ on $\widetilde{C}'$ is non-special. If we assume the opposite, then by the Clifford inequality we obtain $\frac{d}{n} - 2r \geq 0$. Therefore $d \geq 2nr \geq 4r$, which contradicts to the assumption $d < 3r$. By the Riemann-Roch theorem, $\widetilde{g}' = \frac{d}{n} - r$, where $\widetilde{g}'$ is genus of $\widetilde{C}'$. From it and from $d - 3r < 0$, we get $\widetilde{g}' = \frac{d}{n} - r < \frac{3r}{n} - r = (3-n)\frac{r}{n}$, and since $n \geq 2$, we conclude that $n = 2$. Hence, $\deg \phi = \deg \Phi_{g^r_d} = 2$. Next, it is easy to see that for g^1_k on C computing the gonality k, the associated mappings $\Phi_{g^1_k} : C \to \mathbb{P}^1$ and $\phi : C \to \widetilde{C}'$ don't factor in a non-trivial way through another curve. If k is odd, this is obvious. If k is even we couldn't have a double covering $C \to \widetilde{C}'$ since in such case $r \geq \frac{d}{2} - \widetilde{g}'$ and applying Proposition 1.2, we get $\widetilde{g}' \geq \frac{d}{2} - r \geq \frac{k-3}{2}$, contrary to the assumptions in the lemma. Applying Castelnuovo-Severi inequality for the maps $\Phi_{g^1_k}$, ϕ and curves C, $\widetilde{C}'$ and $\mathbb{P}^1$, we obtain $g \leq (k-1) + 2\widetilde{g}' = k - 1 + d - 2r$. Since $k \leq [\frac{g+3}{2}]$, or equivalently $-k \geq -\frac{g+3}{2}$, we get $2d \geq 2g - 2k + 4r + 2 \geq 2g - (g+3) + 2 + 4r = g - 1 + 4r \geq g - 1 + r + d + 1$ (here we use assumption $3r \geq d + 1$), or simply:

$$0 \geq -h^1(C, g^r_d) = d - g - r \geq 0.$$

The last would be a contradiction, unless we have equalities everywhere in the above sequence. This forces $k = \frac{g+3}{2}$, $h^1(C, g^r_d) = 0$, $r = d - g$, and $d = 3r - 1$, from where $g = 2r - 1$ and $k = r + 1$. But then we obtain $d - (2g - k) = 3r - 1 - (4r - 2 - r - 1) = 2 > 0$, which contradicts to the assumption $d \leq 2g - k$. □

So, assume that for a given $g^r_d \equiv g^r_{g+1}$ inequality (1) is violated. By the lemma the linear series must be simple. Our next observation is that we must have $d \geq 3r - 2$. To see this assume the opposite, i.e. $d \leq 3r - 3$. Then by the Castelnuovo inequality, i.e. Proposition 1.5, we have $g \leq \pi(d, r) = \binom{m}{2}(r-1) + m\varepsilon$, where $m = [\frac{d-1}{r-1}] = 2$ and $\varepsilon = d - 1 - 2(r-1)$. Hence $d - 1 = g \leq r - 1 + 2(d - 2r + 1) = 2d - 3r + 1$, or $0 \leq d - 3r + 2$ which is of course a contradiction and this proves the claim.

It remains to consider the cases $d = 3r - 2$ and $d = 3r - 1$.

Case $\mathbf{d = 3r - 2}$. Here we obtain for $\pi(d, r)$ that $\pi(3r - 2, r) = 3r - 3 = g$ and so C must be an extremal curve. Then by [3, Theorem 2.5, p.122], it needs to be one of the following:

(1) an image of a smooth plane curve under the Veronese imbedding $\nu_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$;
(2) a non-singular member of $|mH + L|$ on a rational normal scroll, where H is a general hyperplane section, L is the ruling of the scroll and $m = 3$;
(3) a non-singular member of $|(m+1)H - (r - \varepsilon - 2)L|$ on a rational normal scroll, H and L like above, $m = 3$.

None of these options is possible: (1) is impossible because in it $r = 5$, $d = 3r - 2 = 13$ and $g = 12$, which means that C needs to have a smooth plane model of genus 12 which is impossible; (2) and (3) are not possible either because projecting C along the ruling L of the rational normal scroll to a hyperplane section induces either $g^1_m \equiv g^1_3$ or $g^1_{m+1} \equiv g^1_4$, i.e. the gonality of C is at most 4, which is not allowed by assumption.

Case $d = 3r - 1$. First, Proposition 1.2 gives $2r \leq d - (k-3)$, or just $k \leq r+2$. Consider now the dual of g^r_d linear series, which we denote by $g^{r'}_{d'}$. By the Riemann-Roch theorem, $g^{r'}_{d'} \equiv g^{r-2}_{3r-5}$. From this it immediately follows that $g^{r'}_{d'}$ is base point free, because otherwise we would have $g^{r-2}_{3(r-2)}$, with $3(r-2) = g-4$, which is impossible under the assumptions in theorem.

Next we claim that this $g^{r'}_{d'}$ is birationally very ample. Suppose it is not, and assume that it induces a morphism $\Phi_{g^{r'}_{d'}}$ of degree $m \geq 2$ onto the image $C' = \Phi_{g^{r'}_{d'}}(C)$. By the standard Clifford inequality, the induced linear series $g^{r-2}_{\frac{3r-5}{m}}$ is non-special and further it is easily seen that $m = 2$. Since $g^{r'}_{d'}$ is complete, $g^{r-2}_{\frac{3r-5}{2}}$ is also complete, wherefrom we find for the genus g' of C' that $g' = \frac{r-1}{2}$. Then the morphisms $\Phi_{g^1_k} : C \to \mathbb{P}^1$ and $\Phi_{g^{r'}_{d'}} : C \to C'$ can not factor through a proper mapping and we find by the Castelnuovo-Severi inequality that $3r-2 = g \leq 2\frac{r-1}{2} + (k-1)$, i.e. $2r \leq k$. Since $k \leq r+2$, we obtain that in this case $r \leq 2$, which is impossible. So, we conclude that $g^{r'}_{d'}$ must be birationally very ample.

