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HÖLDER CONVERGENCE OF THE WEAK SOLUTION TO AN EVOLUTION EQUATION OF p-GINZBURG-LANDAU TYPE

Yutian Lei

Abstract. The author studies the local Hölder convergence of the solution to an evolution equation of p-Ginzburg-Landau type, to the heat flow of the p-harmonic map, when the parameter tends to zero. The convergence is derived by establishing a uniform gradient estimation for the solution of the regularized equation.

1. Introduction

Let $G \subset R^2$ be a bounded and simply connected domain with smooth boundary ∂G, $g \in C^\infty(\partial G, S^1)$, where $S^1 = \{x \in R^2; |x| = 1\}$, and $\deg(g, \partial G) = d = 0$, $u_0 \in C^\infty(\overline{G}, S^1)$ and $u_0(x) = g(x)$ as $x \in \partial G$. Denote $G_T = G \times (0, T]$ with $T \in (0, \infty)$. We are concerned with the asymptotic behavior of the weak solution u_ε of the following problem when $\varepsilon \to 0$,

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \frac{1}{\varepsilon^p}u(1-|u|^2), \quad \text{a.e. on} \quad G_T \tag{1.1}$$

$$u|_{\partial G \times R^+} = g(x), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in G. \tag{1.3}$$

Recall that $u \in C_{loc}(0,T; L^2(G,R^2)) \cap L^p_{loc}(0,T; W^{1,p}(G,R^2))$ is a weak solution of (1.1), if u satisfies

$$\begin{aligned} \int_{G_T} u\phi_t dxdt &= \int_{G_T} |\nabla u|^{p-2}\nabla u \nabla \phi dxdt \\ &\quad - \tfrac{1}{\varepsilon^p}\int_{G_T} u\phi(1-|u|^2)dxdt, \quad \forall \phi \in C_0^\infty(G_T). \end{aligned} \tag{1.4}$$

For the fixed $\varepsilon > 0$, the weak solution of (1.1)-(1.3) exists. In fact, it can be derived as the limit of the solution u_ε^n of the regularized problem

$$u_t = \operatorname{div}[(|\nabla u|^2 + \frac{1}{n})^{(p-2)/2}\nabla u] + \frac{1}{\varepsilon^p}u(1-|u|^2), \quad \text{on} \quad G_T \tag{1.5}$$

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$$u|_{\partial G\times R^+} = g(x),$$
$$u(x,0) = u_0(x), \quad x \in G,$$

by letting $n \to \infty$ (cf. [9]). The uniqueness of the weak solution will be proved in §2.

When $p = 2$, F. Bethuel, H. Brezis and F. Helein discussed the stationary case. They obtained the asymptotic behavior of the minimizer of the simplified Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2 dx$$

under the assumption of $\deg(g, \partial G) = 0$ (see [1]). In particular, it was proved that the Ginzburg-Landau minimizer converges to the harmonic map with boundary data g on G.

In the evolutionary case and under some general assumption on $u(x,\ 0)$, F. H. Lin, R. Jerrard and H. Soner studied the dynamics of the zeros of the solution u_ε, i.e., the dynamical properties of Ginzburg-Landau vortices (see [6] and [8]). Under the special assumption of $u_0(x) \in C^\infty(\overline{G}, S^1)$, some uniform estimates and the Hölder convergence of u_ε is derived in [5]. The work shows that the limit of the solution u_ε is a heat flow of the harmonic map.

In this paper, under the assumption of $u_0(x) \in C^\infty(\overline{G}, S^1)$ as in [5], we will present the local Hölder convergence of the p-Ginzburg-Landau solution (the weak solution of (1.1)-(1.3)) when $p > 2$, by establishing a series of uniform estimations of the solution u_ε^n of (1.5), (1.2) and (1.3). The reason to consider the estimation of u_ε^n is that, comparing with the case of $p = 2$, the system (1.1) is degenerate and nonlinear in the main part. It leads to the difficulty when setting up the regularity estimation of u_ε.

Besides the results in [5], the study is also motivated by the research work in [3]. The authors proved that the heat flow of the p-harmonic map u_p can be approached by the weak solution u_ε of (1.1)-(1.3) in some weak sense, when $\varepsilon \to 0$. This work proffered a new idea to study the p-harmonic maps. Namely, we can firstly research the properties of weak solutions u_ε to some p-Laplace system with a penalization term, such as p-Ginzburg-Landau system (1.1). Next, letting $\varepsilon \to 0$, and investigating the asymptotic behavior of u_ε to obtain the corresponding properties of the limit p-harmonic map. We hope the convergence of u_ε to the p-harmonic map u_p can be improved in more strong sense (cf. Theorems 5.2 and 5.3). Thus, the relation between u_ε and u_p will be more clear.

The limit function u_p of u_ε when $\varepsilon \to 0$ can be introduced as follows. If $u_p \in L^\infty_{loc}(0,T; L^2(G,S^1)) \cap L^p_{loc}(0,T; W^{1,p}(G,S^1))$ is a weak solution of the system

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u|\nabla u|^p, \tag{1.6}$$

then it is called the heat flow of the p-harmonic map. By $\deg(g, \partial G) = 0$, there exists $\theta \in L^\infty_{loc}(0,T; L^2(G,R)) \cap L^p_{loc}(0,T; W^{1,p}(G,R))$, $\theta_1 \in C^\infty(\partial G, R)$

and $\theta_2 \in C^\infty(\overline{G}, R)$ such that $u_p = (\cos\theta, \sin\theta)$, $g = (\cos\theta_1, \sin\theta_1)$, $u_0 = (\cos\theta_2, \sin\theta_2)$ and

$$\theta|_{\partial G\times R^+} = \theta_1, \tag{1.7}$$

$$\theta|_{G\times\{t=0\}} = \theta_2. \tag{1.8}$$

It follows from (1.6) that

$$\theta_t = \operatorname{div}(|\nabla\theta|^{p-2}\nabla\theta). \tag{1.9}$$

From Lemma 1.2 in [3] it is easy to see that there exists a unique solution

$$\Theta \in L^\infty_{loc}(0,T; L^2(G,R)) \cap L^p_{loc}(0,T; W^{1,p}(G,R))$$

satisfying the problem (1.7)-(1.9) in the weak sense. Hence the heat flow of the p-harmonic map $u_p = (\cos\Theta, \sin\Theta)$ is also unique.

We shall prove a series of preliminary propositions in Section 2. By setting up the gradient estimations of u^n_ε locally in Section 3 and 4, we will present the Hölder convergence of the weak solution u_ε and its gradient ∇u_ε in §5 (Theorem 5.2 and 5.3).

