HÖLDER CONVERGENCE OF THE WEAK SOLUTION TO AN EVOLUTION EQUATION OF p-GINZBURG-LANDAU TYPE

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Abstract. The author studies the local Hölder convergence of the solution to an evolution equation of p-Ginzburg-Landau type, to the heat flow of the p-harmonic map, when the parameter tends to zero. The convergence is derived by establishing a uniform gradient estimation for the solution of the regularized equation.

1. Introduction

Let $G \subset R^2$ be a bounded and simply connected domain with smooth boundary $\partial G$, $g \in C^\infty(\partial G, S^1)$, where $S^1 = \{x \in R^2; |x| = 1\}$, and deg($g, \partial G$) = $d = 0$, $u_0 \in C^\infty(\overline{G}, S^1)$ and $u_0(x) = g(x)$ as $x \in \partial G$. Denote $G_T = G \times (0, T]$ with $T \in (0, \infty)$. We are concerned with the asymptotic behavior of the weak solution $u_\varepsilon$ of the following problem when $\varepsilon \rightarrow 0$,

\begin{equation}
(1.1) \quad u_t = \text{div}(|\nabla u|^{p-2}\nabla u) + \frac{1}{\varepsilon^p} u(1 - |u|^2), \quad \text{a.e. on } G_T
\end{equation}

\begin{equation}
(1.2) \quad u|_{\partial G \times R^+} = g(x),
\end{equation}

\begin{equation}
(1.3) \quad u(x, 0) = u_0(x), \quad x \in G.
\end{equation}

Recall that $u \in C_{loc}(0, T; L^2(G, R^2)) \cap L^p_{loc}(0, T; W^{1, p}(G, R^2))$ is a weak solution of (1.1), if $u$ satisfies

\begin{equation}
(1.4) \quad \int_{G_T} u \phi_t \, dx \, dt = \int_{G_T} |\nabla u|^p \nabla u \nabla \phi \, dx \, dt - \frac{1}{\varepsilon^p} \int_{G_T} u \phi (1 - |u|^2) \, dx \, dt, \quad \forall \phi \in C_0^\infty(G_T).
\end{equation}

For the fixed $\varepsilon > 0$, the weak solution $u_\varepsilon$ of (1.1)-(1.3) exists. In fact, it can be derived as the limit of the solution $u_\varepsilon$ of the regularized problem

\begin{equation}
(1.5) \quad u_\varepsilon = \text{div}((|\nabla u|^{p-2}/n) \nabla u) + \frac{1}{\varepsilon^p} u(1 - |u|^2) \quad \text{on } G_T.
\end{equation}

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\[
u|_{\partial G \times R^+} = g(x),
\]
\[
u(x, 0) = u_0(x), \quad x \in G,
\]
by letting \(n \to \infty\) (cf. [9]). The uniqueness of the weak solution will be proved in \(\S 2\).

When \(p = 2\), F. Bethuel, H. Brezis and F. Helein discussed the stationary case. They obtained the asymptotic behavior of the minimizer of the simplified Ginzburg-Landau functional
\[
E_\epsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 dx + \frac{1}{4\epsilon^2} \int_G (1 - |u|^2)^2 dx
\]
under the assumption of \(\text{deg}(g, \partial G) = 0\) (see [1]). In particular, it was proved that the Ginzburg-Landau minimizer converges to the harmonic map with boundary data \(g\) on \(G\).

In the evolutionary case and under some general assumption on \(u(x, 0), F. H. Lin, R. Jerrard\) and H. Soner studied the dynamics of the zeros of the solution \(u_\epsilon\), i.e., the dynamical properties of Ginzburg-Landau vortices (see [6] and [8]). Under the special assumption of \(u_0(x) \in C^\infty(\overline{G}, S^1)\), some uniform estimates and the Hölder convergence of \(u_\epsilon\) is derived in [5]. The work shows that the limit of the solution \(u_\epsilon\) is a heat flow of the harmonic map.

In this paper, under the assumption of \(u_0(x) \in C^\infty(\overline{G}, S^1)\) as in [5], we will present the local Hölder convergence of the \(p\)-Ginzburg-Landau solution (the weak solution of (1.1)-(1.3)) when \(p > 2\), by establishing a series of uniform estimations of the solution \(u_\epsilon^p\) of (1.5), (1.2) and (1.3). The reason to consider the estimation of \(u_\epsilon^p\) is that, comparing with the case of \(p = 2\), the system (1.1) is degenerate and nonlinear in the main part. It leads to the difficulty when setting up the regularity estimation of \(u_\epsilon\).

Besides the results in [5], the study is also motivated by the research work in [3]. The authors proved that the heat flow of the \(p\)-harmonic map \(u_p\) can be approached by the weak solution \(u_\epsilon\) of (1.1)-(1.3) in some weak sense, when \(\epsilon \to 0\). This work proffered a new idea to study the \(p\)-harmonic maps. Namely, we can firstly research the properties of weak solutions \(u_\epsilon\) to some \(p\)-Laplace system with a penalization term, such as \(p\)-Ginzburg-Landau system (1.1). Next, letting \(\epsilon \to 0\), and investigating the asymptotic behavior of \(u_\epsilon\) to obtain the corresponding properties of the limit \(p\)-harmonic map. We hope the convergence of \(u_\epsilon\) to the \(p\)-harmonic map \(u_p\) can be improved in more strong sense (cf. Theorems 5.2 and 5.3). Thus, the relation between \(u_\epsilon\) and \(u_p\) will be more clear.

The limit function \(u_p\) of \(u_\epsilon\) when \(\epsilon \to 0\) can be introduced as follows. If \(u_p \in L^\infty_{\text{loc}}(0, T; L^2(G, S^1)) \cap L^p_{\text{loc}}(0, T; W^{1,p}(G, S^1))\) is a weak solution of the system
\[
(1.6) \quad u_t = \text{div}(|\nabla u|^{p-2}\nabla u) + u|\nabla u|^p,
\]
then it is called the heat flow of the \(p\)-harmonic map. By \(\text{deg}(g, \partial G) = 0\), there exists \(\theta \in L^\infty_{\text{loc}}(0, T; L^2(G, R)) \cap L^p_{\text{loc}}(0, T; W^{1,p}(G, R)), \theta_1 \in C^\infty(\partial G, R)\)
and $\theta_2 \in C^\infty(\mathcal{G}, R)$ such that $u_\varepsilon = (\cos \theta, \sin \theta)$, $g = (\cos \theta_1, \sin \theta_1)$, $u_0 = (\cos \theta_2, \sin \theta_2)$ and

\begin{equation}
\theta|_{\partial G \times R^+} = \theta_1,
\end{equation}

\begin{equation}
\theta|_{G \times \{t=0\}} = \theta_2.
\end{equation}

It follows from (1.6) that

\begin{equation}
\theta_t = \text{div}(|\nabla \theta|^{p-2}\nabla \theta).
\end{equation}

From Lemma 1.2 in [3] it is easy to see that there exists a unique solution

$$
\Theta \in L^\infty_{\text{loc}}(0,T;L^2(G,R)) \cap L^p_{\text{loc}}(0,T;W^{1,p}(G,R))
$$

satisfying the problem (1.7)-(1.9) in the weak sense. Hence the heat flow of the $p$-harmonic map $u_\varepsilon = (\cos \Theta, \sin \Theta)$ is also unique.

We shall prove a series of preliminary propositions in Section 2. By setting up the gradient estimations of $u_\varepsilon$ locally in Section 3 and 4, we will present the Hölder convergence of the weak solution $u_\varepsilon$ and its gradient $\nabla u_\varepsilon$ in §5 (Theorem 5.2 and 5.3).

