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DISCRETE CONDITIONS FOR THE HOLONOMY GROUP OF A PAIR OF PANTS

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ABSTRACT. A pair of pants $\Sigma(0,3)$ is a building block of oriented surfaces. The purpose of this paper is to determine the discrete conditions for the holonomy group π of hyperbolic structure of a pair of pants. For this goal, we classify the relations between the locations of principal lines and entries of hyperbolic matrices in $\mathbf{PSL}(2,\mathbb{R})$. In the level of the matrix group $\mathbf{SL}(2,\mathbb{R})$, we will show that the signs of traces of hyperbolic elements play a very important role to determine the discreteness of holonomy group of a pair of pants.

1. Introduction

Let M be a compact connected smooth surface with $\chi(M) < 0$ and $\pi = \pi_1(M)$ the fundamental group of M. A *hyperbolic* structure on M is a representation of M as a quotient Ω/Γ of a convex domain $\Omega \subset \mathbb{H}^2$ by a discrete group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ acting properly and freely. For a given hyperbolic structure on M, the action of π on the universal covering space $\tilde{M}$ of M produces a homomorphism $h : \pi \to \mathbf{PSL}(2,\mathbb{R})$ called the *holonomy homomorphism* and it is well-defined up to conjugation in $\mathbf{PSL}(2,\mathbb{R})$. Since the holonomy homomorphism $h : \pi \to \mathbf{PSL}(2,\mathbb{R})$ is isomorphic to its image $\Gamma = h(\pi)$ called the *holonomy group*, the generators of π can be presented by the matrices in $\mathbf{PSL}(2,\mathbb{R})$ up to conjugation. (For more detail, see Thurston [6].) Therefore giving a hyperbolic structure on M is equivalent to finding a discrete subgroup Γ of $\mathbf{PSL}(2,\mathbb{R})$ up to conjugation. (For more detail, see Matsuzaki and Taniguchi [5].)

Let $M = \Sigma(g,n)$ be a compact connected oriented surface with g-genus and n-boundary components. If $\chi(M) = 2-2g-n < 0$, then there exist $3g-3+n$ nontrivial homotopically-distinct disjoint simply-closed curves on M such that they decompose M as the disjoint union of $2g-2+n$ pairs of pants $\Sigma(0,3)$.

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Thus a pair of pants $\Sigma(0,3)$ is a building block of an oriented surface M. (For more detail, see Wolpert [7].)

Thus the purpose of this paper is to determine the discrete conditions for the holonomy group $\Gamma = h(\pi)$ of a pair of pants $\Sigma(0,3)$. In Section 2, we recall some preliminary definitions and classify the locations of fixed points and principal lines of hyperbolic elements. In Section 3, we calculate the discrete conditions for the holonomy group π of a pair of pants $\Sigma(0,3)$ in terms of $\mathbf{SL}(2,\mathbb{R})$. In Section 4, we present a computer algorithm for deciding the discreteness of a holonomy group.

2. Hyperbolic elements and the locations of principal lines

Suppose G is a Lie group and X is a connected smooth manifold. An action of G on X is called *strongly effective* if $g_1, g_2 \in G$ agree on a nonempty open set of X, then $g_1 = g_2$. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane. The Lie group $\mathbf{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}\,. \tag{2.1}$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2,\mathbb{R}) = \mathbf{SL}(2,\mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$.

The elements of $\mathbf{SL}(2,\mathbb{R})$ are classified into three different types. Since $f(\lambda) = \lambda^2 - t\lambda + 1$ is the characteristic polynomial of $A \in \mathbf{SL}(2,\mathbb{R})$ where $t = \mathrm{tr}(A)$, we classify elements of $\mathbf{SL}(2,\mathbb{R})$ by their two eigenvalues ;

- hyperbolic $\Leftrightarrow$ eigenvalues are distinct real numbers $\Leftrightarrow$ $\mathrm{tr}(A)^2 > 4$
- parabolic $\Leftrightarrow$ eigenvalues are equal $\Leftrightarrow$ $\mathrm{tr}(A)^2 = 4$
- elliptic $\Leftrightarrow$ eigenvalues are conjugate $\Leftrightarrow$ $\mathrm{tr}(A)^2 < 4$.