2. Preliminaries

Proposition 2.1. *The weak solution of* (1.1)-(1.3) *is unique.*

Proof. Set $0 < t_1 < t_2 < T$. By a limit process we see that the test function ϕ in (1.4) can be taken in $W^{1,p}_0(G, R^2)$. Then the weak solution satisfies

$$\begin{aligned}&\int_G (u(x,t_2) - u(x,t_1))\phi(x)dx + \int_{t_1}^{t_2}\int_G |\nabla u|^{p-2}\nabla u\nabla\phi dxdt\\ &= \frac{1}{\varepsilon^p}\int_{t_1}^{t_2}\int_G u\phi(1-|u|^2)dxdt.\end{aligned} \tag{2.1}$$

Fix $\tau \in (0,T)$. Choose h satisfying $0 < \tau < \tau + h < T$. Taking $t_1 = \tau$, $t_2 = \tau + h$ in (2.1) and multiplying with h^{-1} we obtain

$$\begin{aligned}&\int_G (u_h(x,\tau))_\tau\phi(x)dx + \int_G (|\nabla u|^{p-2}\nabla u)_h(x,\tau)\nabla\phi(x)dx\\ &=\frac{1}{\varepsilon^p}\int_G \phi[u(1-|u|^2)]_h dx\end{aligned} \tag{2.2}$$

for any $\phi \in W^{1,p}_0(G, R^2)$, where u_h is the Steklov's mean value of u, i.e., $u_h = \frac{1}{h}\int_t^{t+h} u(x,\tau)d\tau$ as $t \in (0, T-h)$; $u_h = 0$ as $t > T-h$.

Assume that both u_1 and u_2 are weak solutions of (1.1)-(1.3), and write $w = u_1 - u_2$. In virtue of (2.2), we see that for any $\phi \in W^{1,p}_0(G, R^2)$ and any given $\tau \in (0,T)$,

$$\int_G w_{h\tau}\phi dx + \int_G (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h \nabla\phi dxdt$$
$$= \frac{1}{\varepsilon^p}\int_{G_T} \phi[u_1(1-|u_1|^2) - u_2(1-|u_2|^2)]_h dxdt.$$

Taking $\phi(x) = w_h(x,\tau)$ and integrating on $(0,t)$, we see that

$$\int_G w_h^2 dx$$
$$= -\int_0^t \int_G (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h(\nabla u_1 - \nabla u_2)_h dxd\tau$$
$$+ \frac{1}{\varepsilon^p}\int_0^t \int_G w_h^2 dxd\tau - \frac{1}{\varepsilon^p}\int_0^t \int_G (u_1|u_1|^2 - u_2|u_2|^2)_h w_h dxd\tau$$

in view of $\int_G w_h^2(x,0)dx = 0$. Letting $h \to 0$, we have

$$\int_G w^2(x,t)dx$$
$$= -\int_0^t \int_G (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(\nabla u_1 - \nabla u_2)dxd\tau$$
$$+ \frac{1}{\varepsilon^p}\int_0^t \int_G w^2 dxd\tau - \frac{1}{\varepsilon^p}\int_0^t \int_G (u_1|u_1|^2 - u_2|u_2|^2)(u_1 - u_2)dxd\tau.$$

Applying Lemma 1.2 in [3], we know that the first and the third terms of the right hand side of the equality above are not positive. Using Gronwall's inequality, we obtain

$$\int_G |u_1 - u_2|^2 dx = 0, \quad \forall t \in (0,T).$$

This implies our conclusion. □

Proposition 2.2. *Assume u_ε^n solves the problem* (1.5), (1.2) *and* (1.3). *Then $|u_\varepsilon^n| \le 1$ on $\overline{G_T}$.*

Proof. Denote $u = u_\varepsilon^n$. Suppose $\psi = |u|^2$ achieves its maximum on $\partial G \times (0,T]$ or $G \times \{0\}$, then the conclusion can be seen easily. Suppose ψ achieves its maximum at (x_0,t_0) in G_T. Then

$$\nabla\psi(x_0,t_0) = 0, \quad \Delta\psi(x_0,t_0) \le 0, \quad \text{and} \quad \psi_t(x_0,t_0) \ge 0. \tag{2.3}$$

We claim that $\psi \le 1$ over G_T. Otherwise, multiplying (1.5) with u, we have

$$\frac{1}{2}(|u|^2)_t = \operatorname{div}(v^{(p-2)/2}u\nabla u) - v^{(p-2)/2}|\nabla u|^2 + \frac{1}{\varepsilon^p}|u|^2(1-|u|^2),$$

where $v = |\nabla u|^2 + \frac{1}{n}$. It follows from (2.3) that at (x_0, t_0),

$$0 \leq \frac{1}{2}\psi_t = \frac{1}{2}\operatorname{div}(v^{(p-2)/2})\nabla\psi + \frac{1}{2}v^{(p-2)/2}\Delta\psi - v^{(p-2)/2}|\nabla u|^2 + \frac{1}{\varepsilon^p}|u|^2(1-|u|^2) < 0,$$

which is impossible. The proof is complete. □

Proposition 2.3. *Assume u_ε^n solves* (1.5), (1.2) *and* (1.3)*. Then, there exists a constant $C_0 > 0$, which is independent of ε and n, such that*

$$\sup_{t\in(0,T]}\left[\int_0^t\int_G \left|\frac{\partial}{\partial t}u_\varepsilon^n(x,t)\right|^2 dx d\tau + E_\varepsilon^n(u_\varepsilon^n(x,t))\right] \leq C_0. \tag{2.4}$$

Here

$$E_\varepsilon^n(u) = \frac{1}{p}\int_G (|\nabla u|^2 + \frac{1}{n})^{p/2}dx + \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2 dx.$$

Proof. Multiplying (1.5) with u_t and integrating over G, we have

$$\int_G u_t^2 dx = \int_G u_t \operatorname{div}(v^{(p-2)/2}\nabla u)dx + \frac{1}{\varepsilon^p}\int_G u u_t(1-|u|^2)dx,$$

where $v = |\nabla u|^2 + \frac{1}{n}$. Noting $u_t = g_t = 0$ on ∂G, we obtain

$$\int_G u_t^2 dx = -\frac{\partial}{\partial t}E_\varepsilon^n(u)$$

and thus for any $t > 0$,

$$\int_0^t\int_G \left|\frac{\partial}{\partial t}u_\varepsilon^n(x,t)\right|^2 dx d\tau + E_\varepsilon^n(u_\varepsilon^n(x,t)) \leq E_\varepsilon^n(u_0) = C_0.$$

This means that (2.4) holds. □

Proposition 2.4. *For any $\varepsilon \in (0, \varepsilon_0)$ with some fixed small constant ε_0, if $n \to \infty$, then*

$$u_\varepsilon^n \to u_\varepsilon, \quad \text{weakly star in} \quad L^\infty((0,T); W^{1,p}(G, R^2));$$

$$u_\varepsilon^n \to u_\varepsilon, \quad \text{strongly in} \quad L^p((0,T); L^p(G, R^2));$$

$$\frac{\partial}{\partial t}u_\varepsilon^n \to \frac{\partial}{\partial t}u_\varepsilon, \quad \text{weakly in} \quad L^2(0,T; L^2(G, R^2)).$$

In addition, the limit u_ε is just the unique weak solution of (1.1)-(1.3).