2. Preliminaries

**Proposition 2.1.** The weak solution of (1.1)-(1.3) is unique.

**Proof.** Set $0 < t_1 < t_2 < T$. By a limit process we see that the test function $\phi$ in (1.4) can be taken in $W^{1,p}_0(G,R^2)$. Then the weak solution satisfies

\begin{equation}
\int_G (u(x,t_2) - u(x,t_1))\phi(x)dx + \int_{t_1}^{t_2} \int_G |\nabla u|^{p-2}\nabla u \nabla \phi \, dx \, dt
\end{equation}

\begin{equation}
= \frac{1}{\varepsilon^p} \int_{t_1}^{t_2} \int_G u\phi(1 - |u|^2) \, dx \, dt.
\end{equation}

Fix $\tau \in (0,T)$. Choose $h$ satisfying $0 < \tau < \tau + h < T$. Taking $t_1 = \tau$, $t_2 = \tau + h$ in (2.1) and multiplying with $h^{-1}$ we obtain

\begin{equation}
\int_G (u_h(x,\tau))\phi(x)dx + \int_G (|\nabla u|^{p-2}\nabla u)_h(x,\tau) \nabla \phi(x)dx
\end{equation}

\begin{equation}
= \frac{1}{\varepsilon^p} \int_G \phi [u(1 - |u|^2)]_h \, dx
\end{equation}

for any $\phi \in W^{1,p}_0(G,R^2)$, where $u_h$ is the Steklov’s mean value of $u$, i.e., $u_h = \frac{1}{h} \int_t^{t+h} u(x,\tau) \, d\tau$ as $t \in (0,T-h)$; $u_h = 0$ as $t > T-h$.

Assume that both $u_1$ and $u_2$ are weak solutions of (1.1)-(1.3), and write $w = u_1 - u_2$. In virtue of (2.2), we see that for any $\phi \in W^{1,p}_0(G,R^2)$ and any
given $\tau \in (0, T)$,
\[
\int_G w_h \phi dx + \int_G \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right)_h \nabla \phi dx dt
= \frac{1}{\varepsilon^p} \int_{G_{T_\tau}} \phi [u_1 (1 - |u_1|^2) - u_2 (1 - |u_2|^2)]_h dx dt.
\]
Taking $\phi(x) = w_h(x, \tau)$ and integrating on $(0, t)$, we see that
\[
\int_G w_h^2 dx
= - \int_0^t \int_G \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right)_h (\nabla u_1 - \nabla u_2)_h dx d\tau
+ \frac{1}{\varepsilon^p} \int_0^t \int_G w_h^2 dx d\tau - \frac{1}{\varepsilon^p} \int_0^t \int_G (u_1|u_1|^2 - u_2|u_2|^2)_h w_h dx d\tau
\]
in view of $\int_G w_h^2(x, 0) dx = 0$. Letting $h \to 0$, we have
\[
\int_G w^2(x, t) dx
= - \int_0^t \int_G \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right)_h (\nabla u_1 - \nabla u_2)_h dx d\tau
+ \frac{1}{\varepsilon^p} \int_0^t \int_G w^2 dx d\tau - \frac{1}{\varepsilon^p} \int_0^t \int_G (u_1|u_1|^2 - u_2|u_2|^2)(u_1 - u_2) dx d\tau.
\]
Applying Lemma 1.2 in [3], we know that the first and the third terms of the right hand side of the equality above are not positive. Using Gronwall’s inequality, we obtain
\[
\int_G |u_1 - u_2|^2 dx = 0, \quad \forall t \in (0, T).
\]
This implies our conclusion. \qed

**Proposition 2.2.** Assume $u^n$ solves the problem (1.5), (1.2) and (1.3). Then $|u^n| \leq 1$ on $\bar{G}_T$.

**Proof.** Denote $u = u^n$. Suppose $\psi = |u|^2$ achieves its maximum on $\partial G \times (0, T)$ or $G \times \{0\}$, then the conclusion can be seen easily. Suppose $\psi$ achieves its maximum at $(x_0, t_0)$ in $G_T$. Then
\[
(2.3) \quad \nabla \psi(x_0, t_0) = 0, \quad \Delta \psi(x_0, t_0) \leq 0, \quad \text{and} \quad \psi_t(x_0, t_0) \geq 0.
\]
We claim that $\psi \leq 1$ over $G_T$. Otherwise, multiplying (1.5) with $u$, we have
\[
\frac{1}{2} (|u|^2)_t = \text{div}(u^{(p-2)/2} u \nabla u) - u^{(p-2)/2} |\nabla u|^2 + \frac{1}{\varepsilon^p} |u|^2 (1 - |u|^2),
\]
where \( v = |\nabla u|^2 + \frac{1}{n} \). It follows from (2.3) that at \((x_0, t_0)\),
\[
0 \leq \frac{1}{2} \psi = \frac{1}{2} \text{div}(v^{(p-2)/2}) \nabla \psi + \frac{1}{2} v^{(p-2)/2} \Delta \psi - v^{(p-2)/2} |\nabla u|^2
\]
\[
+ \frac{1}{\varepsilon} |u|^2 (1 - |u|^2) < 0,
\]
which is impossible. The proof is complete. \(\square\)

**Proposition 2.3.** Assume \( u^n_\varepsilon \) solves (1.5), (1.2) and (1.3). Then, there exists a constant \( C_0 > 0 \), which is independent of \( \varepsilon \) and \( n \), such that
\[
(2.4) \quad \sup_{t \in (0, T]} \int_0^t \int_G \frac{1}{\partial t} u^n_\varepsilon(x, t)^2 dx d\tau + E^n_\varepsilon(u^n_\varepsilon(x, t)) \leq C_0.
\]

Here
\[
E^n_\varepsilon(u) = \frac{1}{p} \int_G (|\nabla u|^2 + \frac{1}{n})^{p/2} dx + \frac{1}{4\varepsilon} \int_G (1 - |u|^2)^2 dx.
\]

**Proof.** Multiplying (1.5) with \( u_t \) and integrating over \( G \), we have
\[
\int_G u_t^2 dx = \int_G u_t \text{div}(v^{(p-2)/2} \nabla u) dx + \frac{1}{\varepsilon} \int_G uu_t (1 - |u|^2) dx,
\]
where \( v = |\nabla u|^2 + \frac{1}{n} \). Noting \( u_t = g_t = 0 \) on \( \partial G \), we obtain
\[
\int_G u_t^2 dx = -\frac{\partial}{\partial t} E^n_\varepsilon(u)
\]
and thus for any \( t > 0 \),
\[
\int_0^t \int_G \frac{\partial}{\partial t} u^n_\varepsilon(x, t)^2 dx d\tau + E^n_\varepsilon(u^n_\varepsilon(x, t)) \leq E^n_\varepsilon(u_0) = C_0.
\]
This means that (2.4) holds. \(\square\)

**Proposition 2.4.** For any \( \varepsilon \in (0, \varepsilon_0) \) with some fixed small constant \( \varepsilon_0 \), if \( n \to \infty \), then
\[
u^n_\varepsilon \to u_\varepsilon, \quad \text{weakly star in } L^\infty((0, T); W^{1, p}(G, R^2));
\]
\[
u^n_\varepsilon \to u_\varepsilon, \quad \text{strongly in } L^p((0, T); L^p(G, R^2));
\]
\[
\frac{\partial}{\partial t} u^n_\varepsilon \to \frac{\partial}{\partial t} u_\varepsilon, \quad \text{weakly in } L^2(0, T; L^2(G, R^2)).
\]

In addition, the limit \( u_\varepsilon \) is just the unique weak solution of (1.1)-(1.3).