Let A be an element of $\mathbf{PSL}(2,\mathbb{R})$. Though the trace and eigenvalues of A are not defined, the absolute value of trace and eigenvalues are well-defined. We classify the elements of $\mathbf{PSL}(2,\mathbb{R})$ by their fixed points in the closure of $\mathbb{H}^2$. The number and locations of fixed points of A closely relate to the absolute value of $\mathrm{tr}(A)$. For $A \in \mathbf{PSL}(2,\mathbb{R})$,

- hyperbolic $\Leftrightarrow$ two distinct fixed points on $\partial\mathbb{H}^2$ $\Leftrightarrow$ $|\mathrm{tr}(A)| > 2$
- parabolic $\Leftrightarrow$ unique fixed point on $\partial\mathbb{H}^2$ $\Leftrightarrow$ $|\mathrm{tr}(A)| = 2$
- elliptic $\Leftrightarrow$ unique fixed point on $\mathbb{H}^2$ $\Leftrightarrow$ $|\mathrm{tr}(A)| < 2$.

A hyperbolic element A of $\mathbf{PSL}(2,\mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \overset{\text{let}}{=} \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{2.2}$$

via an $\mathbf{SL}(2,\mathbb{R})$-conjugation where $\lambda > 1$. The following theorem is due to Kuiper [4].

Theorem 2.1. *Suppose M is a compact connected oriented hyperbolic surface. Then every nontrivial element of holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ is hyperbolic.*

The goal of this paper is to determine the discrete conditions for the holonomy group $\Gamma = h(\pi)$ of a hyperbolic structure of a pair of pants $\Sigma(0,3)$. Since every nontrivial element of holonomy group Γ is hyperbolic, we shall consider some properties of hyperbolic matrix. The *principal line* of a hyperbolic element $A \in \mathbf{SL}(2,\mathbb{R})$ or $\mathbf{PSL}(2,\mathbb{R})$ is the A-invariant unique geodesic in $\mathbb{H}^2$. It is the line joining two fixed points of A. Since the principal line has a distinct direction, one of the fixed points of A is called the *repelling* fixed point and the other is called the *attracting* fixed point. For more easy understanding, see Figure 1, 2, and 3 or Beardon's book [1]. We now consider the relations between the location of the principal line of A and the entries of A. They are some extended results compared with author's paper [3].

Lemma 2.2. *Let z_a, z_r be the attracting and repelling fixed points of a hyperbolic element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{PSL}(2,\mathbb{R})$. Suppose both fixed points are finite (not infinite). Then we have*

(1) $z_a + z_r > 0 \iff (a-d)\,c > 0,$
(2) $z_a + z_r < 0 \iff (a-d)\,c < 0,$
(3) $z_a \cdot z_r > 0 \iff b\,c < 0,$
(4) $z_a \cdot z_r < 0 \iff b\,c > 0,$
(5) $z_a < z_r \iff (a+d)\,c < 0,$
(6) $z_a > z_r \iff (a+d)\,c > 0.$

Proof. Since z_a, z_r are the fixed points of the hyperbolic transformation $A(z) = \frac{az+b}{cz+d}$, they are the roots of the equation

$$c\,z^2 + (d-a)\,z - b = 0. \tag{2.3}$$

We claim that $c \neq 0$. If $c = 0$, then $1 = \det(A) = ad$. Thus $d = a^{-1}$ and $A(z) = a^2 z + ab$. This yields that ∞ is a fixed point of $A(z)$ since $a \neq 0$. It contradicts the assumption that both fixed points are finite. Since $z_a + z_r = \frac{a-d}{c}$ and $z_a \cdot z_r = \frac{-b}{c}$, it proves 1, 2, 3, and 4.