Proof. From Proposition 2.2 and (2.4), we see that there exists $C = C(T) > 0$ (independent of ε and n), such that

$$\|u_\varepsilon^n\|_{L^\infty(0,T;W^{1,p}(G,R^2))} \leq C(T), \tag{2.5}$$

$$\left\|\frac{\partial}{\partial t}u_\varepsilon^n\right\|_{L^2(0,T;L^2(G,R^2))} \leq C(T). \tag{2.6}$$

So for any $\varepsilon \in (0, \varepsilon_0)$, there exists a subsequence $u_\varepsilon^{n_k}$ of u_ε^n such that as $k \to \infty$,

$$u_\varepsilon^{n_k} \to w_\varepsilon, \quad \text{weakly star in} \quad L^\infty((0,T); W^{1,p}(G, R^2)); \tag{2.7}$$

$$u_\varepsilon^{n_k} \to w_\varepsilon, \quad \text{strongly in} \quad L^p((0,T); L^p(G, R^2)); \tag{2.8}$$

$$\frac{\partial}{\partial t} u_\varepsilon^{n_k} \to \frac{\partial}{\partial t} w_\varepsilon, \quad \text{weakly in} \quad L^2(0,T; L^2(G, R^2)) \tag{2.9}$$

by applying Lemma 1.4 and the proof of Theorem 2.1 in [3]. Assume K is any compact subset of G_T. Set $\zeta \in C_0^\infty(G_T, R)$ and $\zeta = 1$ on K. Multiplying (1.5) by $u\zeta$ and integrating over G_T we obtain

$$\frac{1}{\varepsilon^p} \int_K (1 - |u|^2)|u|^2 dxdt \leq C(1 + \int_{G_T} v^{p/2} dxdt + \int_{G_T} |u_t|^2 dxdt) \leq C(T),$$

by applying the Hölder inequality, (2.5) and (2.6). Here $v = |\nabla u|^2 + \frac{1}{n}$. Combining this with $\frac{1}{\varepsilon^p} \int_{G_T} (1 - |u|^2)^2 dxdt \leq C(T)$ (which is implied by (2.4)) yields

$$\frac{1}{\varepsilon^p} \int_K (1 - |u|^2) dxdt \leq C.$$

Hence, using Theorem 2.1 in [3] yields that for any $\varepsilon \in (0, \varepsilon_0)$, as $n_k \to \infty$,

$$v^{(p-2)/2} \nabla u_\varepsilon^{n_k} \to |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon, \quad \text{in} \quad L^1(K). \tag{2.10}$$

Obviously, u_ε^n satisfies

$$\begin{aligned} \int_{G_T} u\phi_t dxdt &= \int_{G_T} v^{(p-2)/2} \nabla u \nabla \phi dxdt \\ &\quad - \tfrac{1}{\varepsilon^p} \int_{G_T} u\phi(1 - |u|^2) dxdt, \quad \forall \phi \in C_0^\infty(G_T). \end{aligned}$$

Letting $n_k \to \infty$, and using (2.7)-(2.10), we get

$$\begin{aligned} \int_{G_T} w_\varepsilon \phi_t dxdt &= \int_{G_T} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla \phi dxdt \\ &\quad - \tfrac{1}{\varepsilon^p} \int_{G_T} w_\varepsilon \phi(1 - |w_\varepsilon|^2) dxdt, \quad \forall \phi \in C_0^\infty(G_T). \end{aligned}$$

This, together with the uniqueness of the weak solution u_ε of (1.1)-(1.3), implies $w_\varepsilon = u_\varepsilon$. Our proposition can be completed by (2.7), (2.8), (2.9) and the uniqueness of u_ε. □

Proposition 2.5. *Assume u_ε is the weak solution of* (1.1)-(1.3). *Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_p, \quad \textit{weakly star in} \quad L^\infty(0,T; W^{1,p}(G, R^2)),$$

$$\frac{\partial}{\partial t} u_\varepsilon \to \frac{\partial}{\partial t} u_p, \quad \textit{weakly in} \quad L^2(0,T; L^2(G, R^2)),$$

$$u_\varepsilon \to u_p, \quad \textit{strongly in} \quad L^p(0,T; L^p(G, R^2)).$$

Proof. By (2.5), (2.6) and Proposition 2.4, we can find a positive constant C (independent of ε), such that

$$\|u_\varepsilon\|_{L^\infty(0,T;W^{1,p}(G,R^2))} \leq C, \tag{2.11}$$

$$\|\frac{\partial}{\partial t}u_\varepsilon\|_{L^2(0,T;L^2(G,R^2))} \leq C. \tag{2.12}$$

Hence, there is a subsequence ε_k such that as $\varepsilon_k \to 0$,

$$\begin{aligned} u_{\varepsilon_k} &\to u_p, \quad \text{weakly in} \quad L^\infty(0,T;W^{1,p}(G,R^2)) \\ \frac{\partial}{\partial t}u_{\varepsilon_k} &\to \frac{\partial}{\partial t}u_p, \quad \text{weakly in} \quad L^2(0,T;L^2(G,R^2)) \\ u_{\varepsilon_k} &\to u_p, \quad \text{strongly in} \quad L^p(0,T;L^p(G,R^2)). \end{aligned}$$

Using Lemma 1.4 and Theorem 2.2 in [3], we see that u_p is just the unique heat flow of the p-harmonic map. Thus, the convergence above can be generalized to all ε instead of the subsequence ε_k. □

Proposition 2.6. *Assume u_ε is the weak solution of* (1.1)-(1.3). *Then for any compact subset $K \subset G$, there exists a constant $C > 0$, which is independent of ε, such that for any $x_1, x_2 \in K$ and $t \in (0,T]$,*

$$|u(x_1,t) - u(x_2,t)| \leq C|x_1 - x_2|.$$

Proof. Assume $R > 0$ and $t_0 \in (0,T)$. Let J_η be the mollification operator, such that

$$u_\eta(x,t) = J_\eta u(x,t) = \int_0^T \int_{R^2} j_\eta(x-y,t-\tau)u(y,\tau)dyd\tau,$$

where $0 < \eta < t_0 < t < T - \eta$. Denote u_ε by u. For any $x_1, x_2 \in B_R$, there holds

$$\begin{aligned} & u_\eta(x_1,t) - u_\eta(x_2,t) \\ = & \textstyle\int_0^T \int_{R^2} \int_0^1 \frac{d}{ds} j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)dsdyd\tau \\ = & \textstyle(x_1 - x_2)\int_0^T \int_{R^2} \int_0^1 \nabla_x j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)dsdyd\tau \\ = & \textstyle-(x_1 - x_2)\int_0^1 \int_{R^2} \int_0^T \nabla_y j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)d\tau dyds \\ = & \textstyle(x_1 - x_2)\int_0^1 \int_{R^2} \int_0^T j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)\nabla_y u(y,\tau)d\tau dyds. \end{aligned}$$

Hence, applying Hölder inequality and (2.11) we obtain

(2.13)

$$\begin{aligned} & |u_\eta(x_1,t) - u_\eta(x_2,t)| \\ \leq & \textstyle|x_1 - x_2| \int_0^1 \int_{R^2} \int_0^T |j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)||\nabla_y u(y,\tau)|d\tau dyds \\ \leq & C|x_1 - x_2|. \end{aligned}$$

Letting $\eta \to 0$ we derive

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|, \tag{2.14}$$

which implies our consequence. □

Proposition 2.7. *For any $n > 0$, as $\varepsilon \to 0$, there holds $|u_\varepsilon^n| \to 1$ in $C_{loc}(G_T)$.*