Now applying Proposition 1.6 for $\pi_1(d', r')$ we find $\pi_1(3r-5, r-2) = \binom{3}{2}(r-2)+3 = 3r-3$, while $p_a(C') \geq g = 3r-2$ and so we obtain that C' lies on a surface S' of degree $r-3$ in $\mathbb{P}^{r-2}$. By Proposition 1.4, S' is either a rational normal scroll in $\mathbb{P}^{r-2}$ or the Veronese surface in $\mathbb{P}^5$, the last only possible for $r = 7$. Next we consider those two possibilities.

1. Let $C' \hookrightarrow S'$, S' a rational normal surface in $\mathbb{P}^{r-2}$.

1.1. Assume first that $p_a(C') = g = 3r-2$, i.e. C' is a smooth curve. It follows then by the Segre formula, i.e. Proposition 1.7, that

$$3r - 2 = (n-1)(\deg C' - 1 - \frac{1}{2}n \deg S') + ng_0,$$

where n is the degree of the morphism from C' to the hyperplane section of S' and g_0 is the sectional genus of S'. Since $g_0 = 0$, we obtain for n

$$(r-3)n^2 - (7r-15)n + 12r - 16 = 0.$$

This gives $n_{1,2} = \frac{7r-15\pm\sqrt{r^2-2r+33}}{2(r-3)}$. For $r > 8$ we get $n < 5$. Since C' is a $n:1$ covering of a rational curve, it follows that the gonality of C is less than 5, contrary to the assumptions in the theorem. For $r = 3, \ldots, 7$, the equation has no integer solutions. For $r = 8$, we obtain $n = 5$, i.e. C is 5-gonal curve at worst and we might have $g^r_d \equiv g^8_{23}$. Thus, for $r = 8$ the linear series g^6_{19} induces an embedding of C into a rational normal scroll. Since $n = 5$, C needs to be a $5:1$ covering of the hyperplane section, or equivalently C has a 5-secant line. Projecting from such a line gives g^4_{14} on C. It is easy to see (applying again the Castelnuovo genus-bound formula) that this g^4_{14} is base point free and complete.

Further, we claim that it is not compounded. To see this assume the opposite. Then C can be only a $2:1$ covering of another curve, lets say Γ of genus γ. The covering induces a linear series g_7^4 on Γ, which is obviously non-special, therefore $\gamma \leq 7-4=3$. Since C is assumed to be 5-gonal, we get by the Castelnuovo-Severi inequality $22 = g \leq (5-1)(2-1) + 2.3 = 11$, which is a contradiction. Thus, it remains for g_{14}^4 to be birationally very ample. Calculating now $\pi(14,4)$, we find $g = 22 = \pi(14,4)$, or in other words $C' := \Phi_{g_{14}^4}(C)$ is an extremal curve, which implies that C' is smooth or equivalently, g_{14}^4 is very ample.

1.2. Assume now that $3r-2 = g < p_a(C')$. By the Segre's formula we obtain

$$3r - 1 \leq p_a(C') = (n-1)(3(r-2) - \frac{1}{2}n(r-3)),$$

which is equivalent to

$$(r-3)n^2 - (7r-15)n + 12r - 14 \leq 0,$$

or also

$$(r-3)(n - \frac{4r-6}{r-3})(n-3) \leq -4.$$

For $r \geq 8$ we obtain $n < 5$, so need to check only the cases $r = 4, \ldots, 7$. For $r = 5,6,7$, we obtain that $n < 7$. For $r = 4$, we get $g_{d'}^{r'} = g_7^2$, i.e., C' is a plane model of C of degree 7 while $g = 10$, and projecting from a singular point of the plane model yields g_5^1. So, C is 5-gonal at worst.

2. Let $C' \hookrightarrow S'$, where S' is the Veronese surface in $\mathbb{P}^5$. This can occur only if $r = 7$. In this case $\deg C' = d' = 3r - 5 = 16$ and $p_a(C') \geq g = 3r - 2 = 19$. As it is well known, see for instance [13, Ex.13, p.13], every curve on S' is the intersection of S' with a hypersurface F in $\mathbb{P}^5$. Since $\deg C' = 16$, we get that F is of degree 4. For a hyperplane section H of S' we have $K_{S'}.H = -H^2 + 2p_a(H) - 2 = -6$, and from $C' \sim 4H$, we get by the adjunction formula $p_a(C') = 21$. So, pulling-back C' down to $\mathbb{P}^2$ gives a singular plane model of C of degree 8. Then a projection from one of the singular points to $\mathbb{P}^1$ gives a mapping of degree at most 6, and due to the assumption for odd-gonality, C must be of at most 5-gonal.

To finish the proof, it remains only to see that in all cases $g_d^r \equiv g_{3r-1}^r$ is very ample. For this, we need first to rule out the possibility of C being like the curve in case (3) of Proposition 1.1. The only possible exception that we need to resolve is for $g_d^r \equiv g_{3r-1}^r \equiv g_{17}^6$. Since in this case there exists a very ample g_{15}^5 on C, then we have for that g_{15}^5 and the dual g_{13}^4 of g_{17}^6 that $|g_{15}^5 - g_{13}^4| = \emptyset$. As it was already explained, g_{13}^4 is base point free and simple. Then we can apply [3, Ex. B-6, p.138] (in its correct form!), to get that $\dim |g_{15}^5 + g_{13}^4| \geq 14$, but by classical Clifford inequality this can happen only if C is hyperelliptic, which is impossible under the assumptions in the theorem. Finally, recall that our $g_d^r \equiv g_{3r-1}^r$, which we want to prove very ample, is necessarily base point free and birationally very ample. Assume that it is not very ample. Then there exist points P and Q on C, such that $\dim |g_{3r-1}^r - P - Q| \geq \dim |g_{3r-1}^r| - 1$, or in other words we get $g^{r-1}_{3(r-1)}$ on C. But such a possibility has already been ruled out in the assumptions of the theorem as it is one of the exceptional cases in the Martens classification. This completes the proof of Theorem 1.

2.2. Existence of the sporadic cases (2) and (3) from Theorem 1

It is worth confirming that curves as the ones described in (2) and (3) of Theorem 1 do exist.