Since we have $c \neq 0$, the roots z_a, z_r of the Equation (2.3) can be expressed by

$$z_a,\ z_r = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2\,c}. \tag{2.4}$$

Let $w = \frac{z_a + z_r}{2} = \frac{a-d}{2c}$ be the mid point of the fixed points z_a and z_r. Then the condition $z_a < z_r$ is equivalent to $A(w) < w$. (See Figure 1.) Since we can compute

$$A(w) - w = \frac{(a+d)^2 - 4}{2(a+d)c} \tag{2.5}$$

and $(a+d)^2 > 4$, it proves $z_a < z_r$ if and only $(a+d)\,c < 0$. Similarly $z_a > z_r$ if and only $(a+d)\,c > 0$. This completes the proof. □

Proposition 2.3. *Let $A \in \mathbf{PSL}(2,\mathbb{R})$ representing a hyperbolic transformation of $\mathbb{H}^2$ with the finite fixed points z_r, z_a. Then we have the following relations.*

(1) $0 < z_a < z_r \iff a^2 < d^2,\, b\,c < 0,\, b\,d > 0$
(2) $0 < z_r < z_a \iff a^2 > d^2,\, b\,c < 0,\, a\,c > 0$
(3) $z_a < z_r < 0 \iff a^2 > d^2,\, b\,c < 0,\, a\,c < 0$
(4) $z_r < z_a < 0 \iff a^2 < d^2,\, b\,c < 0,\, b\,d < 0$
(5) $z_a < 0 < z_r \iff b\,c > 0\ ,\, a\,c < 0,\, b\,d < 0$
(6) $z_r < 0 < z_a \iff b\,c > 0\ ,\, a\,c > 0,\, b\,d > 0$