Proof. Assume K is a compact subset of G_T. For any $(x_0,t_0) \in K$, denote $\gamma = |u_\varepsilon(x_0,t_0)|$. Similar to the proof of (2.14), for u_ε^n we also have

$$|u_\varepsilon^n(x,t_0) - u_\varepsilon^n(x_0,t_0)| \le C_0|x - x_0|$$

by using (2.5). Here C_0 is independent of ε and n. Hence $|u_\varepsilon^n(x,t_0)| \le \gamma + \varepsilon$ as $x \in B(x_0, C_0^{-1}\varepsilon)$ and

$$\int_{B(x_0,C_0^{-1}\varepsilon)} (1 - |u_\varepsilon^n(x,t_0)|^2)^2 dx \ge C_0^{-2}\pi(1-\gamma-\varepsilon)^2\varepsilon^2.$$

Combining this with (2.4) we obtain $(1-\gamma-\varepsilon)^2 \le C\varepsilon^{p-2}$ or $1-\gamma \le C\varepsilon^{(p-2)/2} + \varepsilon$. Noting the arbitrariness of (x_0,t_0), we see our consequence. □

Proposition 2.7 implies that, there exists $\varepsilon_0 > 0$ sufficiently small, such that as $\varepsilon \in (0,\varepsilon_0)$,

$$|u_\varepsilon^n| \ge 1/2 \tag{2.15}$$

on any compact subset of G_T.

3. Estimate for $\|\nabla u_\varepsilon^n\|_{L^l_{loc}}$

Proposition 3.1. *Assume u_ε^n is a solution of (1.5), (1.2) and (1.3). If $p \ge 4$, then for any compact subset $K \subset G$, $t \in (0,T)$ and $l > 1$, there exists a constant $C > 0$, which is independent of n and ε, such that*

$$\sup_{\tau\in[t,T]} \int_K |\nabla u_\varepsilon^n|^l dx + \|\nabla u_\varepsilon^n\|_{L^l(K_T,R^2)} \le C = C(K_T,l),$$

where $K_T = K \times [t,T]$.

Proof. Obviously $u = u_\varepsilon$ satisfies (1.5). Denote $v = |\nabla u|^2 + \frac{1}{n}$. Differentiate (1.5) with respect to x_j,

$$\frac{\partial u_{x_j}}{\partial t} = (v^{(p-2)/2}u_{x_i})_{x_ix_j} + \frac{1}{\varepsilon^p}[u(1-|u|^2)]_{x_j}. \tag{3.1}$$

Let $\zeta \in C^\infty(\overline{B_{2\rho}} \times [0,T],[0,1])$ satisfy

$$0 \le \zeta(x,\tau) \le 1, \quad (x,\tau) \in B_{2\rho} \times [0,T];$$

$$\zeta(x,\tau) = 1, \quad (x,\tau) \in B_\rho \times (t,T);$$

$$\zeta(x,\tau) = 0, \quad (x,\tau) \in \partial B_{2\rho} \times [0,T] \quad \text{or} \quad \tau \le t/2;$$

$$|\nabla\zeta| \le C\rho^{-1}, \quad |\zeta_t| \le Ct^{-1}.$$

Multiplying (3.1) by $\zeta^2 v^b u_{x_j} (b \geq 0)$ and integrating over $B_{2\rho} \times [0, t_0]$, we obtain

(3.2)
$$\begin{aligned}
&\tfrac{1}{2(b+1)} \int_{B_{2\rho}} \zeta^2 v^{b+1}(x, t_0)dx + \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 (v^b u_{x_j})_{x_i} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau \\
= &\tfrac{1}{b+1} \int_0^{t_0} \int_{B_{2\rho}} \zeta \zeta_t v^{b+1} dx d\tau - 2 \int_0^{t_0} \int_{B_{2\rho}} \zeta \zeta_{x_i} v^b u_{x_j} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau \\
&+ \tfrac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2\rho}} (1 - |u|^2) \zeta^2 v^b u_{x_j}^2 dx d\tau - \tfrac{1}{2\varepsilon^p} \int_0^{t_0} \int_{B_{2\rho}} [(|u|^2)_{x_j}]^2 \zeta^2 v^b dx d\tau.
\end{aligned}$$

Applying (1.5) and (2.15) we see that

$$\begin{aligned}
&\tfrac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2\rho}} (1 - |u|^2) \zeta^2 v^b u_{x_j}^2 dx d\tau \\
\leq &\int_0^{t_0} \int_{B_{2\rho}} |u|^{-1} [|u_t| + |\operatorname{div}(v^{(p-2)/2} \nabla u)|] \zeta^2 v^{b+1} dx d\tau.
\end{aligned}$$

Using Young's inequality and (2.4) to estimate the right hand side of the inequality above, we may obtain that for any $\delta \in (0, 1)$,

$$\begin{aligned}
&\tfrac{1}{\varepsilon^p} \int_0^{t_0} \int_{B_{2\rho}} (1 - |u|^2) \zeta^2 v^b u_{x_j}^2 dx d\tau \\
\leq &\delta \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau \\
&+ C(\delta) (\int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau + \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dx d\tau) + C I_0,
\end{aligned}$$

where $I_0 = \int_0^{t_0} \int_{B_{2\rho}} \zeta v^{2b+2} dx d\tau$, and

$$\begin{aligned}
&\int_0^{t_0} \int_{B_{2\rho}} \zeta \zeta_{x_i} v^b u_{x_j} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau \\
\leq &C(\delta) \int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau \\
&+ \delta (\int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-2)/2} \textstyle\sum_{j=1}^2 |\nabla u_{x_j}|^2 dx d\tau \\
&+ \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau).
\end{aligned}$$

Substituting these (with δ sufficiently small) into (3.2), and calculating its left hand side, we deduce that

(3.3)
$$\begin{aligned}
&\frac{1}{2(b+1)} \int_{B_{2\rho}} \zeta^2 v^{b+1}(x, t_0) dx + \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-2)/2} \sum_{j=1}^2 |\nabla u_{x_j}|^2 dx d\tau \\
&+ \frac{p+2b-2}{4} \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{\frac{p+2b-4}{2}} |\nabla v|^2 dx d\tau \\
&+ \frac{b(p-2)}{2} \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{\frac{p+2b-6}{2}} (\nabla u \nabla v)^2 dx d\tau \\
\leq &C (\int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} (|\nabla \zeta|^2 + 1) dx d\tau + \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dx d\tau) + C I_0.
\end{aligned}$$

To estimate the last term of the right hand side of (3.3), we take

$$f = \zeta^{2/q} v^{(p+2b+2)/(2q)}$$

in the embedding inequality

$$(3.4)\quad \int_{G_T} |f(x,t)|^q dxdt \le C(\int_{G_T} |\nabla f(x,t)|^{h_1} dxdt)(\sup_t \int_G |f(x,t)|^{h_2} dx)^{h_1/2}$$

for $f \in L^\infty(0,T;L^q(G,R^2)) \cap L^p(0,T;W_0^{1,p}(G,R^2))$, where $h_1, h_2 \ge 1$, $q = \frac{h_1(h_2+2)}{2}$, C only depends on h_1 and h_2. Set $h_1 = 4/3$, $h_2 = 1$, thus $q = 2$. Using Hölder inequality we obtain

$$(3.5)\quad \begin{aligned} & \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dxd\tau \\ \le\ & C(\sup_{0<t<t_0} \int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} dx)^{2/3} \\ & \cdot[\int_0^{t_0} \int_{B_{2\rho}} |\nabla\zeta|^{4/3} v^{(p+2b+2)/3} dxd\tau \\ & +(\int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dxd\tau)^{2/3} (\int_0^{t_0} \int_{B_{2\rho}} v^2 dxd\tau)^{1/3}]. \end{aligned}$$