Case (2) of Theorem 1. The examples involve curves on smooth del-Pezzo surfaces. Recall first that a smooth del-Pezzo surface $S_r \hookrightarrow \mathbb{P}^r$, $r = 4, \dots, 7$ has the following properties (see [13, V.4]) :

- it is anti-canonically embedded in $\mathbb{P}^r$ as a surface of degree r and can be regarded as the blow-up of $\mathbb{P}^2$ at $9 - r$ points in general position;
- its Picard group is generated by the transform l of a line in $\mathbb{P}^2$ and e_i, $i = 1, \dots, 9 - r$, where $e_1, \dots, e_{9-r}$ are the exceptional divisors;
- the intersection numbers are: $l^2 = -e_i^2 = 1$, and $l.e_i = e_i.e_j = 0$, for $i \neq j = 1, \dots, 9 - r$.

Thus, a divisor $D \in \mathrm{Pic}(S_r)$ decomposes as

$$D \sim a.l - \sum_{i=1,\dots,9-r} b_i.e_i,$$

where $a, b_1, \dots, b_{9-r}$ are integers, and for such D we have

$$\deg D = 3a - \sum_{i=1,\dots,9-r} b_i, \tag{2}$$

and for its arithmetic genus

$$p_a(D) = \frac{1}{2}(K_{S_r}.D + D^2) + 1 = \frac{1}{2}(a^2 - 3a - \sum_{i=1,\dots,9-r} (b_i^2 - b_i)) + 1, \tag{3}$$

where $K_{S_r} = -3l + \sum_{i=1,\dots,9-r} e_i$ denotes the canonical divisor of S_r.

Remark 2.2. By [14] and [8], it is known that if $b_1 \geq \cdots \geq b_{9-r} \geq 0$ and $a \geq b_1 + \cdots + b_h$, with $h = \min\{3, h\}$, then the linear series $|D|$ on S_r contains a smooth, connected curve, with the exception of $D \sim a(l - e_1)$, $a \geq 2$.

To show the existence of curves and linear series with the prescribed properties, we consider:

For $r = 4$, $g = 10$ and g^4_{11} , the curves on S_4 in the linearly equivalence class of

$$7l - \sum_{i=1,\dots,5} 2e_i.$$

By Remark 2.2, it follows that we can choose a smooth curve C in this class. By formulas (2) and (3), we find that $\deg C = 21 - 10 = 11$ and for its genus $p_a(C) = \frac{1}{2}(49 - 21 - 5(4 - 2)) + 1 = 10$. The linear system g^4_{11} on C cut by the hyperplanes in $\mathbb{P}^4$ is easily seen to be complete. Finally, by [3, Exer. I.20, p.56] C is necessarily 5-gonal.

For $r = 5$, $g = 13$ and g^5_{14}, the curves on S_5 in the equivalence class of

$$8l - 3e_1 - 3e_2 - 2e_3 - 2e_4.$$

Again by Remark 2.2, we can choose a smooth C in this class. By (2) and (3), we find that $\deg C = 24 - 10 = 14$ and for its genus $p_a(C) = \frac{1}{2}(64 - 24 - (18 + 8 - 6 - 4)) + 1 = 13$. The linear system g^5_{14} on C cut by the hyperplanes in $\mathbb{P}^5$ is complete. Blowing down to $\mathbb{P}^2$ we get a plane model of degree 8 for C that has two singular points of multiplicity 3 and two singular points of multiplicity 2. Projecting from a singular point with multiplicity 3 to $\mathbb{P}^1$ yields g^1_5 on C. Further, there is no g^1_m with $m \leq 4$ on C, because otherwise we would get by the Castelnuovo-Severi inequality $13 = g \leq (5 - 1)(4 - 1) = 12$ which is impossible. Thus, C must be 5-gonal.

For $r = 6$, $g = 16$ and g^6_{17}, the curves on S_6 in the equivalence class of

$$8l - 3e_1 - 2e_2 - 2e_3.$$

Remark 2.2 guarantees the existence of a smooth C in this class. Using again (2) and (3), we find $\deg C = 24-7 = 17$ and $p_a(C) = \frac{1}{2}(64-24-(9+8-3-4))+1 = 16$. The linear system g^6_{17} on C cut by the hyperplanes in $\mathbb{P}^6$ is complete. Blowing down to $\mathbb{P}^2$, we get a plane model of degree 8 for C that has one singular point of multiplicity 3 and two singular points of multiplicity 2. Projecting from the singular point of multiplicity 3 to $\mathbb{P}^1$ gives g^1_5 on C. Just like in the previous paragraph it follows that C is 5-gonal.

For $r = 7$, $g = 19$ and g^7_{20}, the curves on S_7 in the equivalence class of

$$9l - 4e_1 - 3e_2.$$

The check is similar to the ones carried above.

Case (3) of Theorem 1. The example involves extremal curves on smooth rational surface scroll. According to [5, Theorem p.353], the Hilbert scheme $H_{d,g,r}$ of smooth irreducible curves of maximal genus in $\mathbb{P}^r$ has a component of curves $C \sim (m+1)H - (r - 2 - \varepsilon)L$ found on a smooth rational surface scroll, where H is the hyperplane section and L the ruling of the surface scroll, and $m = [\frac{d-1}{r-1}]$, $\varepsilon = d - 1 - m(r-1)$. Taking a smooth curve C in such a component of $H_{14,22,4}$, we get $m = 4$ and thus $C \sim 5H - L$ on the corresponding surface scroll. The morphism $\Phi_L : C \to \mathbb{P}^1$ induced by the ruling L gives a pencil g^1_5 on C. It is easy to see that C is precisely 5-gonal. Further, consider the very ample linear series g^4_{14} on C induced by $C \hookrightarrow \mathbb{P}^4$. By [10, 5.2] or [7, Lemma 1.8, p.39], we find $\dim |g^4_{14} + g^1_5| \geq 6$, and next it is not difficult to check that $g^6_{19} = |g^4_{14} + g^1_5|$, and also that this g^6_{19} must be very ample. Taking its residual gives a linear series g^8_{23} on the smooth curve C of genus 22.