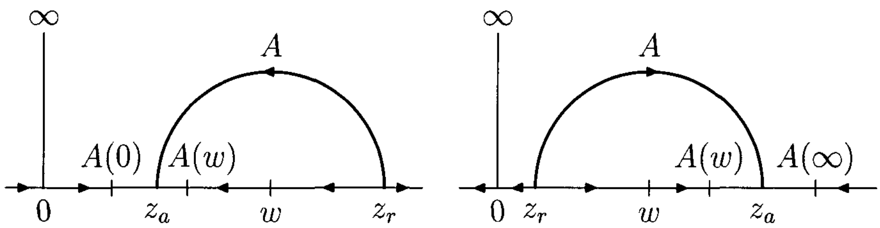

FIGURE 1. Fixed points with $0 < z_a < z_r$ and $0 < z_r < z_a$

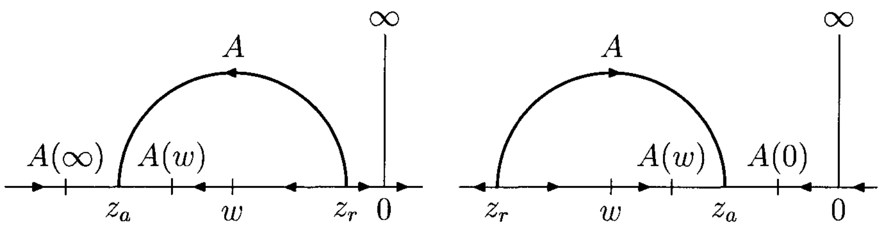

FIGURE 2. Fixed points with $z_a < z_r < 0$ and $z_r < z_a < 0$

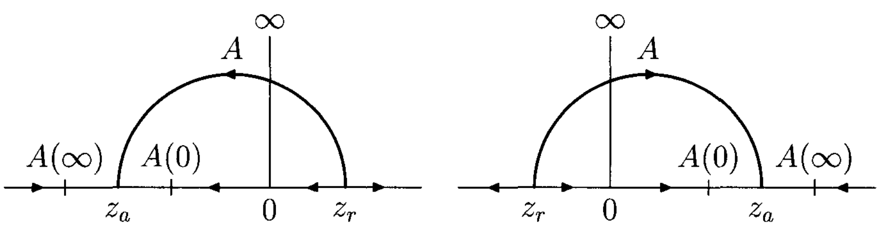

FIGURE 3. Fixed points with $z_a < 0 < z_r$ and $z_r < 0 < z_a$

Proof. Suppose $0 < z_a < z_r$. From Lemma 2.2, it is equivalent to the relations $(a-d)\,c > 0$, $b\,c < 0$ and $(a+d)\,c < 0$. Thus $(a-d)(a+d)c^2 = (a^2-d^2)c^2 < 0$ implies $a^2 < d^2$. Since $z_a < z_r$, the image of the origin under A should be positive as in the Figure 1. That means $A(0) = b/d > 0$. Thus we have $b\,d > 0$. Conversely suppose $a^2 < d^2$, $b\,c < 0$, $b\,d > 0$. Then the signs of c and d should be opposite. Since the condition $a^2 < d^2$ is equivalent to $-|d| < a < |d|$, if $c > 0$ and $d < 0$, then $a^2 < d^2$ implies $d < a < -d$. Thus $(a-d)\,c > 0$ and $(a+d)\,c < 0$. If $c < 0$ and $d > 0$, then $a^2 < d^2$ implies $-d < a < d$. Thus we have $(a-d)\,c > 0$ and $(a+d)\,c < 0$ again. It proves the case 1. Similarly we can prove the cases 2, 3, and 4.

Suppose $z_a < 0 < z_r$. From Lemma 2.2, it is equivalent to the relations $b\,c > 0$ and $(a+d)\,c < 0$. Since $z_a < 0 < z_r$, the image of the origin and infinite under A should be negative as in the Figure 3. That means $A(0) = b/d < 0$ and $A(\infty) = a/c < 0$. Thus we have $b\,d < 0$ and $a\,c < 0$. Conversely suppose $b\,c > 0$, $a\,c < 0$, $b\,d < 0$. Since $a\,d = b\,c + 1$, the condition $b\,c > 0$ implies $a\,d > 1$. If $c > 0$, then we have the signs $a < 0, b > 0$ and $d < 0$. Therefore $(a+d)\,c < 0$. If $c < 0$, then we have the signs $a > 0, b < 0$ and $d > 0$. Therefore $(a+d)\,c < 0$ again. It proves the case 5. Similarly we can prove the case 6. □

3. Holonomy group of a pair of pants $\Sigma(0,3)$

Recall that a pair of pants $M = \Sigma(0,3)$ is a sphere with three holes. Suppose M is equipped with a hyperbolic structure. Since the holonomy homomorphism $h : \pi \to \mathbf{PSL}(2,\mathbb{R})$ is isomorphic to its image $\Gamma = h(\pi)$, the fundamental group π of M will be identified with

$$\pi = \langle\, A, B, C \in \mathbf{PSL}(2,\mathbb{R}) \mid R = CBA = I \,\rangle.$$

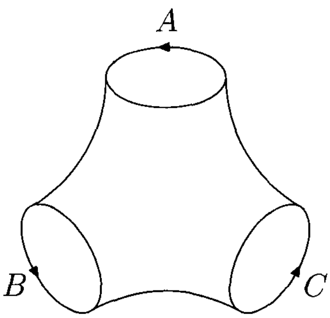

FIGURE 4. A pair of pants $M = \Sigma(0,3)$

The following is a more effectively modified method for representing generators of holonomy group of a pair of pants compared with author's paper [3]. Let $A, B, C \in \mathbf{PSL}(2,\mathbb{R})$ represent the boundary components of M. We will find the expression of the generators A, B and C of π in terms of $\mathbf{SL}(2,\mathbb{R})$