From (2.4) it is led to $\int_G v^{p/2} dx \le pC_0$. By using the reverse Hölder inequality, we see that there is a small constant $\kappa \in (0,1)$, such that

$$\int_{B_{2\rho}} v^{(p+2\kappa)/2} dx \le [C(\int_G v^{p/2} dx)^{2/p} + (pC_0)^{1/p}]^{(p+2\kappa)/2} \le C(p, C_0, \kappa).$$

Substituting this into

$$\int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} dx \le (\int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} dx)^{\frac{1}{2}} (\int_{B_{2\rho}} v^{\frac{2\kappa+p}{2}} \zeta^{\frac{2\kappa}{b+1}} dx)^{\frac{1}{2}},$$

which is implied by Hölder inequality, we have

$$\int_{B_{2\rho}} \zeta v^{(p+2b+2)/4} dx \le C(\int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} dx)^{\frac{1}{2}}.$$

Substitute this into (3.5). Since $p \ge 4$, we see $2 \le p/2$ and $\frac{p+2b+2}{3} \le \frac{p+2b}{2}$. Using $\int_G v^{p/2} dx \le pC_0$, we derive that for any $\delta_1 \in (0,1)$,

$$\begin{aligned} & \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dxd\tau \\ \le\ & C(\sup_t \int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} dx)^{\frac{1}{3}} [(\int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} dxd\tau)^{\frac{2(p+2b+2)}{3(p+2b)}} \\ & +C(\int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dxd\tau)^{2/3}] \\ \le\ & \delta_1 \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dxd\tau \\ & +C(\delta_1) \sup_t \int_{B_{2\rho}} \zeta^{2\frac{b+1-\kappa}{b+1}} v^{b+1-\kappa} dx \\ & +C(\delta_1)(\int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} dxd\tau)^C \end{aligned}$$

with C independent of n and ε. Using Hölder inequality again, we see that for any $\delta_2 \in (0,1)$,

$$\begin{aligned} &\int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b+2)/2} dx d\tau \\ \leq \; & \delta_1 \int_0^{t_0} \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau \\ & + \delta_2 \sup_t \int_{B_{2\rho}} \zeta^2 v^{b+1} dx + C(\delta_1, \delta_2)[1 + (\int_0^{t_0} \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau)^C]. \end{aligned}$$

Similarly, we also have the same upper bound for I_0 in (3.3). Substitute this into (3.3) and choose δ_1 and δ_2 sufficiently small. Noting that it always true for all $t_0 \in (0,T)$, we obtain

$$\begin{aligned} & \sup_{0<\tau<T} \int_{B_{2\rho}} \zeta^2 v^{b+1}(x,\tau) dx + \int_0^T \int_{B_{2\rho}} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 dx d\tau \\ \leq \; & C + C(\int_0^T \int_{B_{2\rho}} v^{(p+2b)/2} dx d\tau)^C. \end{aligned} \tag{3.7}$$

First, take $b = 0$. Using (2.4) and (2.5), we have

$$\sup_{0<\tau<T} \int_{B_{2\rho}} (\zeta^{2/\lambda} w)^\lambda (x,\tau) dx + \int_0^T \int_{B_{2\rho}} (\zeta^{2/\lambda} |\nabla w|)^2 dx d\tau \leq C,$$

where $w = v^{(p+2b)/4}$ and $\lambda = \frac{4(b+1)}{p+2b}$. For $\zeta^{2/\lambda} w$, using the embedding inequality (3.4), we have $\int_0^T \int_{B_{2\rho}} (\zeta^{2/\lambda} w)^r dx d\tau \leq C$. It implies $\int_t^T \int_{B_\rho} w^r dx d\tau \leq C$, where $r = 2 + \frac{4}{p}$. This means $|\nabla u| \in L^{p+2}(B_\rho \times (t,T))$ and

$$\int_t^T \int_{B_\rho} |\nabla u|^{p+2} dx d\tau \leq C. \tag{3.8}$$

Next, take $b = 1$. Applying (3.8) and proceeding in the same way above, we can see that $|\nabla u| \in L^{p+6}(B_\rho \times (t,T))$ and

$$\int_t^T \int_{B_\rho} |\nabla u|^{p+6} dx d\tau \leq C.$$

For a given $l > 0$, by discussing inductively, we may find s_k (for some k) such that $p + s_k > l$, and deduce that $|\nabla u| \in L^{p+s_k}(B_\rho \times (t,T))$,

$$\int_t^T \int_{B_\rho} |\nabla u|^{p+s_k} dx d\tau \leq C.$$

Substituting this into (3.7) with some sufficiently large b, we see that

$$\sup_{t<\tau<T} \int_{B_{2\rho}} |\nabla u|^l dx \leq C.$$

Thus Proposition 3.1 is proved. □

4. Estimate for $\|\nabla u_\varepsilon\|_{L^\infty_{loc}}$

By means of the Moser iteration, from the estimate Proposition 3.1, we can further prove the following:

Proposition 4.1. *Assume $p \geq 4$. Then for any compact subset $K \subset G_T$, there exists a constant $C > 0$ (independent of n and ε), such that*

$$\|\nabla u_\varepsilon^n\|_{L^\infty(K,R^2)} \leq C = C(K).$$

Proof. Assume $R > 0$, $t_1 \in (0,T)$. Let $t_0 \in (t_1, T)$, $t_0 > 4t_1$. Define

$$Q(\rho, t_0) = B_\rho(x_0) \times (t_0, T), \quad T_m = \tfrac{t_0}{2} - \tfrac{t_0}{2^{m+2}},$$

$$\rho_m = \rho + \tfrac{\rho}{2^m}, \quad \overline{\rho}_m = \tfrac{1}{2}(\rho_m + \rho_{m+1}) = \rho + \tfrac{3\rho}{2^{m+2}},$$

$$B_m = B(x_0, \rho_m), \quad B'_m = B(x_0, \overline{\rho}_m),$$

$$Q_m = B_m \times (T_m, T), \quad Q'_m = B'_m \times (T_{m+1}, T),$$

and assume $\zeta_m(x,t) \in C_0^1(Q_m)$ satisfying

$$\zeta_m = 0, \forall t \leq T_m; \quad \zeta_m = 1, \forall (x,t) \in Q'_m$$

$$|\nabla \zeta_m| \leq \rho^{-1} 2^{m+2}, \quad 0 \leq \zeta_{mt} \leq t_0^{-1} 2^{m+3}.$$