3. Further improvements

The results in the previous section suggest that the dimension of the subset in the moduli space of curves of genus g formed by curves for which there is g^r_d

with $3r > d$ increases with the increase of d with respect to g. The next two results offer some bounds for its dimension.

Proposition 3.1. *Let C be a smooth odd gonal curve of genus g and gonality $\operatorname{gon}(C) \geq 5 + 2l$, where l is an integer $0 \leq l \leq [\frac{g}{4}] - 2$. Then for any linear series g^r_d on C for which $0 \leq d \leq g + l$ we have*

$$d \geq 3r - 1$$

as equality is possible for at most finitely many (possibly none) linear series on C.

Proof. The proof is similar to the proof of the main theorem in [4]. We may assume $d = g + l$ since otherwise the claim would follow by induction on l. All we have to do is show that are there at most finitely many linear series g^r_d such that $3r - 1 \geq d$ and none for $3r - 2 \geq d$. So, assume that there exists a complete linear series g^r_d on C such that $3r > d$, where $d \leq g + l$. Let $g^{r'}_{d'}$ be its dual. Therefore $\dim W^{r'}_{d'}(C) = \dim W^r_d(C) \geq 0$. Let $W \subset \dim W^{r'}_{d'}(C)$ be an irreducible component of maximal dimension. Let $\alpha \geq 0$ be the degree of the base locus of a general $g^{r'}_{d'} \in W$. Then, $\dim W = \dim W^{r'}_{d'}(C) \leq \dim U + \alpha$, where $U \subset \dim W^{r'}_{d'-\alpha}(C)$. Then we consider separately the following two cases:

(i) When a general $g^{r'}_{d'-\alpha} \in U$ is simple;
(ii) When a general $g^{r'}_{d'-\alpha} \in U$ is compounded.

(i) In this case using [3, Ex-2, p.1999], we get for a general $E \in U \subset W^{r'}_{d'-\alpha}(C)$

$$\begin{aligned} 0 &\leq \dim U \\ &\leq h^0(C, \mathcal{O}_C(2E)) - 3r' + \alpha \\ &= h^1(C, \mathcal{O}_C(2E)) + 2(d' - \alpha) - g + 1 - 3r' + \alpha \\ &= h^1(C, \mathcal{O}_C(2E)) + (d - 3r) - \alpha + d - g - r + r' + 1 \\ &= h^1(C, \mathcal{O}_C(2E)) + (d - 3r) - \alpha. \end{aligned}$$

If $h^1(C, \mathcal{O}_C(2E)) \geq 2$, then the linear series $|2E|$ contributes to the Clifford index of C (that $h^0(C, \mathcal{O}_C(2E)) \geq 2$ follows easily from $h^0(C, \mathcal{O}_C(E)) = r - d + g \geq \frac{d+1}{3} - d + g \geq \frac{g-2l+1}{3} \geq \frac{g/2+5}{3}$) and for its Clifford index we have $\operatorname{Cliff}(|2E|) = \operatorname{Cliff}(|K - 2E|) = 2g - 2 - 2(d' - \alpha) - 2(h^1(C, \mathcal{O}_C(2E)) - 1)$. Now if $d - 3r \leq -2$ or $d - 3r = -1$ but $0 < h^1(C, \mathcal{O}_C(2E)) + (d - 3r) - \alpha$, we get $-2(h^1(C, \mathcal{O}_C(2E)) - 1)) \leq -2 - 2\alpha$ and therefore $\operatorname{Cliff}(|2E|) \leq 2g - 2 - 2(d' - \alpha) - 2 - 2\alpha = d - d' - 2 = 2l$, as it is assumed that $d = g + l$. As it is known from [6], $\operatorname{gon}(C) - 3 \leq \operatorname{Cliff}(C)$, wherefrom we get $2l + 5 \leq \operatorname{gon}(C) \leq \operatorname{Cliff}(2E) + 3 \leq 2l + 3$, which is a contradiction. Alternatively (still in the case $h^1(C, \mathcal{O}_C(2E)) \geq 2$), we can have only $0 = h^1(C, \mathcal{O}_C(2E)) + (d - 3r) - \alpha$ and $d = 3r - 1$ which implies $\dim U = 0$ and so we have only finitely many g^r_d's such that $d < 3r$ and for them $d = 3r - 1$.

If $h^1(C, \mathcal{O}_C(2E)) \leq 1$ we immediately obtain $h^1(C, \mathcal{O}_C(2E)) = 1$, $d = 3r - 1$ and $\alpha = 0$ in which case $\dim U \leq 0$, and exactly as above we could have at most finitely many such g^r_d's.

(ii) Consider now the case in which the general $g^{r'}_{d'-\alpha} \in U \subset W^{r'}_{d'-\alpha}$ is compounded. Remark that by upper semi-continuity U can not contain any simple linear series. Let $C' = \Phi_{g^{r'}_{d'-\alpha}}(C)$ be the image of C under the morphism determined by the linear series $g^{r'}_{d'-\alpha}$. Let $n \geq 2$ be the degree of the morphism. In this way we get a linear series $g^{r'}_{\frac{d'-\alpha}{n}}$ on C'. Just like in the proof of Lemma 2.1 we split the study into two subcases, the first one when $g^{r'}_{\frac{d'-\alpha}{n}}$ is special, and the second one when it is not. If $g^{r'}_{\frac{d'-\alpha}{n}}$ is special, we have that $4r' \leq d' - \alpha \leq d'$, and the assumption $d \leq 3r - 1$ gives: $1 \leq 3r - d = 4r' - 2d' + d - r \leq d - d' - r = 2l + 2 - r \leq \text{Cliff}(C) - r \leq \text{Cliff}(C) - \text{Cliff}(g^r_d) - 1 < 0$, which is a contradiction. Assume now that $g^{r'}_{\frac{d'-\alpha}{n}}$ is non-special. Then we have $r' = \frac{d'-\alpha}{n} - \gamma$, where γ is the genus of C'. By the Riemann-Roch theorem, $\gamma = \frac{d'-\alpha}{n} - r'$. If $n = 2$ we apply Castelnuovo inequality to obtain