instead of $\mathbf{PSL}(2,\mathbb{R})$ because $\mathbf{SL}(2,\mathbb{R})$ is more convenient to compute and understand than $\mathbf{PSL}(2,\mathbb{R})$. Since the matrices $A, B, C \in \mathbf{SL}(2,\mathbb{R})$ are hyperbolic and represented up to conjugate, without loss of generality, we can assume

$$B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \tag{3.1}$$

with $\mu > 1$. Thus

$$B(z) = \frac{\mu \cdot z + 0}{0 \cdot z + \mu^{-1}} = \mu^2 z$$

and 0 is the repelling fixed point and ∞ is the attracting fixed point of B. Suppose 0 and ∞ are not a fixed point of a hyperbolic element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have the conditions $b \neq 0$ and $c \neq 0$. Because if $b = 0$, then $A(0) = \frac{b}{d} = 0$ and if $c = 0$, then $A(\infty) = \frac{a}{c} = \infty$. Suppose $\operatorname{tr}(A) = \lambda + \lambda^{-1}$. Without loss of generality, we can assume $\lambda > 1$. If $\operatorname{tr}(A) < -2$, then we exchange A to $-A$. Since $a + d = \operatorname{tr}(A) = \lambda + \lambda^{-1}$, we have $d = -a + \lambda + \lambda^{-1}$. Since $\det(A) = ad - bc = 1$, we obtain

$$bc = ad - 1 = a(-a + \lambda + \lambda^{-1}) - 1 = -(a - \lambda)(a - \lambda^{-1}).$$

Thus we have $b = -(a - \lambda)(a - \lambda^{-1})c^{-1}$ since $c \neq 0$. Therefore

$$A = \begin{pmatrix} a & -(a-\lambda)(a-\lambda^{-1})c^{-1} \\ c & -a + \lambda + \lambda^{-1} \end{pmatrix} \quad \text{with } \lambda > 1. \tag{3.2}$$

Proposition 3.1. *Let z_r, z_a be the repelling and attracting fixed points of the hyperbolic element A in (3.2). Then*

(1) $0 < z_a < z_r \iff c < 0,\ a < \lambda^{-1}$
(2) $0 < z_r < z_a \iff c > 0,\ a > \lambda$
(3) $z_a < z_r < 0 \iff c < 0,\ a > \lambda$
(4) $z_r < z_a < 0 \iff c > 0,\ a < \lambda^{-1}$
(5) $z_a < 0 < z_r \iff c < 0,\ \lambda^{-1} < a < \lambda$
(6) $z_r < 0 < z_a \iff c > 0,\ \lambda^{-1} < a < \lambda$.

Proof. We will prove the cases 1 and 5. The other cases are can be proven similarly. Suppose $0 < z_a < z_r$. From Proposition 2.3, it is equivalent to the relations $a^2 < d^2$, $bc < 0$ and $bd > 0$. Since $b = -(a - \lambda)(a - \lambda^{-1})c^{-1}$ and $d = -a + \lambda + \lambda^{-1}$, we have

$$\begin{aligned} d^2 - a^2 &= (\lambda + \lambda^{-1})(\lambda + \lambda^{-1} - 2a) > 0 \\ -bc &= (a - \lambda)(a - \lambda^{-1}) > 0 \\ -bd &= (a - \lambda)(a - \lambda^{-1})c^{-1}(-a + \lambda + \lambda^{-1}) < 0. \end{aligned}$$

The condition $\lambda > 1$ implies $(\lambda + \lambda^{-1} - 2a) > 0$. We claim that $-bc > 0$ induces $(a - \lambda) < 0$ and $(a - \lambda^{-1}) < 0$. If $(a - \lambda) > 0$ and $(a - \lambda^{-1}) > 0$, then it produces $(2a - \lambda - \lambda^{-1}) > 0$. It contradicts to $(\lambda + \lambda^{-1} - 2a) > 0$. Since

$(-a+\lambda+\lambda^{-1}) > (-a+\lambda) > 0$, the condition $-b\,d < 0$ induces $c < 0$. Therefore we get the results $c < 0$ and $a < \lambda^{-1}$. Conversely suppose we have $c < 0$ and $a < \lambda^{-1}$. Then $\lambda^{-1} < 1 < \lambda$ induces $a < \lambda$. Through easy computations, we can show $a^2 < d^2$, $b\,c < 0$ and $b\,d > 0$. It proves the case 1.