Without loss of generality, we suppose $x_0 = 0$. Take $\rho > 0$ sufficiently small such that $B_{2\rho} \subset B_R$ and $|Q_m| \leq 1$. Multiplying (3.1) with $\zeta_m^2 v^b u_{x_j}$ $(b \geq 1)$ and integrating over $B_{2\rho} \times (t_0/4, t)$, we obtain

$$\begin{aligned}
&\frac{1}{2(b+1)} \int_{B_{2\rho}} \zeta_m^2 v^{b+1}(x,t)dx \\
&+ \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m^2 (v^b u_{x_j})_{x_i} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau \\
= &\frac{1}{b+1} \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m \zeta_{m\tau} v^{b+1} dx d\tau \\
&- 2 \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m \zeta_{mx_i} v^b u_{x_j} (v^{(p-2)/2} u_{x_i})_{x_j} dx d\tau \\
&+ \frac{1}{\varepsilon^p} \int_{t_0/4}^t \int_{B_{2\rho}} (1 - |u|^2) \zeta_m^2 v^b (u_{x_j})^2 dx d\tau \\
&- \frac{1}{2\varepsilon^p} \int_{t_0/4}^t \int_{B_{2\rho}} \zeta_m^2 v^b ((|u|^2)_{x_j})^2 dx d\tau.
\end{aligned}$$

Similar to the derivation of (3.3), we may also get

$$
\begin{aligned}
&\frac{1}{2(b+1)}\int_{B_{2\rho}} \zeta_m^2 v^{b+1}(x,t_0)dx \\
&+\int_{t_0/4}^{t}\int_{B_{2\rho}} \zeta_m^2 v^{(p+2b-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2 dxd\tau \\
&+\frac{p+2b-2}{4}\int_{t_0/4}^{t}\int_{B_{2\rho}} \zeta_m^2 v^{\frac{p+2b-4}{2}}|\nabla v|^2 dxd\tau \\
&+\frac{b(p-2)}{2}\int_{t_0/4}^{t}\int_{B_{2\rho}} \zeta_m^2 v^{\frac{p+2b-6}{2}}(\nabla u\nabla u)^2 dxd\tau \\
\leq\ & C\Bigg(\int_{t_0/4}^{t}\int_{B_{2\rho}} |\zeta_{m\tau}|v^{b+1}dxd\tau + \int_{t_0/4}^{t}\int_{B_{2\rho}} v^{(p+2b)/2}|\nabla\zeta_m|^2 dxd\tau \\
&+\frac{p+2b-2}{2}\int_{t_0/4}^{t}\int_{B_{2\rho}} \zeta_m^2\left(v^{(p+2b+2)/2}+v^{2b+2}\right)dxd\tau\Bigg)
\end{aligned}
\tag{4.1}
$$

with the constant C independent of n, ε, b and m. To estimate

$$
\int_{t_0/4}^{t}\int_{B_{2\rho}} \zeta_m^2 v^{(p+2b+2)/2}dxd\tau,
$$

we take $f=\zeta_m^{2/q}v^{(p+2b+2)/(2q)}$ in (3.4) with $h_1=1$, $h_2=2(q-1)$ and $q\in(\frac{3}{2},2)$. Then, using Hölder inequality, and noticing $|Q_m|\leq 1$, we obtain

$$
\int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2}dxd\tau \leq CI_3^{\frac{1}{2}}\left[\frac{2^m}{\rho}I_4^{\frac{p+2b+2}{q(p+2b)}}+\frac{p+2b+2}{2q}I_1^{\frac{1}{2}}I_4^{\frac{p+2b+2}{q(p+2b)}-\frac{1}{2}}\right]
\tag{4.2}
$$

(in which we replace t_0 by $t\in(T_m,T)$, since (4.2) still holds for all $t_0\in(T_m,T)$), where

$$
\begin{aligned}
&I_1=\textstyle\int_{Q_m}\zeta_m^2 v^{(p+2b-4)/2}|\nabla v|^2 dxd\tau,\\
&I_3=\sup_{T_m<t<T}\textstyle\int_{B_m}\zeta_m^{4(q-1)/q}v^{(1-1/q)(p+2b+2)}dx,\\
&I_4=\textstyle\int_{Q_m} v^{(p+2b)/2}dxd\tau.
\end{aligned}
$$

Substituting (4.2) into (4.1) and similar to the derivation of (3.7), we obtain

$$\begin{aligned} & I_1+I_2 \\ \leq & C\left\{\frac{2^m}{t_0} I_4^{\frac{2(b+1)}{p+2b}}\right. \\ & \left.+\left(\frac{2^m}{\rho}\right)^2 I_4+\frac{p+2b-2}{2} I_3^{\frac{1}{2}}\left[\frac{2^m}{\rho} I_4^{\frac{p+2b+2}{q(p+2b)}}+\frac{p+2b+2}{2q} I_1^{\frac{1}{2}} I_4^{\frac{p+2b+2}{q(p+2b)}-\frac{1}{2}}\right]\right\}, \end{aligned}$$

where $I_2=\sup_{T_m<t<T} \int_{B_m} \zeta_m^2 v^{b+1} dx$. Applying Hölder inequality and (2.4) we have

$$I_3 \leq I_2^{2(q-1)/q}(\sup_t \int_{Q_m} v^{\frac{p(q-1)}{2-q}} dx)^{2/q-1} \leq C I_2^{2(q-1)/q}.$$

Hence, by using Young's inequality, we can see that

$$\begin{aligned} I_1+I_2 \quad \leq C & \left[\tfrac{2^m}{t_0} I_4^{\frac{2(b+1)}{p+2b}}+(\tfrac{2^m}{\rho})^2 I_4+(\tfrac{p+2b-2}{2})^\lambda(\tfrac{2^m}{\rho})^q I_4^{\frac{p+2b+2}{p+2b}}\right. \\ & \left.+(\tfrac{p+2b+2}{2q})^\lambda I_4^{(\frac{p+2b+2}{q(p+2b)}-\frac{1}{2})(\frac{2q}{2-q})}\right], \end{aligned}$$

where $\lambda>0$ only depends on q. Since

$$\frac{2(b+1)}{p+2b} \leq 1 \leq \frac{p+2b+2}{p+2b} \leq (\frac{p+2b+2}{q(p+2b)}-\frac{1}{2})(\frac{2q}{2-q}),$$

from Young's inequality it follows that

$$I_1+I_2 \leq C_1 C_2^m\left(1+I_4^{1+\frac{C_3}{(p-2)/2+2^{m-1}}}\right),$$

where $C_1>0$; C_2 depends on t_0, ρ, q and λ; $2^{m-1}=b+1$ and $C_3=\frac{2}{2-q}$. Denote $w=v^{(p+2b)/4}$, $\sigma=\frac{4(b+1)}{p+2b}$. Then

$$\sup_{T_m<\tau<T} \int_{B_m}(\zeta_m^{2/\sigma} w)^\sigma(x, \tau) dx+\int_{Q_m}(\zeta_m^{2/\sigma}|\nabla w|)^2 dx d\tau \leq I_1+I_2.$$

Applying the embedding inequality (3.4) we have

$$\int_{Q_{m+1}} w^{2+\sigma} dx d\tau \leq (I_1+I_2)^2.$$

Since $b=2^{m-1}-1$, the inequality above implies

$$\begin{aligned} & \int_{Q_{m+1}} v^{(p-2)/2+2^m} dx d\tau \\ \leq & (C_1 C_2^m)^2\left(1+\int_{Q_m} v^{(p-2)/2+2^{m-1}} dx d\tau\right)^{2(1+C_3/((p-2)/2+2^{m-1}))} \end{aligned} \tag{4.3}$$