**Proof.** From Proposition 2.2 and (2.4), we see that there exists \( C = C(T) > 0 \) (independent of \( \varepsilon \) and \( n \)), such that
\[
(2.5) \quad \|u^n_\varepsilon\|_{L^\infty(0, T; W^{1, p}(G, R^2))} \leq C(T),
\]
\[
(2.6) \quad \|\frac{\partial}{\partial t} u^n_\varepsilon\|_{L^2(0, T; L^2(G, R^2))} \leq C(T).
\]
So for any $\varepsilon \in (0, \varepsilon_0)$, there exists a subsequence $u^{n_k}_\varepsilon$ of $u^n_\varepsilon$ such that as $k \to \infty$,

$$
(2.7) \quad u^{n_k}_\varepsilon \rightharpoonup w_\varepsilon, \quad \text{weakly in } L^\infty((0, T); W^{1,p}(G, R^2));
$$

$$
(2.8) \quad u^{n_k}_\varepsilon \to w_\varepsilon, \quad \text{strongly in } L^p((0, T); L^p(G, R^2));
$$

$$
(2.9) \quad \frac{\partial}{\partial t} u^{n_k}_\varepsilon \to \frac{\partial}{\partial t} w_\varepsilon, \quad \text{weakly in } L^2(0, T; L^2(G, R^2))
$$

by applying Lemma 1.4 and the proof of Theorem 2.1 in [3]. Assume $K$ is any compact subset of $G_T$. Set $\zeta \in C^\infty_0(G_T, R)$ and $\zeta = 1$ on $K$. Multiplying (1.5) by $u\zeta$ and integrating over $G_T$ we obtain

$$
\frac{1}{\varepsilon^p} \int_K (1 - |u|^2)|u|^p dx dt \leq C(1 + \int_{G_T} v^{p/2} dx dt + \int_{G_T} |u| dx dt) \leq C(T),
$$

by applying the Hölder inequality, (2.5) and (2.6). Here $v = |\nabla u| + \frac{1}{n}$. Combining this with $\frac{1}{\varepsilon^p} \int_{G_T} (1 - |u|^2) dx dt \leq C(T)$ (which is implied by (2.4)) yields

$$
\frac{1}{\varepsilon^p} \int_K (1 - |u|^2) dx dt \leq C.
$$

Hence, using Theorem 2.1 in [3] yields that for any $\varepsilon \in (0, \varepsilon_0)$, as $n_k \to \infty$,

$$
(2.10) \quad v^{(p-2)/2} \nabla u^{n_k}_\varepsilon \rightharpoonup |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon, \quad \text{in } L^1(K).
$$

Obviously, $u^n_\varepsilon$ satisfies

$$
\int_{G_T} u \phi_t dx dt = \int_{G_T} v^{(p-2)/2} \nabla u \nabla \phi dx dt
$$

$$
- \frac{1}{\varepsilon^p} \int_{G_T} u\phi(1 - |u|^2) dx dt, \quad \forall \phi \in C^\infty_0(G_T).
$$

Letting $n_k \to \infty$, and using (2.7)-(2.10), we get

$$
\int_{G_T} w_\varepsilon \phi_t dx dt = \int_{G_T} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla \phi dx dt
$$

$$
- \frac{1}{\varepsilon^p} \int_{G_T} w_\varepsilon \phi_1 - |w_\varepsilon|^2 dx dt, \quad \forall \phi \in C^\infty_0(G_T).
$$

This, together with the uniqueness of the weak solution $u_\varepsilon$ of (1.1)-(1.3), implies $w_\varepsilon = u_\varepsilon$. Our proposition can be completed by (2.7), (2.8), (2.9) and the uniqueness of $u_\varepsilon$.

**Proposition 2.5.** Assume $u_\varepsilon$ is the weak solution of (1.1)-(1.3). Then as $\varepsilon \to 0$,

$$
u_\varepsilon \rightharpoonup u_p, \quad \text{weakly star in } L^\infty(0, T; W^{1,p}(G, R^2)),
$$

$$
\frac{\partial}{\partial t} u_\varepsilon \rightharpoonup \frac{\partial}{\partial t} u_p, \quad \text{weakly in } L^2(0, T; L^2(G, R^2)),
$$

$$
\quad u_\varepsilon \to u_p, \quad \text{strongly in } L^p(0, T; L^p(G, R^2)).
$$
Proof. By (2.5), (2.6) and Proposition 2.4, we can find a positive constant $C$ (independent of $\varepsilon$), such that

\begin{equation}
\|u_\varepsilon\|_{L^\infty(0,T;W^{1,p}(G;R^2))} \leq C, \tag{2.11}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} u_\varepsilon \rightarrow \frac{\partial}{\partial t} u_p, \quad \text{weakly in } L^2(0,T;L^2(G;R^2)). \tag{2.12}
\end{equation}

Hence, there is a subsequence $\varepsilon_k$ such that as $\varepsilon_k \rightarrow 0$,

\begin{align*}
&u_{\varepsilon_k} \rightarrow u_p, \quad \text{weakly in } L^\infty(0,T;W^{1,p}(G;R^2)) \\
&\frac{\partial}{\partial t} u_{\varepsilon_k} \rightarrow \frac{\partial}{\partial t} u_p, \quad \text{weakly in } L^2(0,T;L^2(G;R^2)) \\
&u_{\varepsilon_k} \rightarrow u_p, \quad \text{strongly in } L^p(0,T;L^p(G;R^2)).
\end{align*}

Using Lemma 1.4 and Theorem 2.2 in [3], we see that $u_p$ is just the unique heat flow of the p-harmonic map. Thus, the convergence above can be generalized to all $\varepsilon$ instead of the subsequence $\varepsilon_k$. \hfill \Box

**Proposition 2.6.** Assume $u_\varepsilon$ is the weak solution of (1.1)-(1.3). Then for any compact subset $K \subset G$, there exists a constant $C > 0$, which is independent of $\varepsilon$, such that for any $x_1, x_2 \in K$ and $t \in (0,T)$,

\[ |u(x_1,t) - u(x_2,t)| \leq C|x_1 - x_2|. \]

**Proof.** Assume $R > 0$ and $t_0 \in (0,T)$. Let $J_\eta$ be the mollification operator, such that

\[ u_\eta(x,t) = J_\eta u(x,t) = \int_0^T \int_{R^2} j_\eta(x - y, t - \tau)u(y, \tau)dyd\tau, \]

where $0 < \eta < t_0 < t < T - \eta$. Denote $u_\varepsilon$ by $u$. For any $x_1, x_2 \in B_R$, there holds

\[ u_\eta(x_1,t) - u_\eta(x_2,t) \]

\[ = \int_0^T \int_{R^2} \int_0^1 \frac{d}{ds}j_\eta(sx_1 + (1-s)x_2 - y, t - \tau)u(y, \tau)dsdyd\tau \]

\[ = (x_1 - x_2) \int_0^T \int_{R^2} \int_0^1 \nabla_x j_\eta(sx_1 + (1-s)x_2 - y, t - \tau)u(y, \tau)dsdyd\tau \]

\[ = -(x_1 - x_2) \int_0^1 \int_{R^2} \int_0^T \nabla_y j_\eta(sx_1 + (1-s)x_2 - y, t - \tau)u(y, \tau)d\tau dyds \]

\[ = (x_1 - x_2) \int_0^1 \int_{R^2} \int_0^T j_\eta(sx_1 + (1-s)x_2 - y, t - \tau)\nabla_y u(y, \tau)d\tau dyds. \]

Hence, applying H"{o}lder inequality and (2.11) we obtain

\begin{equation}
|u_\eta(x_1,t) - u_\eta(x_2,t)| \leq |x_1 - x_2| \int_0^1 \int_{R^2} \int_0^T |j_\eta(sx_1 + (1-s)x_2 - y, t - \tau)||\nabla_y u(y, \tau)|d\tau dyds \leq C|x_1 - x_2|. \tag{2.13}
\end{equation}
Letting $\eta \to 0$ we derive
\begin{equation}
|u(x_1, t) - u(x_2, t)| \leq C|x_1 - x_2|,
\end{equation}
which implies our consequence. \hfill \Box

**Proposition 2.7.** For any $n > 0$, as $\varepsilon \to 0$, there holds $|u^\varepsilon_n| \to 1$ in $C_{\text{loc}}(G_T)$.