Suppose $z_a < 0 < z_r$. From Proposition 2.3, it is equivalent to the relations $b\,c > 0$, $a\,c < 0$ and $b\,d < 0$. We claim that $-b\,c < 0$ induces $(a-\lambda) < 0$ and $(a-\lambda^{-1}) > 0$. If $(a-\lambda) > 0$ and $(a-\lambda^{-1}) < 0$, then we have $\lambda < a < \lambda^{-1}$. It contradicts to $\lambda > 1$. The condition $\lambda^{-1} < a$ induces $a > 0$. Thus c should be negative. Therefore we get the results $c < 0$ and $\lambda^{-1} < a < \lambda$. Through easy computations we can show the converse is also true. □

Now we consider the positions of fixed points of the matrix A.

Proposition 3.2. *Let A be the hyperbolic element in (3.2). Then the positions of repelling and attracting fixed points z_r, z_a of A are*

(1) $0 < z_a < z_r \implies z_a = \frac{(\lambda^{-1}-a)}{(-c)}, \quad z_r = \frac{(\lambda-a)}{(-c)}$

(2) $0 < z_r < z_a \implies z_r = \frac{(a-\lambda)}{c}, \quad z_a = \frac{(a-\lambda^{-1})}{c}$

(3) $z_a < z_r < 0 \implies z_a = \frac{-(a-\lambda^{-1})}{(-c)}, \quad z_r = \frac{-(a-\lambda)}{(-c)}$

(4) $z_r < z_a < 0 \implies z_r = \frac{-(\lambda-a)}{c}, \quad z_a = \frac{-(\lambda^{-1}-a)}{c}$

(5) $z_a < 0 < z_r \implies z_a = \frac{-(a-\lambda^{-1})}{(-c)}, \quad z_r = \frac{(\lambda-a)}{(-c)}$

(6) $z_r < 0 < z_a \implies z_r = \frac{-(\lambda-a)}{c}, \quad z_a = \frac{(a-\lambda^{-1})}{c},$

where the inside of each parentheses ( ) is positive.

Proof. By the Equation (2.4),

$$\begin{aligned} z_a,\, z_r &= \frac{(2a-\lambda-\lambda^{-1}) \pm \sqrt{(\lambda+\lambda^{-1})^2-4}}{2\,c} \\ &= \frac{(2a-\lambda-\lambda^{-1}) \pm (\lambda-\lambda^{-1})}{2\,c} \\ &= \frac{a-\lambda^{-1}}{c} \text{ or } \frac{a-\lambda}{c}. \end{aligned}$$

(Case 1.) Suppose $0 < z_a < z_r$. From Proposition 3.1, we have $c < 0$, $a < \lambda^{-1}$. Since $0 < (\lambda^{-1}-a) < (\lambda-a)$, the attracting fixed point z_a of A is $(\lambda^{-1}-a)/(-c)$ and the repelling fixed point z_r of A is $(\lambda-a)/(-c)$.

(Case 2.) Suppose $0 < z_r < z_a$, then we have $c > 0$, $a > \lambda$. Since $(a-\lambda^{-1}) > (a-\lambda) > 0$, the repelling fixed point z_r of A is $(a-\lambda)/c$ and the attracting fixed point z_a of A is $(a-\lambda^{-1})/c$.