If there exists a subsequence of positive integers $\{m_i\}$ with $m_i \rightarrow \infty$, such that

$$I_4=\int_{Q_{m_i}} v^{(p-2)/2+2^{m_i}} dx d\tau<1,$$

then letting $m_i \to \infty$ yields immediately

$$\|v\|_{L^\infty(Q_\infty,R)} \leq C. \tag{4.4}$$

Otherwise, there must be a positive integer m_0 such that

$$I_4 = \int_{Q_m} v^{\frac{p-2}{2}+2^m} dxd\tau \geq 1 \quad \text{for} \quad m \geq m_0.$$

This, together with (4.3), implies

$$\int_{Q_{m+1}} v^{(p-2)/2+2^m} dxd\tau$$

$$\leq (C_1 C_2^m)^2 \left(\int_{Q_m} v^{(p-2)/2+2^{m-1}} dxd\tau \right)^{2(1+C_3/((p-2)/2+2^{m-1}))}$$

Using Proposition 3.1 and an iteration lemma (Proposition 2.3 in [7]), we can also deduce the estimation (4.4). Thus the proof of Proposition 4.1 is complete. □

5. Hölder convergence

Once the uniform estimate $\|\nabla u_\varepsilon\|_{L^\infty(\Omega_R)}$ is established, it is not difficult to prove

Proposition 5.1. *Assume $p \geq 4$. Let $\psi_\varepsilon = \frac{1}{\varepsilon^p}(1 - |u_\varepsilon|^2)$. Then for any compact subset K of G_T, there exists a constant $C > 0$ (independent of ε), such that*

$$\|\psi_\varepsilon\|_{L^\infty(K,R)} \leq C. \tag{5.1}$$

Proof. Consider the inner product of the both sides of (1.5) with $u = u_\varepsilon^n$,

$$uu_t - \operatorname{div}(v^{(p-2)/2}\nabla u)u = \frac{1}{\varepsilon^p}|u|^2(1-|u|^2) = |u|^2\psi,$$

where $\psi = \frac{1}{\varepsilon^p}(1 - |u_\varepsilon^n|^2)$. Combining this and $\nabla\psi = -\frac{2}{\varepsilon^p} u \cdot \nabla u$ with

$$\operatorname{div}(v^{(p-2)/2}\nabla u)u = \operatorname{div}(v^{(p-2)/2}u \cdot \nabla u) - v^{(p-2)/2}|\nabla u|^2$$

yields

$$|u|^2\psi = v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}\operatorname{div}(v^{(p-2)/2}\nabla\psi) - \frac{\varepsilon^p}{2}\psi_t.$$

Using (2.15) we further obtain

$$\frac{1}{4}\psi \leq v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}\operatorname{div}(v^{(p-2)/2}\nabla\psi) - \frac{\varepsilon^p}{2}\psi_t, \ \forall\varepsilon \in (0,\varepsilon_0). \tag{5.2}$$

Since at the point (x_0, t_0), where ψ achieves its maximum, $\nabla\psi = 0$, $\Delta\psi \leq 0$, $\psi_t \geq 0$ and

$$\operatorname{div}(v^{(p-2)/2}\nabla\psi) = v^{(p-2)/2}\Delta\psi + \frac{p-2}{2}v^{(p-4)/2}\nabla v\nabla\psi \leq 0$$

we derive that, from (5.2),

$$\|\psi_\varepsilon^n\|_{L^\infty(K,R)} \le C \tag{5.3}$$

by using the local estimate Proposition 4.1. Letting $n \to \infty$ and applying proposition 2.4, we can see that (5.1) holds. □

Theorem 5.2. *Assume $p \ge 4$. For any $\alpha \in (0,1)$, there holds*

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_p, \quad in \quad C_{loc}^{\alpha,\frac{\alpha}{2}}(G_T, R^2).$$

Proof. Assume $0 < \eta < t_0 < t_1 < t_2 < T$. Set $B(\Delta t) = B(x_0, (\Delta t)^{1/2})$, where $x_0 \in B_R$, $\Delta t = t_2 - t_1$. Take $\phi \in C_0^1(B(\Delta t))$, and denote u_ε^n by u. Then,

$$\begin{aligned} &\int_{B(\Delta t)} \phi(x)(u_\eta(x,t_2) - u_\eta(x,t_1))dx \\ &= \int_{B(\Delta t)} \phi(x) \int_0^1 \frac{d}{ds} u_\eta(x, st_2 + (1-s)t_1) ds dx \\ &= \Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \int_0^T \int_{R^2} j_{\eta t}(x-y, st_2 + (1-s)t_1 - \tau) \\ &\quad u(y,\tau) dy d\tau ds dx \\ &= -\Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \int_0^T \int_{R^2} j_{\eta\tau}(x-y, st_2 + (1-s)t_1 - \tau) \\ &\quad u(y,\tau) dy d\tau ds dx. \end{aligned} \tag{5.4}$$

For given $(x,t) \in G_T$ with t satisfying $0 < \eta < t_0 < t < T - \eta$, we have $J_\eta(x-y, t-\tau) \in C_0^1(G_T)$. Since u is the solution of (1.5), it must satisfy

$$\begin{aligned} &\int_0^T \int_{R^2} j_{\eta\tau}(x-y, st_2 + (1-s)t_1 - \tau) u(y,\tau) dy d\tau \\ = &\int_0^T \int_{R^2} v_y^{(p-2)/2} \nabla_y u \nabla_y j_\eta(x-y, st_2 + (1-s)t_1 - \tau) dy d\tau \\ &- \frac{1}{\varepsilon^p} \int_0^T \int_{R^2} u(y,\tau)(1 - |u(y,\tau)|^2) j_\eta(x-y, st_2 + (1-s)t_1 - \tau) dy d\tau, \end{aligned}$$

where $v_y = |\nabla_y u|^2 + \frac{1}{n}$. Substituting this into (5.4) yields

$$\begin{aligned} &\int_{B(\Delta t)} \phi(x)(u_\eta(x,t_2) - u_\eta(x,t_1))dx \\ = &-\Delta t \int_{B(\Delta t)} \phi(x) \int_0^1 \Big[\int_0^T \int_{R^2} v_y^{(p-2)/2} \nabla_y u \nabla_y j_\eta(x-y, \\ &st_2 + (1-s)t_1 - \tau) dy d\tau \\ &- \frac{1}{\varepsilon^p} \int_0^T \int_{R^2} u(y,\tau)(1 - |u(y,\tau)|^2) j_\eta(x-y, \\ &st_2 + (1-s)t_1 - \tau) dy d\tau \Big] ds dx \end{aligned}$$

$$
\begin{aligned}
(5.5)\quad &= -\Delta t \int_0^1 \int_0^T \int_{R^2} v_y^{(p-2)/2} \nabla_y u (\int_{B(\Delta t)} \nabla_x \phi j_\eta (x-y, \\
& \quad st_2 + (1-s)t_1 - \tau) dx) dy d\tau ds \\
& \quad + \Delta t \int_{B(\Delta)} \phi(x) \int_0^1 J_\eta (\tfrac{1}{\varepsilon^p} u(1-|u|^2))(x, st_2 + (1-s)t_1) ds dx \\
&= -\Delta t \int_0^1 \int_{B(\Delta t)} \nabla_x \phi (\int_0^T \int_{R^2} j_\eta (x-y, st_2 + (1-s)t_1 - \tau) \\
& \quad v_y^{(p-2)/2} \nabla_y u dy d\tau) dx ds \\
& \quad + \Delta t \int_{B(\Delta)} \phi(x) \int_0^1 J_\eta (\tfrac{1}{\varepsilon^p} u(1-|u|^2))(x, st_2 + (1-s)t_1) ds dx \\
&= -\Delta t \int_0^1 \int_{B(\Delta t)} \nabla_x \phi J_\eta (v^{(p-2)/2} \nabla u)(x, st_2 + (1-s)t_1) dx ds \\
& \quad + \Delta t \int_0^1 \int_{B(\Delta)} \phi(x) J_\eta (\tfrac{1}{\varepsilon^p} u(1-|u|^2))(x, st_2 + (1-s)t_1) dx ds.
\end{aligned}
$$