**Proof.** Assume $K$ is a compact subset of $G_T$. For any $(x_0, t_0) \in K$, denote $\gamma = |u_\varepsilon(x_0, t_0)|$. Similar to the proof of (2.14), for $u^\varepsilon_n$ we also have
\[ |u^\varepsilon_n(x, t_0) - u^\varepsilon_n(x_0, t_0)| \leq C_0|x - x_0| \]
by using (2.5). Here $C_0$ is independent of $\varepsilon$ and $n$. Hence $|u^\varepsilon_n(x, t_0)| \leq \gamma + \varepsilon$
as $x \in B(x_0, C_0^{-1}\varepsilon)$ and
\[ \int_{B(x_0, C_0^{-1}\varepsilon)} (1 - |u^\varepsilon_n(x, t_0)|)^2 dx \geq C_0^{-2}\pi(1 - \gamma - \varepsilon)^2\varepsilon^2. \]
Combining this with (2.4) we obtain $(1 - \gamma - \varepsilon)^2 \leq C\varepsilon^{p-2}$ or $1 - \gamma \leq C\varepsilon^{(p-2)/2} + \varepsilon$. Noting the arbitrariness of $(x_0, t_0)$, we see our consequence. \hfill \Box

Proposition 2.7 implies that, there exists $\varepsilon_0 > 0$ sufficiently small, such that as $\varepsilon \in (0, \varepsilon_0)$,
\begin{equation}
|u^\varepsilon_n| \geq 1/2
\end{equation}
on any compact subset of $G_T$.

3. **Estimate for $\|\nabla u^\varepsilon_n\|_{L^1_{\text{loc}}}$**

**Proposition 3.1.** Assume $u^\varepsilon_n$ is a solution of (1.5), (1.2) and (1.3). If $p \geq 4$, then for any compact subset $K \subset G$, $t \in (0, T)$ and $l > 1$, there exists a constant $C > 0$, which is independent of $n$ and $\varepsilon$, such that

\[ \sup_{\tau \in [t, T]} \int_K |\nabla u^\varepsilon_n|^l\, dx + \|\nabla u^\varepsilon_n\|_{L^l(K_T, R^2)} \leq C = C(K_T, l), \]

where $K_T = K \times [t, T]$.

**Proof.** Obviously $u = u_\varepsilon$ satisfies (1.5). Denote $v = |\nabla u|^2 + \frac{1}{n}$. Differentiate (1.5) with respect to $x_j$,
\begin{equation}
\frac{\partial u_{x_j}}{\partial t} = (v^{(p-2)/2} u_{x_i})_{x_i x_j} + \frac{1}{\varepsilon^p}[u(1 - |u|^2)]_{x_j}.
\end{equation}

Let $\zeta \in C^\infty(\overline{B_{2\rho}} \times [0, T], [0, 1])$ satisfy
\begin{align*}
0 &\leq \zeta(x, \tau) \leq 1, \quad (x, \tau) \in B_{2\rho} \times [0, T]; \\
\zeta(x, \tau) = 1, &\quad (x, \tau) \in B_{\rho} \times (t, T); \\
\zeta(x, \tau) = 0, &\quad (x, \tau) \in \partial B_{2\rho} \times [0, T] \quad \text{or} \quad \tau \leq t/2; \\
|\nabla \zeta| &\leq C\rho^{-1}, \quad |\zeta_t| \leq C t^{-1}.
\end{align*}
Multiplying (3.1) by $\zeta^2 v^bu_{x_j} (b \geq 0)$ and integrating over $B_{2^p} \times [0, t_0]$, we obtain

\begin{equation}
\frac{1}{2(b+1)} \int_{B_{2^p}} \zeta^2 v^{b+1}(x, t_0)dx + \int_0^{t_0} \int_{B_{2^p}} \zeta^2 (v^bu_{x_j})_x(v^{(p-2)/2}u_{x_i})_x dxdt \\
= \frac{1}{b+1} \int_0^{t_0} \int_{B_{2^p}} \zeta \zeta v^{b+1}dxdt - 2 \int_0^{t_0} \int_{B_{2^p}} \zeta \zeta x_i v^bu_{x_j}(v^{(p-2)/2}u_{x_i})_x dxdt \\
+ \frac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2^p}} (1 - |u|^2)^2 \zeta^2 v^b u_{x_j}^2 dxdt - \frac{1}{2\varepsilon^p} \int_0^{t_0} \int_{B_{2^p}} [(|u|^2)_{x_j}]^2 \zeta^2 v^b dxdt.
\end{equation}

Applying (1.5) and (2.15) we see that

\begin{equation}
\frac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2^p}} (1 - |u|^2)^2 \zeta^2 v^b u_{x_j}^2 dxdt \\
\leq \int_0^{t_0} \int_{B_{2^p}} |u|^2 \frac{1}{|u|^1} | |u|^1 + \text{div}(v^{(p-2)/2} \nabla u)| | \zeta^2 v^{b+1} dxdt.
\end{equation}

Using Young’s inequality and (2.4) to estimate the right hand side of the inequality above, we may obtain that for any $\delta \in (0, 1)$,

\begin{equation}
\frac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2^p}} (1 - |u|^2)^2 \zeta^2 v^b u_{x_j}^2 dxdt \\
\leq \delta \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dxdt \\
+ C(\delta) \int_0^{t_0} \int_{B_{2^p}} v^{(p+2b)/2} dxdt + \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b+2)/2} dxdt + CI_0,
\end{equation}

where $I_0 = \int_0^{t_0} \int_{B_{2^p}} \zeta v^{2b+2} dxdt$, and

\begin{equation}
\int_0^{t_0} \int_{B_{2^p}} \zeta x_i v^bu_{x_j}(v^{(p-2)/2}u_{x_i})_x dxdt \\
\leq C(\delta) \int_0^{t_0} \int_{B_{2^p}} v^{(p+2b)/2} dxdt \\
+ \delta \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b-2)/2} \sum_{j=1}^{2} |\nabla u_{x_j}|^2 dxdt \\
+ \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dxdt).
\end{equation}

Substituting these (with $\delta$ sufficiently small) into (3.2), and calculating its left hand side, we deduce that

\begin{equation}
\frac{1}{2(b+1)} \int_{B_{2^p}} \zeta^2 v^{b+1}(x, t_0)dx + \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b-2)/2} \sum_{j=1}^{2} |\nabla u_{x_j}|^2 dxdt \\
+ \frac{p + 2b - 2}{4} \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{p+2b-4/2} |\nabla v|^2 dxdt \\
+ \frac{b(p - 2)}{2} \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{p+2b-6/2} (\nabla u v)^2 dxdt \\
\leq C(\int_0^{t_0} \int_{B_{2^p}} v^{(p+2b)/2} (|\nabla v|^2 + 1) dxdt + \int_0^{t_0} \int_{B_{2^p}} \zeta^2 v^{(p+2b+2)/2} dxdt) + CI_0.
\end{equation}

To estimate the last term of the right hand side of (3.3), we take

\begin{equation}
f = \zeta^2 v^{(p+2b+2)/(2q)}
\end{equation}
in the embedding inequality

\[(3.4) \quad \int_{G_T} |f(x, t)|^q \, dx \, dt \leq C(\int_G |\nabla f(x, t)|^{h_1} \, dx \, dt)(\sup_t \int_G |f(x, t)|^{h_2} \, dx)^{h_1/2}\]

for \( f \in L^\infty(0, T; L^q(G, R^2)) \cap L^p(0, T; W^{1, p}_0(G, R^2)) \), where \( h_1, h_2 \geq 1, \quad q = \frac{h_1(h_2+2)}{2}, \quad C \) only depends on \( h_1 \) and \( h_2 \). Set \( h_1 = 4/3, \quad h_2 = 1, \) thus \( q = 2 \).