$$\begin{aligned} g &\leq \text{gon}(C) - 1 + 2\gamma \\ &\leq \text{gon}(C) - 1 + d' - 2r' \\ &= \text{gon}(C) - 1 + d - 2r, \end{aligned}$$

and we reach a contradiction exactly like in Lemma 2.1. It remains to consider the case $n \geq 3$. It is trivial to see that the only interesting case is $n = 3$. Further, since there exists a pencil of degree $[\frac{\gamma+3}{2}]$ on C', pulling it back on C and using $\gamma = \frac{d'-\alpha}{n} - r'$ we get a pencil of degree no bigger than $\frac{d'-3r'+9}{6}$ on C'. Therefore $\text{gon}(C) \leq \frac{d'-3r'+9}{2} = \frac{1}{2}(d - 3r + r - r' + 9) \leq \frac{1}{2}(-1 + d - g + 1 + 9) = \frac{1}{2}(l + 9)$. The last contradicts to the assumption $\text{gon}(C) \geq 5 + 2l$.

This completes the proof of the proposition. □

Further in this section we will estimate the dimension of the subspace of curves in the moduli space defined as

$$\mathcal{M}^{(3)} := \{C \in \mathcal{M}_g | \exists g^r_d \text{ on } C, d \leq g + l, d < 3r\},$$

where l is a fixed integer $0 \leq l \leq \frac{g}{2} - 3$. The idea is to bound the dimension of the varieties of line bundles $\mathcal{W}^{r,d}_{\mathcal{C}/S}$ defined for a family of curves $\pi : \mathcal{C} \to S$. For this, let's recall briefly their definition. In analogy with $W^r_d(C)$, defined in the case of a single curve C, $\mathcal{W}^{r,d}_{\mathcal{C}/S}$ is by definition the variety representing the functor $\mathfrak{F}_{\mathcal{W}^{r,d}_{\mathcal{C}/S}} : (\mathfrak{Schemes} \,/\, S) \to (\mathfrak{Sets})$

$$T \mapsto \left\{ \begin{array}{c} \textit{equivalence classes of families } \mathcal{L} \textit{ of line bun-} \\ \textit{dles of degree } d \textit{ on } \phi : \mathcal{C} \times_S T \to T \textit{ such that} \\ \textit{the Fitting rang of } R^1_{\phi_*}\mathcal{L} \textit{ is at least } g - d + r \end{array} \right\}$$

as a fine moduli space. Unlike the case of a single curve, such a fine moduli space doesn't exist in general (see [20] for a criterion in the case of $\pi : \mathcal{C}^0 \to \mathcal{M}_g^0$, i.e. the universal family of curves without automorphisms) and depends on the family $\pi : \mathcal{C} \to S$. More precisely, its existence depends on the existence of a fine moduli space representing the functor $\mathfrak{F}_{\mathcal{P}ic^d_{\mathcal{C}/S}} : (\mathfrak{Schemes} \ / \ S) \to (\mathfrak{Abelian\ groups})$

$$T \mapsto \left\{ \begin{array}{c} \textit{equivalence classes of families } \mathcal{L} \textit{ of line} \\ \textit{bundles of degree } d \textit{ on } \phi : \mathcal{C} \times_S T \to T \end{array} \right\} = (\operatorname{Pic}(\mathcal{C} \times_S T)/\operatorname{Pic}(T))_d.$$

The remedy is that for any given curve $C_0 \in \mathcal{M}_g$ there exist a *Kuranishi family* $\pi : \mathcal{C} \to S$, where $S \subset \mathbb{C}^{3g-3}$ is a polydisc centered at $C_{s_0} \simeq C_0$ and a finite ramified covering $m : S \to U$ of an open analytic neighborhood $C_0 \in U \subset \mathcal{M}_g$ such that the functor $\mathfrak{F}_{\mathcal{P}ic^d_{\mathcal{C}/S}}$ is represented by a fine moduli space $\mathcal{P}ic^d_{\mathcal{C}/S}$. As soon as one gets the universal family on $\mathcal{C} \times_S \mathcal{P}ic^d_{\mathcal{C}/S}$, the construction of the varieties $\mathcal{W}^{r,d}_{\mathcal{C}/S}$ and $\mathcal{G}^{r,d}_{\mathcal{C}/S}$ proceeds exactly like the construction of their analogs $W^r_d(C)$ and $G^r_d(C)$ for the case a of fixed curve C. Formally put, $\eta_P : \mathcal{P}ic^d_{\mathcal{C}/S} \to S$, $\eta_W : \mathcal{W}^{r,d}_{\mathcal{C}/S} \to S$ and $\eta_G : \mathcal{G}^{r,d}_{\mathcal{C}/S} \to S$ are proper varieties over S whose fibers over each point $s \in S$ are isomorphic to $\operatorname{Pic}^d(m(s))$, $W^r_d(m(s))$ and $G^r_d(m(s))$ correspondingly. Informally, a closed point of $\mathcal{P}ic^d_{\mathcal{C}/S}$ is a line bundle L_s of degree d on the curve $C_s \simeq m(s) \equiv m(\eta_P(L_s))$, a closed point of $\mathcal{W}^{r,d}_{\mathcal{C}/S}$ is line bundle L_s on $C_s \simeq m(s)$ for which $h^0(C_s, L_s) \geq r+1$, and a closed point of $\mathcal{G}^{r,d}_{\mathcal{C}/S}$ is $g^r_{d,s} = (V_s, L_s)$ where $V_s \subset H^0(C_s, L_s)$ such that $\dim V_s = r+1$. At the expense of being imprecise but in order to avoid some cumbersome notations, we will denote in the remaining part of this section the closed points of $\mathcal{W}^{r,d}_{\mathcal{C}/S}$ by (C_s, L_s), where in fact $C_s = \pi^{-1}(\eta_W(L_s))$. For additional details on the definition and the properties of $\mathcal{P}ic^d_{\mathcal{C}/S}, \mathcal{W}^{r,d}_{\mathcal{C}/S}$ and $\mathcal{G}^{r,d}_{\mathcal{C}/S}$, the interested reader may refer to [1] and [2].