Take $\delta \in C_0^1(R, R^+)$ satisfying $\delta(s) = 0$ as $|s| \geq 1$ and $\int_R \delta(s) ds = 1$. For $h > 0$ define $\delta_h(s) = \frac{1}{h}\delta(\frac{s}{h})$. By a limit process, we see that (5.5) still holds for $\phi \in W_0^{1,1}(B(\Delta t))$. Hence, taking

$$
\phi = \phi_h(x) = \int_{-h}^{(\Delta t)^{1/2} - |x-x_0| - 2h} \delta_h(s) ds
$$

in (5.5), we have

$$
\begin{aligned}
& \int_{B(\Delta t)} \phi_h(x)(u_\eta(x,t_2) - u_\eta(x,t_1)) dx \\
= \; & \Delta t \int_0^1 \int_{B(\Delta t)} \delta_h((\Delta t)^{1/2} - |x-x_0| - 2h) \tfrac{x_i - x_{0i}}{|x-x_0|} J_\eta (v^{(p-2)/2} u_{x_i}) \\
& (x, st_2 + (1-s)t_1) dx ds \\
& + \Delta t \int_0^1 \int_{B(\Delta)} \phi_h(x) J_\eta (\tfrac{1}{\varepsilon^p} u(1-|u|^2))(x, st_2 + (1-s)t_1) dx ds.
\end{aligned}
$$

Clearly, $\lim_{h\to 0} \phi_h(x) = 1$ as $x \in B(\Delta t)$; $\delta((\Delta t)^{1/2} - |x-x_0| - 2h) = 0$ as $|x - x_0| < (\Delta t)^{1/2} - h$; $\delta_h \leq Ch^{-1}$ and $|B(\Delta) \setminus B(x_0, (\Delta)^{1/2} - h)| \leq Ch(\Delta t)^{1/2}$. Using Proposition 4.1 and (5.3), we derive

$$
|\int_{B(\Delta t)} \phi_h(x)(u_\eta(x,t_2) - u_\eta(x,t_1)) dx| \leq C(\Delta t)^{3/2}.
$$

Letting $h \to 0$ yields for all $n > 0$,

$$
|\int_{B(\Delta t)} (u_\eta(x,t_2) - u_\eta(x,t_1)) dx| \leq C(\Delta t)^{3/2}.
$$

Letting $n \to \infty$, and using Proposition 2.4, we see that

$$
|\int_{B(\Delta t)} (u_{\varepsilon\eta}(x,t_2) - u_{\varepsilon\eta}(x,t_1)) dx| \leq C(\Delta t)^{3/2}.
$$

By the mean value theorem, there exists $x^* \in B(\Delta t)$ such that

$$|u_{\varepsilon\eta}(x^*, t_2) - u_{\varepsilon\eta}(x^*, t_1))| \leq C(\Delta t)^{1/2}.$$

Applying (2.13) we have

$$\begin{aligned} & |u_{\varepsilon\eta}(x_0, t_2) - u_{\varepsilon\eta}(x_0, t_1))| \\ \leq \quad & |u_{\varepsilon\eta}(x_0, t_2) - u_{\varepsilon\eta}(x^*, t_2))| + |u_{\varepsilon\eta}(x^*, t_2) - u_{\varepsilon\eta}(x^*, t_1))| \\ & +|u_{\varepsilon\eta}(x^*, t_1) - u_{\varepsilon\eta}(x_0, t_1))| \leq C(\Delta t)^{1/2}. \end{aligned}$$

Combining this with (2.13) and letting $\eta \to 0$, we see that u_ε satisfies

$$\|u_\varepsilon\|_{C^{0+1,\frac{1}{2}}(B_R \times (t_0,T),R^2)} \leq C.$$

There exists a subsequence u_{ε_k} such that as $k \to \infty$,

$$u_{\varepsilon_k} \to u^*, \quad \text{in} \quad C^{\alpha,\frac{\alpha}{2}}(B_R \times (t_0, T), R^2),$$

where $\alpha \in (0, 1)$. From Proposition 2.5, it follows that $u^* = u_p$. In view of the uniqueness of u_p, the convergence is true not only for subsequence u_{ε_k}, but also for all u_ε. □

Theorem 5.3. *Assume $p \geq 4$. For some $\alpha \in (0,1)$, there holds*

$$\lim_{\varepsilon \to 0} \nabla u_\varepsilon = \nabla u_p, \quad in \quad C^{\alpha,\frac{\alpha}{2}}_{loc}(G_T, R^2). \tag{5.6}$$

Proof. Now, according to Proposition 5.1, the right hand side of (1.1) is bounded on the compact subset K of G_T uniformly in $\varepsilon \in (0, \varepsilon_0)$. Thus, applying Theorem 1.1 in [4] yields that, for some $\beta \in (0,1)$, we have

$$\|\nabla u_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}(K,R^2)} \leq C, \tag{5.7}$$

where the constant does not depend on $\varepsilon \in (0, \varepsilon_0)$. From this it follows that, there exist a function u_* and a subsequence $u_{\varepsilon_k}(\varepsilon_k \to 0$, as $k \to \infty)$ of u_ε, such that

$$\lim_{k \to \infty} \nabla u_{\varepsilon_k} = u_* \quad \text{in } C^{\alpha,\frac{\alpha}{2}}(K, R^2), \quad \alpha \in (0, \beta). \tag{5.8}$$

Combining (5.7) with Theorem 5.2, we see that $u_* = \nabla u_p$. By the fact that, any subsequence of u_ε contains a subsequence converging in $C^{\alpha,\frac{\alpha}{2}}(K, R^2)$ sense, and the limit is the same function u_p, we may assert that (5.8) is also true for all u_ε. Theorem is proved. □

Remark 5.4. It is only obtained that the weak solution u_ε of (1.1)-(1.3) converges to u_p locally, since I have not found any regularity (near the boundary) of the weak solution to the initial-boundary value problem for the p-Laplace system, except the results of that in [2]. However, the results in [2] is not sufficient to deduce the convergence (5.6) near the boundary. This is an open problem.

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Institute of Mathematics
School of Mathematics and Computer Sciences
Nanjing Normal University
Nanjing 210097, P. R. China
E-mail address: lythxl@163.com