Using Hölder inequality we obtain

\[(3.5) \quad \int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} \, dx \, dt \leq C(\sup_{0 < t < 0} \int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} \, dx)^{2/3} \]

\[\int_0^t \int_{B_{2\rho}} |\nabla \zeta|^{4/3} v^{(p+2b+2)/3} \, dx \, dt \]

\[+ (\int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \, dx \, dt)^{2/3} \left( \int_0^t \int_{B_{2\rho}} v^2 \, dx \, dt \right)^{1/3}].\]

From (2.4) it is led to \( \int_G v^{p/2} \, dx \leq pC_0 \). By using the reverse Hölder inequality, we see that there is a small constant \( \kappa \in (0, 1) \), such that

\[\int_{B_{2\rho}} v^{(p+2\kappa)/2} \, dx \leq C(\int_G v^{p/2} \, dx)^{2/p} + (pC_0)^{1/p}(p+2\kappa)/2 \leq C(p, C_0, \kappa).\]

Substituting this into

\[\int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} \, dx \leq (\int_{B_{2\rho}} \zeta^2 v^{b+1-\kappa} \, dx)^{1/2} \left( \int_{B_{2\rho}} v^{2b+2} v^{2\kappa} \, dx \right)^{1/2},\]

which is implied by Hölder inequality, we have

\[\int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} \, dx \leq C(\int_{B_{2\rho}} \zeta^2 v^{b+1-\kappa} \, dx)^{1/2}.\]

Substitute this into (3.5). Since \( p \geq 4 \), we see \( 2 \leq p/2 \) and \( \frac{p+2b+2}{3} \leq \frac{p+2b}{2} \).

Using \( \int_G v^{p/2} \, dx \leq pC_0 \), we derive that for any \( \delta_1 \in (0, 1) \),

\[\int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} \, dx \, dt \leq C(\sup_t \int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} \, dx)^{1/2} \left( \int_0^t \int_{B_{2\rho}} v^{(p+2b)/2} \, dx \, dt \right)^{2/3} \left( \int_0^t \int_{B_{2\rho}} v^{2b+2} \, dx \, dt \right)^{1/3}\]

\[+ C(\int_0^t \int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} \, dx \, dt)^{2/3} \left( \int_0^t \int_{B_{2\rho}} v^{2b+2} \, dx \, dt \right)^{1/3}\]

\[\leq \delta_1 \int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \, dx \, dt \]

\[+ C(\delta_1) \sup_t \int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} \, dx \]

\[+ C(\delta_1)(\int_0^t \int_{B_{2\rho}} v^{(p+2b)/2} \, dx \, dt)^C.\]
with $C$ independent of $n$ and $\varepsilon$. Using Hölder inequality again, we see that for any $\delta_2 \in (0, 1)$,
\[
\int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dx d\tau \\
\leq \delta_1 \int_0^t \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau \\
+ \delta_2 \sup_t \int_{B_{2\rho}} \zeta^2 v^{b+1} dx + C(\delta_1, \delta_2)[1 + (\int_0^t \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau)^C].
\]

Similarly, we also have the same upper bound for $I_0$ in (3.3). Subtract this into (3.3) and choose $\delta_1$ and $\delta_2$ sufficiently small. Noting that it always true for all $t \in (0, T)$, we obtain
\[
\sup_{0 < \tau < T} \int_{B_{2\rho}} \zeta^2 v^{b+1}(x, \tau) dx + \int_0^T \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau \\
\leq C + C(\int_0^T \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau)^C.
\]  

(3.7)

First, take $b = 0$. Using (2.4) and (2.5), we have
\[
\sup_{0 < \tau < T} \int_{B_{2\rho}} (\zeta^{2/\lambda} w)^{\lambda}(x, \tau) dx + \int_0^T \int_{B_{2\rho}} (\zeta^{2/\lambda} |\nabla w|)^2 dx d\tau \leq C,
\]
where $w = v^{(p+2b)/4}$ and $\lambda = \frac{4(b+1)}{p+2b}$. For $\zeta^{2/\lambda} w$, using the embedding inequality (3.4), we have $\int_0^T \int_{B_{2\rho}} (\zeta^{2/\lambda} w)^r dx d\tau \leq C$. It implies $\int_t^T \int_{B_\rho} w^r dx d\tau \leq C$, where $r = 2 + \frac{4}{p}$. This means $|\nabla u| \in L^{p+2}(B_\rho \times (t, T))$ and
\[
\int_t^T \int_{B_\rho} |\nabla u|^{p+2} dx d\tau \leq C.
\]

(3.8)

Next, take $b = 1$. Applying (3.8) and proceeding in the same way above, we can see that $|\nabla u| \in L^{p+6}(B_\rho \times (t, T))$ and
\[
\int_t^T \int_{B_\rho} |\nabla u|^{p+6} dx d\tau \leq C.
\]

For a given $l > 0$, by discussing inductively, we may find $s_k$ (for some $k$) such that $p + s_k > l$, and deduce that $|\nabla u| \in L^{p+s_k}(B_\rho \times (t, T))$,
\[
\int_t^T \int_{B_\rho} |\nabla u|^{p+s_k} dx d\tau \leq C.
\]

Substituting this into (3.7) with some sufficiently large $b$, we see that
\[
\sup_{t < \tau < T} \int_{B_{2\rho}} |\nabla u|^{l} dx \leq C.
\]

Thus Proposition 3.1 is proved. \qed
4. Estimate for $\|\nabla u_\varepsilon\|_{L^\infty_{loc}}$

By means of the Moser iteration, from the estimate Proposition 3.1, we can further prove the following:

**Proposition 4.1.** Assume $p \geq 4$. Then for any compact subset $K \subset G_T$, there exists a constant $C > 0$ (independent of $n$ and $\varepsilon$), such that

$$\|\nabla u_\varepsilon^2\|_{L^\infty(K, R^2)} \leq C = C(K).$$

**Proof.** Assume $R > 0$, $t_1 \in (0, T)$. Let $t_0 \in (t_1, T)$, $t_0 > 4t_1$. Define

$$Q(\rho, t_0) = B_{\rho}(x_0) \times (t_0, T), \quad T_m = \frac{t_0}{2} - \frac{t_0}{2^{m+1}},$$

$$\rho_m = \rho + \frac{\rho}{2^m}, \quad \overline{\rho}_m = \frac{1}{2}(\rho_m + \rho_{m+1}) = \rho + \frac{3\rho}{2^{m+2}},$$

$$B_m = B(x_0, \rho_m), \quad B'_m = B(x_0, \overline{\rho}_m),$$

$$Q_m = B_m \times (T_m, T), \quad Q'_m = B'_m \times (T_{m+1}, T),$$

and assume $\zeta_m(x, t) \in C^1_0(Q_m)$ satisfying

$$\zeta_m = 0, \forall t \leq T_m; \quad \zeta_m = 1, \forall (x, t) \in Q'_m$$

$$|\nabla \zeta_m| \leq \rho^{-1}2^{m+2}, \quad 0 \leq \zeta_{mt} \leq t_0^{-1}2^{m+3}.$$