For the proof of the second main theorem in this paper we need the following two preparatory lemmas.

Lemma 3.2. *Let e, g and t be positive integers such that $t \geq 2$ and $g-e+t \geq 2$. Let $\mathcal{C}/S$ be a smooth family of complex integral curves of genus g and $\mathcal{W} \subset \mathcal{W}^{t,e}_{\mathcal{C}/S}$ be an irreducible component for whose general element $(C_s, L_s) \in \mathcal{W}$ the degree of the base locus of L_s is α. Then*

$$\dim \mathcal{W} \leq \begin{cases} 3e+g+1-5t-2\alpha, & \textit{if } r \geq 3 \textit{ and the moving part of } L_s \\ & \textit{from a general } (C_s, L_s) \in \mathcal{W} \textit{ is very ample;} \\ 3e+g-1-4t-2\alpha, & \textit{if the moving part of } L_s \textit{ from a} \\ & \textit{general } (C_s, L_s) \in \mathcal{W} \textit{ is birationally very ample;} \\ 2g-1+e-2t, & \textit{otherwise.} \end{cases}$$

Proof. See [9] and [17] or alternatively [15] for a detailed proof. Remark that, as it is pointed in [9], the bound given in the very ample case actually applies for linear series producing birationally unramified mappings, but we do not pursue such generality here. □

Lemma 3.3. *Let $\mathcal{C} \to S$ be a smooth family of smooth integral complex curves of genus g and $\mathcal{W}^{t,e}_{\mathcal{C}/S}$ be the corresponding variety of linear series, where e, g and t are positive integers such that $t \geq 2$ and $g-e+t \geq 3$. Assume that $\mathcal{W} \subset \mathcal{W}^{t,e}_{\mathcal{C}/S}$ is an irreducible component for whose general element $(C_s, L_s) \in \mathcal{W}$ the line bundle L_s is strictly birationally very ample, $h^0(C_s, L_s) = t+1$ and the moving part of $K_{C_s} \otimes L_s^{-1}$ is birationally very ample. Then, there exists an irreducible component $\mathcal{U} \subset \mathcal{W}^{t-1,e-2}_{\mathcal{C}/S}$ such that*

$$\dim \mathcal{W} \leq \dim \mathcal{U}.$$

Proof. This lemma is almost identical to [15, Lemma 2.2], but for a convenience of the reader we give its complete proof below supplying even a few extra details. To prove the lemma it is enough to show that there exists a component $\mathcal{U} \subset \mathcal{W}^{t-1,e-2}_{\mathcal{C}/S}$ such that $\dim \eta_{\mathcal{W}}(\mathcal{U}) \geq \dim \eta_{\mathcal{W}}(\mathcal{W})$ and $\dim \mathcal{U}_s \geq \dim \mathcal{W}_s$ for the closed general fibers of $\mathcal{W}$ and $\mathcal{U}$. The first observation is that since the general element of $\mathcal{W} \subset \mathcal{W}^{t,e}_{\mathcal{C}/S}$ is strictly birationally very ample, for any $L_s \in \mathcal{W}$ there exist $p_s, q_s \in C_s$ such that $h^0(C_s, L_s(-p_s - q_s)) \geq t$. In particular, for any $s \in \eta_W(\mathcal{W})$ the fiber $\mathcal{W}^{t-1,e-2}_s$ is nonempty, therefore $\dim \eta_W(\mathcal{W}^{t-1,e-2}_{\mathcal{C}/S}) \geq \dim \eta_W(\mathcal{W})$. Next, since a general $L_s \in \mathcal{W}$ is strictly birationally very ample and $h^0(C_s, L_s) = t+1$, there exist finitely many points $p_s, q_s \in C_s$ such $h^0(C_s, L_s(-p_s - q_s)) = h^0(C_s, L_s) - 1 = t$. We need to bound $\dim \mathcal{W}$ with the dimension of an appropriately chosen component $\mathcal{U}$ of $\mathcal{W}^{t-1,e-2}_{\mathcal{C}/S}$. The fiber of $\mathcal{W} \subset \mathcal{W}^{t,e}_{\mathcal{C}/S}$ over the image of C_s in $\mathcal{M}_g$ consists of finite number of copies of $W^t_e(C_s)$ and similarly for the fiber of $\mathcal{W}^{t-1,e-2}_{\mathcal{C}/S}$, as we just saw that $W^{t-1}_{e-2}(C_s) \neq \varnothing$. Consider the mapping ρ defined as

$$\begin{aligned} \rho : W \times C_{s\,2} &\to W^{t-2}_{e-2}(C_s) \\ (M_s, p_s + q_s) &\mapsto M_s(-p_s - q_s), \end{aligned}$$

where $W \subset W^t_e(C_s)$ is an irreducible component of maximal dimension containing L_s. Clearly, we may assume that for general $(C_s, L_s) \in \mathcal{W}$ the component W is built-up generically of strictly birationally very ample linear series (due to our assumption that the general $L_s \in \mathcal{W}$ is strictly birationally very ample). Further, let ρ_Z be its restriction of ρ to Z, where $Z := \{(M_s, p_s + q_s) \in W \times C_{s\,2} | h^0(C_s, M_s(-p_s - q_s)) \geq t\}$. Clearly, $\dim Z = \dim W$ as we can have only finitely many points $p_s, q_s \in C_s$ with the property $h^0(C_s, M_s(-p_s - q_s)) \geq t$ for general $M_s \in W$. Consider now the restriction map

$$\rho_Z : Z \to W^{t-1}_{e-2}(C_s).$$

It is easy to see that this is a finite mapping. For this, assume that $M, M' \in W$ are general and for some points $p_s, q_s, p'_s, q'_s \in C_s$ we have $M(-p_s - q_s) \simeq M'(-p'_s - q'_s) \in W^{t-2}_{e-2}(C_s)$. A simple calculation gives

$$\begin{aligned} & h^0(C_s, K_{C_s} \otimes M^{-1}(p_s + q_s - p'_s - q'_s)) \\ &= h^0(C_s, K_{C_s} \otimes M'^{-1}) \\ &= h^0(C_s, K_{C_s} \otimes M^{-1}) \\ &= h^0(C_s, K_{C_s} \otimes M^{-1}(p_s + q_s)) - 1. \end{aligned}$$