Without loss of generality, we suppose $x_0 = 0$. Take $\rho > 0$ sufficiently small such that $B_{2\rho} \subset B_R$ and $|Q_m| \leq 1$. Multiplying (3.1) with $\zeta_m^2 v^b u_{x_j} (b \geq 1)$ and integrating over $B_{2\rho} \times (t_0/4, t)$, we obtain

$$\frac{1}{2(b+1)} \int_{B_{2\rho}} \zeta_m^2 v^{b+1}(x, t) dx$$

$$+ \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m^2 v^b u_{x_j} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau$$

$$= \frac{1}{b+1} \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m \zeta_{m\tau} v^{b+1} dx d\tau$$

$$- 2 \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m \zeta_{mx_j} v^b u_{x_j} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau$$

$$+ \frac{1}{\varepsilon^p} \int_{t_0/4}^t \int_{B_{2\rho}} (1 - |u|^2) \zeta_m^2 v^b (u_{x_j})^2 dx d\tau$$

$$- \frac{1}{2\varepsilon^p} \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m^2 v^b ((|u|^2)_{x_j})^2 dx d\tau.$$
Similar to the derivation of (3.3), we may also get

$$
\frac{1}{2(b+1)} \int_{B_{2\rho}} \zeta_m^2 v^{b+1}(x, t_0) \, dx \\
+ \int_{t_0/4}^{t} \int_{B_{2\rho}} \zeta_m^2 v^{(p+2b-2)/2} \sum_{j=1}^{2} |\nabla u_{x_j}|^2 \, dx \, dt \\
+ \frac{p + 2b - 2}{4} \int_{t_0/4}^{t} \int_{B_{2\rho}} \zeta_m^2 v^{\frac{p+2b-4}{2}} |\nabla v|^2 \, dx \, dt \\
+ \frac{b(p-2)}{2} \int_{t_0/4}^{t} \int_{B_{2\rho}} \zeta_m^2 v^{\frac{p+2b-6}{2}} (\nabla u \nabla u)^2 \, dx \, dt \\
\leq C \left( \int_{t_0/4}^{t} \int_{B_{2\rho}} |\zeta_m|^2 u^{b+1} \, dx \, dt + \int_{t_0/4}^{t} \int_{B_{2\rho}} v^{(p+2b)/2} |\nabla \zeta_m|^2 \, dx \, dt \\
+ \frac{p + 2b - 2}{2} \int_{t_0/4}^{t} \int_{B_{2\rho}} \zeta_m^2 (v^{(p+2b+2)/2} + v^{2b+2}) \, dx \, dt \right)
$$

(4.1)

with the constant $C$ independent of $n$, $\varepsilon$, $b$ and $m$. To estimate

$$
\int_{t_0/4}^{t} \int_{B_{2\rho}} \zeta_m^2 v^{(p+2b+2)/2} \, dx \, dt,
$$

we take $f = \zeta_m^{2/q} v^{(p+2b+2)/(2q)}$ in (3.4) with $h_1 = 1$, $h_2 = 2(q-1)$ and $q \in \left( \frac{3}{2}, 2 \right)$. Then, using Hölder inequality, and noticing $|Q_m| \leq 1$, we obtain

$$
(4.2) \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} \, dx \, dt \leq C I_3 \left[ \frac{9m}{\rho I_4^{(p+2b)}} + \frac{p + 2b + 2}{2q} I_1^{\frac{p+2b+2}{4(p+2b)}} - \frac{1}{4} \right]
$$

(in which we replace $t_0$ by $t \in (T_m, T)$, since (4.2) still holds for all $t_0 \in (T_m, T)$), where

$$
I_1 = \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \, dx \, dt, \\
I_3 = \sup_{T_m < t < T} \int_{B_m} \zeta_m^4 (q-1)/q, (1-1/q)(p+2b+2) \, dx, \\
I_4 = \int_{Q_m} v^{(p+2b)/2} \, dx \, dt.
$$
Substituting (4.2) into (4.1) and similar to the derivation of (3.7), we obtain
\[
I_1 + I_2 \leq C \left\{ \frac{2^m}{t_0} I_4^{\frac{2(b+1)}{p+2b}} + \left( \frac{2^m}{\rho} \right)^2 I_4 + \frac{p + 2b - 2}{2} I_3 \left[ \frac{2^m}{\rho} I_4^{\frac{p+2b+2}{p(r+2b)}} + \frac{p + 2b + 2}{2q} I_1 I_4^{\frac{p+2b+2}{p(r+2b)} - \frac{1}{2}} \right] \right\},
\]
where \( I_2 = \sup_{T_m < t < T} \int_{B_m} \zeta_m^2 v^{b+1} dx \). Applying Hölder inequality and (2.4) we have
\[
I_3 \leq I_2^{2(q-1)/q} (\sup_t \int_{Q_m} v^{\frac{p(q-1)}{2-q}} dx)^{2/q-1} \leq CI_2^{2(q-1)/q}.
\]
Hence, by using Young’s inequality, we can see that
\[
I_1 + I_2 \leq C \left\{ \frac{2^m}{t_0} I_4^{\frac{2(b+1)}{p+2b}} + \left( \frac{2^m}{\rho} \right)^2 I_4 + \left( \frac{p + 2b - 2}{2q} \right)^\lambda I_4^{\frac{p+2b+2}{p(r+2b)} - \frac{1}{2}} \right\},
\]
where \( \lambda > 0 \) only depends on \( q \). Since
\[
\frac{2(b+1)}{p+2b} \leq 1 \leq \frac{p + 2b + 2}{p + 2b} \leq \left( \frac{p + 2b + 2}{q(p + 2b)} - \frac{1}{2} \right) \left( \frac{2q}{2-q} \right),
\]
from Young’s inequality it follows that
\[
I_1 + I_2 \leq C_1 C_2^m \left( 1 + I_4^{1 + \frac{C_3}{(p-2)/(2+2m-1)}} \right),
\]
where \( C_1 > 0; \ C_2 \) depends on \( t_0, \rho, q \) and \( \lambda; \ 2m-1 = b + 1 \) and \( C_3 = \frac{2}{2-q} \).