By assumption, the moving part of $K_{C_s} \otimes M^{-1}$ is birationally very ample, hence the moving part of $K_{C_s} \otimes M^{-1}(p_s + q_s)$ will also be birationally very ample. Therefore, on the given C_s there could exist at most finitely many pairs $(p'_s, q'_s) \in C^2_s$ such that the above equality holds. But giving such a pair (p'_s, q'_s) would determine immediately the line bundle M' up to isomorphism by

$$M(-p_s - q_s + p'_s + q'_s) \simeq M'.$$

Therefore there could exist at most finitely many line bundles $M' \in W$ with the above property. From here we conclude that

$$\dim W^{t-1}_{e-2}(C_s) \geq \dim Z = \dim W.$$

Since $\mathcal{W}^{t-1,e-2}_{C/S}$ has finitely many components, and as we saw $\dim \eta_W(\mathcal{W}^{t-1,e-2}_{C/S}) \geq \dim \eta_W(\mathcal{W})$, varying the curve C_s within the family $\mathcal{C} \to S$, we can chose a component $\mathcal{U} \subset \mathcal{W}^{t-1,e-2}_{C/S}$ such that $\dim \eta_W(\mathcal{U}) \geq \dim \eta_W(\mathcal{W})$ and $\dim \mathcal{U}_s \geq \dim \mathcal{W}_s$ for a general $s \in \eta_W(\mathcal{U}) \cap \eta_W(\mathcal{W})$, or in particular $\dim \mathcal{W} \leq \dim \mathcal{U}$. This completes the proof of the lemma. □

Theorem 2. *Let $\mathcal{M}^{(3)} \subset \mathcal{M}_g$ be defined as above where it is assumed that g and l are integers such that $g \geq 7$ and $0 \leq l \leq \frac{g}{2} - 3$. Then*

$$\dim \mathcal{M}^{(3)} \leq 2g - 1 + \frac{1}{3}(g + 2l + 1).$$

Proof. Consider fixed integers d and r such that $0 < d \leq g + l$, $d < 3r$ and let $\pi : \mathcal{C} \to S$ be a smooth family of smooth integral complex curves of genus g for which the variety $\mathcal{W}^{r,d}_{C/S}$ is well defined. To prove the theorem it is enough to show that the dimension of the natural projection of $\pi_W = m \circ \eta_W : \mathcal{W}^{r,d}_{C/S} \to \mathcal{M}_g$ doesn't exceed $2g - 1 + \frac{1}{3}(g + 2l + 1)$. Since the fiber of π_W over a point C in the image $\pi_W(\mathcal{W}^{r,d}_{C/S})$ consists of finite number of copies of $W^r_d(C)$, to bound the dimension $\dim \mathcal{W}^{r,d}_{C/S}$ we will combine the direct estimation of the dimension of its components with evaluation of the dimension of the fibers $W^r_d(C)$ in order to reduce the problem to estimating varieties which are better understood. Remark that we may assume for d that $g \leq d \leq g + l$ since for $d < g$ the curves of genus g for which there exist such g^r_d's are either Castelnuovo (extremal) curves with $d = 3r - 1$ or double covers, [6, Proposition 2.4.1]. By

[5, Theorem 1.4, p.353] it is easy to calculate that the former form a subset in the moduli space of dimension $\leq \frac{5}{3}g+3$, while by [18] the dimension of subset $\mathcal{M}_g(\gamma,n) \subset \mathcal{M}_g$ of curves possessing rational mappings of degree $n \geq 2$ into curves of genus $\gamma \geq 1$ is

$$\dim \mathcal{M}_g(\gamma,n) = 2g-2-(2n-3)(\gamma-1),$$

and in both cases the bound is even better than the one in the theorem. For the proof of the theorem, it is enough to consider $d = g+l$ and find an appropriate bound for the dimension of $\mathcal{W}^{r',d'}_{\mathcal{C}/S}$, where $d' = 2g-2-d = g-l-2$ and $r' = g-d+r-1 = r-l-1$. Remark that we may assume $r' \geq 2$ since otherwise we would have $r = r'+d-g+1 \leq l+2$ and the assumption $d \leq 3r-1$ would give us $g+l \leq 3l+5$ i.e., $\frac{g}{2}-\frac{5}{2} \leq l$, which is impossible under our condition $l \leq \frac{g}{2}-3$.

As it was said, $\dim \mathcal{W}^{r,d}_{\mathcal{C}/S} = \dim \mathcal{W}^{r',d'}_{\mathcal{C}/S}$ and further we will estimate the dimension of the variety $\mathcal{W}^{r',d'}_{\mathcal{C}/S}$, or more precisely the dimension of an arbitrary irreducible component $\mathcal{W} \subset \mathcal{W}^{r',d'}_{\mathcal{C}/S}$. The construction and the proof that follow are similar to the ones carried out in [15, Theorem 3.2]. First, it is easy to see that the dimension of $\mathcal{W} \subset \mathcal{W}^{r',d'}_{\mathcal{C}/S}$, can be bounded with the dimension of $\mathcal{W}_0 \subset \mathcal{W}^{r',d'-\alpha_0}_{\mathcal{C}/S}$, where α_0 is the degree of the base locus of L_s for a general $(C_s, L_s) \in \mathcal{W} \subset \mathcal{W}^{r',d'}_{\mathcal{C}/S}$, as the degree of the base locus is constant by upper semi-continuity. Namely,

$$\dim \mathcal{W} \leq \dim \mathcal{W}_0 + \alpha_0,$$

where for the general element $(C_s, L_{s,\alpha_0}) \in \mathcal{W}_0$ the line bundle L_{s,α_0} is base point free. Remark that if $U \subset W^{r'}_{d'-\alpha_0}(C_s)$ is an irreducible component over the image of such C_s in $\mathcal{M}_g$, then for its dual $\overline{U} \subset W^{r+\alpha_0}_{d+\alpha_0}(C_s)$, we have that the moving part of its general element $K_{C_s} \otimes L^{-1}_{s,\alpha_0}$ is birationally very ample. Further, consider the different possibilities for L_{s,α_0}:

1. For the general element $(C_s, L_{s,\alpha_0}) \in \mathcal{W}_0 \subset \mathcal{W}^{r',d'-\alpha_0}_{\mathcal{C}/S}$, the line bundle L_{s,α_0} is very ample. Then

$$\begin{aligned}
\dim \mathcal{W}_0 &\leq 3d'+g+1-5r'-2\alpha_0 \\
&\leq g+1+3(d-2r)+r' \\
&= g+1+2(d-3r)+d+r' \\
&\leq 2g-1+l+r' \\
&\leq 2g-1+l+\frac{d'}{3} \\
&\leq 2g-1+l+\frac{g-l-2}{3}
\end{aligned}$$

$$= 2g - 1 + \frac{g + 2l - 2}{3}.$$

2. For general $(C_s, L_{s,\alpha_0}) \in \mathcal{W}_0 \subset \mathcal{W}^{r',d'-\alpha_0}_{C/S}$, the morphism associated to the line bundle L_{s,α_0} is multiple covering of C_s to its image. In this case

$$\begin{aligned}\dim \mathcal{W}^{r',d'}_{C/S} &\le 2g - 1 + d' - 2r' \\ &= 2g - 1 + d - 3r + r \\ &\le 2g - 2 + l + r' + 1 \\ &\le 2g - 1 + \frac{g + 2l + 1}{3},\end{aligned}$$

as it is easy to see using that $r' \le \frac{d'}{3}$.

3. For general $(C_s, L_{s,\alpha_0}) \in \mathcal{W}_0$, the line bundle is strictly birationally very ample. If $r' = 2$, we get

$$\begin{aligned}\dim \mathcal{W} &\le 3d' + g - 1 - 8 - 2\alpha_0 \\ &\le g + 3(d' - 4) + 3 \\ &= g + 3(d - 2r) + 3 \\ &= 2g + 1 + l.\end{aligned}$$

Let's remark at this point that we may assume for a general $(C_s, L_{s,\alpha_0}) \in \mathcal{W}_0$ that the moving part of $K_{C_s} \otimes {L_{s,\alpha_0}}^{-1}$ is birationally very ample. Essentially, this due to Lemma 2.1 and the fact that we are interested in the curves for which there exists a g^r_d with $g \le d < 3r$, but it can also be argued as follows: $\dim \mathcal{W}_0 = \dim \widetilde{\mathcal{W}_0}$, where $\widetilde{\mathcal{W}_0} \subset \mathcal{W}^{r+\alpha_0,d+\alpha_0}_{C/S}$ is the "dual" of $\mathcal{W}_0$, and its closed points are $(C_s, K_{C_s} \otimes {L_{s,\alpha_0}}^{-1})$. Then applying Lemma 3.2, we get the desired bound for the dimension of $\mathcal{W}_0$ just like in the case when the line bundle L_{s,α_0} yields a multiple covering. It remains to deal with $r' \ge 3$ and L_{s,α_0} strictly birational. For this we use Lemma 3.3 to proceed estimating $\dim \mathcal{W}_0$ inductively, namely for our $\mathcal{W}_0$ we take $\mathcal{U}_0 \subset \mathcal{W}^{r'-1,d'-\alpha_0-2}_{C/S}$ like in the lemma. Let α_1 be the degree of the base locus for a general $(C_s, M_s) \in \mathcal{U}_0$. Just as before,

$$\dim \mathcal{U}_0 \le \dim \mathcal{W}_1 + \alpha_1,$$

for an irreducible component $\mathcal{W}_1 \subset \mathcal{W}^{r'-1,d'-\alpha_0-\alpha_1-2}_{C/S}$, and next we consider the 3 different cases according to whether a general $(C_s, L_{s,\alpha_1}) \in \mathcal{W}_1$ has a very ample, compounded or strictly birationally very ample line bundle L_{s,α_1} on C_s. The first two options are dealt exactly as before, while in the last one we construct inductively $\mathcal{W}_2 \subset \mathcal{W}^{r'-2,d'-\alpha_0-\alpha_1-\alpha_2-4}_{C/S}$ where the meaning of α_2 is the apparent one, provided that $r' - 1 \ge 3$. In this way, after i steps we will have

$$\dim \mathcal{W}_0 \le \dim \mathcal{W}_i + \alpha,$$

where $\mathcal{W}_i \subset \mathcal{W}_{C/S}^{r'-i,d'-2i-\alpha}$, with $\alpha = \sum_{j=0}^{i} \alpha_j$ and a general $(C_s, L_{s,i}) \in \mathcal{W}_i$ carrying a base point free $L_{s,i}$ on C_s. We again have only 3 options:

(i.1) $L_{s,i}$ to be very ample on C_s;
(i.2) $L_{s,i}$ to produce a multiple covering;
(i.3) $L_{s,i}$ to be strictly birationally very ample on C_s.

In cases (i.1) and (i.2) we obtain the desired bound for $\mathcal{W}_{C/S}^{r',d'}$, while case (i.3) produces $\mathcal{W}_{i+1}$, provided that $r'-i \geq 3$. Thus, if the inductive process doesn't stop earlier due to entering case like (i.1) or (i.2), after $r'-2$ steps we will reach at worst $\mathcal{W}_{r'-2} \subset \mathcal{W}_{C/S}^{2,d'-2(r'-2)-\alpha}$, with general $(C_s, L_{s,r'-2}) \in \mathcal{W}_{r'-2}$ for which $L_{s,r'-2}$ is birationally very ample, where $\alpha = \sum_{j=0}^{r'-2} \alpha_j$. At that final step, we easily calculate

$$\begin{aligned}\dim \mathcal{W}_0 &\leq \dim \mathcal{W}_{C/S}^{2,d'-2(r'-2)-\alpha} + \alpha \\ &\leq 3(d' - 2r' + 4) + g - 1 - 8 \\ &= 2(d - 3r) + d + g + 3 \\ &\leq 2g + 1 + l.\end{aligned}$$

This completes the proof of the theorem. □

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