Denote \( w = v^{(p-2)/4}, \ \sigma = \frac{4(b+1)}{p+2b} \). Then
\[
\sup_{T_m < t < T} \int_{B_m} (\zeta_m^2 \sigma w)^{\sigma} (x, \tau) dx + \int_{Q_m} (\zeta_m^2 \sigma |\nabla w|)^2 dx d\tau \leq I_1 + I_2.
\]
Applying the embedding inequality (3.4) we have
\[
\int_{Q_m+1} w^{2+\sigma} dx d\tau \leq (I_1 + I_2)^2.
\]
Since \( b = 2m-1 - 1 \), the inequality above implies
\[
\int_{Q_m+1} v^{(p-2)/2+2m} dx d\tau \leq (C_1 C_2^m)^2 \left( 1 + \int_{Q_m} v^{(p-2)/2+2m-1} dx d\tau \right)^{2(1+C_3/((p-2)/(2+2m-1)))}
\]
(4.3)
\[
\leq (C_1 C_2^m)^2 \left( 1 + \int_{Q_m} v^{(p-2)/2+2m^*_i} dx d\tau \right)^{2(1+C_3/((p-2)/(2+2m^*_i}))}
\]
If there exists a subsequence of positive integers \( \{m_i\} \) with \( m_i \to \infty \), such that
\[
I_4 = \int_{Q_{m_i}} v^{(p-2)/2+2m_i} dx d\tau < 1,
\]
then letting \( m_i \to \infty \) yields immediately

\[
\|v\|_{L^\infty(Q_\infty, R)} \leq C. 
\]

(4.4)

Otherwise, there must be a positive integer \( m_0 \) such that

\[
I_4 = \int_{Q_m} v^{(p-2)/2+2^m} \ dx \ dt \geq 1 \quad \text{for} \quad m \geq m_0.
\]

This, together with (4.3), implies

\[
\int_{Q_{m+1}} v^{(p-2)/2+2^m} \ dx \ dt \leq (C_1 C_2^m)^2 \left( \int_{Q_m} v^{(p-2)/2+2^m-1} \ dx \ dt \right)^{2(1+C_3/((p-2)/2+2^{m-1}))}
\]

Using Proposition 3.1 and an iteration lemma (Proposition 2.3 in [7]), we can also deduce the estimation (4.4). Thus the proof of Proposition 4.1 is complete.

\[\square\]

5. Hölder convergence

Once the uniform estimate \( \|\nabla u_\varepsilon\|_{L^\infty(\Omega_R)} \) is established, it is not difficult to prove

**Proposition 5.1.** Assume \( p \geq 4 \). Let \( \psi_\varepsilon = \frac{1}{\varepsilon^p} (1 - |u_\varepsilon|^2) \). Then for any compact subset \( K \) of \( G_T \), there exists a constant \( C > 0 \) (independent of \( \varepsilon \)), such that

\[
\|\psi_\varepsilon\|_{L^\infty(K, R)} \leq C.
\]

**Proof.** Consider the inner product of the both sides of (1.5) with \( u = u_\varepsilon^n \),

\[
u u_t - \text{div}(v^{(p-2)/2} \nabla u)u = \frac{1}{\varepsilon^p} |u|^2 (1 - |u|^2) = |u|^2 \psi,
\]

where \( \psi = \frac{1}{\varepsilon^p} (1 - |u_\varepsilon^n|^2) \). Combining this and \( \nabla \psi = -\frac{2}{\varepsilon^p} u \cdot \nabla u \) with

\[
\text{div}(v^{(p-2)/2} \nabla u)u = \text{div}(v^{(p-2)/2} u \cdot \nabla u) - v^{(p-2)/2} |\nabla u|^2
\]

yields

\[
|u|^2 \psi = v^{(p-2)/2} |\nabla u|^2 + \frac{\varepsilon^p}{2} \text{div}(v^{(p-2)/2} \nabla \psi) - \frac{\varepsilon^p}{2} \psi_t.
\]

Using (2.15) we further obtain

\[
\frac{1}{4} \psi \leq v^{(p-2)/2} |\nabla u|^2 + \frac{\varepsilon^p}{2} \text{div}(v^{(p-2)/2} \nabla \psi) - \frac{\varepsilon^p}{2} \psi_t, \ \forall \varepsilon \in (0, \varepsilon_0).
\]

Since at the point \((x_0, t_0)\), where \( \psi \) achieves its maximum, \( \nabla \psi = 0 \), \( \Delta \psi \leq 0 \), \( \psi_t \geq 0 \) and

\[
\text{div}(v^{(p-2)/2} \nabla \psi) = v^{(p-2)/2} \Delta \psi + \frac{p-2}{2} v^{(p-4)/2} \nabla v \nabla \psi \leq 0
\]
we derive that, from (5.2),

\[(5.3) \quad \|\psi_\varepsilon\|_{L^\infty(K, R)} \leq C\]

by using the local estimate Proposition 4.1. Letting \(n \to \infty\) and applying
proposition 2.4, we can see that (5.1) holds. \(\square\)

**Theorem 5.2.** Assume \(p \geq 4\). For any \(\alpha \in (0, 1)\), there holds

\[\lim_{\varepsilon \to 0} u_\varepsilon = u_p, \quad \text{in} \quad C^{\alpha, \frac{\alpha}{2}}_{\text{loc}}(G_T, R^2).\]

**Proof.** Assume \(0 < \eta < t_0 < t_1 < t_2 < T\). Set \(B(\Delta t) = B(x_0, (\Delta t)^{1/2})\), where \(x_0 \in B_R, \Delta t = t_2 - t_1\). Take \(\phi \in C^1_0(B(\Delta t))\), and denote \(u_\varepsilon^0\) by \(u\). Then,

\[(5.4) \quad \begin{align*}
\int_{B(\Delta t)} \phi(x)(u_\eta(x, t_2) - u_\eta(x, t_1))dx \\
= \int_{B(\Delta t)} \phi(x) \int_0^1 \frac{d}{ds} u_\eta(x, st_2 + (1 - s)t_1)dsdx \\
= \Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \int_0^T \int_{R^2} j_{\eta\tau}(x - y, t_2 + (1 - s)t_1 - \tau) \\
\quad u(y, \tau)dyd\tau dsdx \\
= -\Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \int_0^T \int_{R^2} j_{\eta\tau}(x - y, t_2 + (1 - s)t_1 - \tau) \\
\quad u(y, \tau)dyd\tau dsdx.
\end{align*}\]

For given \((x, t) \in G_T\) with \(t\) satisfying \(0 < \eta < t_0 < t < T - \eta\), we have
\(J_\eta(x - y, t - \tau) \in C^1_0(G_T)\). Since \(u\) is the solution of (1.5), it must satisfy

\[\begin{align*}
\int_0^T \int_{R^2} j_{\eta\tau}(x - y, t_2 + (1 - s)t_1 - \tau)u(y, \tau)dyd\tau \\
= \int_0^T \int_{R^2} v_y^{(p-2)/2} \nabla y_u \nabla y_j_\eta(x - y, t_2 + (1 - s)t_1 - \tau)dyd\tau \\
- \frac{1}{\varepsilon^p} \int_0^T \int_{R^2} u(y, \tau)(1 - |u(y, \tau)|^2) j_\eta(x - y, t_2 + (1 - s)t_1 - \tau)dyd\tau,
\end{align*}\]

where \(v_y = |\nabla y_u|^2 + \frac{1}{n}\). Substituting this into (5.4) yields

\[\begin{align*}
\int_{B(\Delta t)} \phi(x)(u_\eta(x, t_2) - u_\eta(x, t_1))dx \\
= -\Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \left[ \int_0^T \int_{R^2} v_y^{(p-2)/2} \nabla y_u \nabla y_j_\eta(x - y, \\
st_2 + (1 - s)t_1 - \tau)dyd\tau \\
- \frac{1}{\varepsilon^p} \int_0^T \int_{R^2} u(y, \tau)(1 - |u(y, \tau)|^2) j_\eta(x - y, \\
st_2 + (1 - s)t_1 - \tau)dyd\tau \right] dtdx.
\end{align*}\]
\[ \begin{align*}
&= -\Delta t \int_0^T \int_{\mathbb{R}^2} v_y^{(p-2)/2} \nabla_y \left( \int_{B(\Delta t)} \nabla_x \phi \eta(x - y, st_2 + (1 - s)t_1 - \tau) \right) dy d\tau ds \\
&+ \Delta t \int_{B(\Delta)} \phi(x) \int_0^1 J_\eta \left( \frac{1}{s} u(1 - |u|^2) \right)(x, st_2 + (1 - s)t_1) ds dx \\
&= -\Delta t \int_0^1 \int_{B(\Delta,t)} \nabla_x \phi \left( \int_T^T \int_{\mathbb{R}^2} j_\eta(x - y, st_2 + (1 - s)t_1) ds dx \\
&+ \Delta t \int_{B(\Delta)} \phi(x) \int_0^1 J_\eta \left( \frac{1}{s} u(1 - |u|^2) \right)(x, st_2 + (1 - s)t_1) ds dx \\
&= -\Delta t \int_0^1 \int_{B(\Delta,t)} \nabla_x \phi \eta(v^{(p-2)/2} \nabla u)(x, st_2 + (1 - s)t_1) ds dx \\
&+ \Delta t \int_0^1 \int_{B(\Delta)} \phi(x) J_\eta \left( \frac{1}{s} u(1 - |u|^2) \right)(x, st_2 + (1 - s)t_1) ds dx.
\end{align*} \tag{5.5} \]

Take \( \delta \in C^1_0(\mathbb{R}, \mathbb{R}^+) \) satisfying \( \delta(s) = 0 \) as \( |s| \geq 1 \) and \( \int_\mathbb{R} \delta(s) ds = 1 \). For \( h > 0 \) define \( \delta_h(s) = \frac{1}{h} \delta(h(s)). \) By a limit process, we see that (5.5) still holds for \( \phi \in W^{1,1}_0(\Delta,t). \) Hence, taking

\[ \phi = \phi_h(x) = \int_{-h}^{-(\Delta t)^{1/2} - |x-x_0| - 2h} \delta_h(s) ds \]

in (5.5), we have

\[ \begin{align*}
\int_{B(\Delta,t)} \phi_h(x)(u_\eta(x, t_2) - u_\eta(x, t_1)) dx \\
&= \Delta t \int_0^1 \int_{B(\Delta,t)} \delta_h((\Delta t)^{1/2} - |x-x_0| - 2h) \frac{x - x_0}{|x-x_0|} J_\eta(v^{(p-2)/2} u_x) \\
&\quad (x, st_2 + (1 - s)t_1) ds dx \\
&+ \Delta t \int_0^1 \int_{B(\Delta)} \phi_h(x) J_\eta \left( \frac{1}{s} u(1 - |u|^2) \right)(x, st_2 + (1 - s)t_1) ds dx.
\end{align*} \]

Clearly, \( \lim_{h \to 0} \phi_h(x) = 1 \) as \( x \in B(\Delta,t); \delta((\Delta t)^{1/2} - |x-x_0| - 2h) = 0 \) as \( |x-x_0| < (\Delta t)^{1/2} - h \); \( \delta_h \leq C h^{-1} \) and \( |B(\Delta) \setminus B(x_0, (\Delta t)^{1/2} - h)| \leq C h(\Delta t)^{1/2}. \)

Using Proposition 4.1 and (5.3), we derive

\[ | \int_{B(\Delta,t)} \phi_h(x)(u_\eta(x, t_2) - u_\eta(x, t_1)) dx | \leq C(\Delta t)^{3/2}. \]

Letting \( h \to 0 \) yields for all \( n > 0, \)

\[ | \int_{B(\Delta,t)} (u_{x\eta}(x, t_2) - u_{x\eta}(x, t_1)) dx | \leq C(\Delta t)^{3/2}. \]

Letting \( n \to \infty \), and using Proposition 2.4, we see that

\[ | \int_{B(\Delta,t)} (u_{x\eta}(x, t_2) - u_{x\eta}(x, t_1)) dx | \leq C(\Delta t)^{3/2}. \]
By the mean value theorem, there exists \( x^* \in B(\Delta t) \) such that
\[
|u_{\varepsilon\eta}(x^*, t_2) - u_{\varepsilon\eta}(x^*, t_1)| \leq C(\Delta t)^{1/2}.
\]
Applying (2.13) we have
\[
|u_{\varepsilon\eta}(x_0, t_2) - u_{\varepsilon\eta}(x_0, t_1)| \\
\leq |u_{\varepsilon\eta}(x_0, t_2) - u_{\varepsilon\eta}(x^*, t_2)| + |u_{\varepsilon\eta}(x^*, t_2) - u_{\varepsilon\eta}(x^*, t_1)| \\
+ |u_{\varepsilon\eta}(x^*, t_1) - u_{\varepsilon\eta}(x_0, t_1)| \leq C(\Delta t)^{1/2}.
\]
Combining this with (2.13) and letting \( \eta \to 0 \), we see that \( u_\varepsilon \) satisfies
\[
\|u_\varepsilon\|_{C^{\alpha+1, \frac{1}{2}}(B_R \times (t_0, T), R^2)} \leq C.
\]
There exists a subsequence \( u_{\varepsilon_k} \) such that as \( k \to \infty \),
\[
u_{\varepsilon_k} \to u^*, \quad \text{in} \quad C^{\alpha, \frac{3}{2}}(B_R \times (t_0, T), R^2),
\]
where \( \alpha \in (0, 1) \). From Proposition 2.5, it follows that \( u^* = u_p \). In view of the uniqueness of \( u_p \), the convergence is true not only for subsequence \( u_{\varepsilon_k} \), but also for all \( u_\varepsilon \). \( \square \)

**Theorem 5.3.** Assume \( p \geq 4 \). For some \( \alpha \in (0, 1) \), there holds
\[
(5.6) \quad \lim_{\varepsilon \to 0} \nabla u_\varepsilon = \nabla u_p, \quad \text{in} \quad C^{\alpha, \frac{3}{2}}_{loc}(G_T, R^2).
\]

**Proof.** Now, according to Proposition 5.1, the right hand side of (1.1) is bounded on the compact subset \( K \) of \( G_T \) uniformly in \( \varepsilon \in (0, \varepsilon_0) \). Thus, applying Theorem 1.1 in [4] yields that, for some \( \beta \in (0, 1) \), we have
\[
(5.7) \quad \|\nabla u_\varepsilon\|_{C^{\alpha, \frac{3}{2}}(K, R^2)} \leq C,
\]
where the constant does not depend on \( \varepsilon \in (0, \varepsilon_0) \). From this it follows that, there exist a function \( u_* \) and a subsequence \( u_{\varepsilon_k}(\varepsilon_k \to 0, \text{as} \ k \to \infty) \) of \( u_\varepsilon \), such that
\[
(5.8) \quad \lim_{k \to \infty} \nabla u_{\varepsilon_k} = u_* \quad \text{in} \quad C^{\alpha, \frac{3}{2}}(K, R^2), \quad \alpha \in (0, \beta).
\]
Combining (5.7) with Theorem 5.2, we see that \( u_* = \nabla u_p \). By the fact that, any subsequence of \( u_\varepsilon \) contains a subsequence converging in \( C^{\alpha, \frac{3}{2}}(K, R^2) \) sense, and the limit is the same function \( u_p \), we may assert that (5.8) is also true for all \( u_\varepsilon \). Theorem is proved. \( \square \)

**Remark 5.4.** It is only obtained that the weak solution \( u_\varepsilon \) of (1.1)-(1.3) converges to \( u_p \) locally, since I have not found any regularity (near the boundary) of the weak solution to the initial-boundary value problem for the p-Laplace system, except the results of that in [2]. However, the results in [2] is not sufficient to deduce the convergence (5.6) near the boundary. This is an open problem.
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