(Case 5.) Suppose $z_a < 0 < z_r$, then we have $c < 0$, $\lambda^{-1} < a < \lambda$. Since $(\lambda^{-1}-a) < 0 < (\lambda-a)$, the attracting fixed point z_a of A is $(\lambda^{-1}-a)/(-c)$ and the repelling fixed point z_r of A is $(\lambda-a)/(-c)$. Similarly we can prove the cases 3, 4 and 6. □

Since $R = CBA = I$, we can get $C = A^{-1}B^{-1}$. Therefore the generators A, B and C of π are expressed by

$$A = \begin{pmatrix} a & -(a-\lambda)(a-\lambda^{-1})c^{-1} \\ c & -a+\lambda+\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{3.3}$$

and

$$C = \begin{pmatrix} \mu^{-1}(-a+\lambda+\lambda^{-1}) & \mu(a-\lambda)(a-\lambda^{-1})c^{-1} \\ -\mu^{-1}c & \mu\, a \end{pmatrix} \tag{3.4}$$

with $\lambda > 1$ and $\mu > 1$. As a result, the trace of C is

$$\mathrm{tr}(C) = \mu^{-1}(-a+\lambda+\lambda^{-1}) + \mu\, a. \tag{3.5}$$

Let A_r, B_r, A_a, B_a be the repelling and attracting fixed points of the hyperbolic matrices A and B respectively. We define $\mathsf{CR}(A,B)$ by the cross ratio of B_a, A_r, A_a, B_r ; that is

$$\mathsf{CR}(A,B) = [B_a, A_r, A_a, B_r] = \frac{(B_a - A_a)(A_r - B_r)}{(B_a - A_r)(A_a - B_r)}.$$

Then $\mathsf{CR}(A,B) = \mathsf{CR}(B,A)$ and it represents relations between fixed points of A and B. Suppose f is a linear fractional transformation such that $f(B_a) = \infty$ and $f(B_r) = 0$. Since the cross ratio is invariant under the linear fractional transformations,

$$\mathsf{CR}(A,B) = \frac{(\infty - z_a)(z_r - 0)}{(\infty - z_r)(z_a - 0)} = \frac{z_r}{z_a},$$

where $z_a = f(A_a)$ and $z_r = f(A_r)$.

Proposition 3.3. *Suppose A, B are hyperbolic elements in $\mathbf{SL}(2,\mathbb{R})$. Then we have the following relations between $\mathsf{CR}(A,B)$ and the locations of fixed points.*

(1) $\mathsf{CR}(A,B) > 1 \iff 0 < z_a < z_r$ *or* $z_r < z_a < 0$
(2) $0 < \mathsf{CR}(A,B) < 1 \iff z_a < z_r < 0$ *or* $0 < z_r < z_a$
(3) $\mathsf{CR}(A,B) < 0 \iff z_r < 0 < z_a$ *or* $z_a < 0 < z_r$.

Proof. If $\mathsf{CR}(A,B) = z_r/z_a > 1$, then both fixed point have the same signs. Thus if $z_a > 0$ then $z_r > z_a > 0$, and if $z_a < 0$ then $z_r < z_a < 0$. Easily we can see that the converse is also true. Other cases can be similarly proved. □

Suppose A, B, C are hyperbolic elements. Then the holonomy group $\pi = \langle A, B, C \mid R = CBA = I \rangle$ is discrete if and only if the locations of the principal lines of A, B, C should be one of the Figure 5 up to conjugation. (For more detail, see Keen [2])

Theorem 3.4. *Suppose that A, B are hyperbolic matrices in $\mathbf{SL}(2,\mathbb{R})$ and $\pi = \langle A, B, C \mid R = CBA = I \rangle$. Then*

(1) *if π is discrete, then* $\mathsf{CR}(A,B) > 1$.
(2) *if* $\mathsf{CR}(A,B) < 1$ *or* $\mathsf{CR}(A,B) = \infty$, *then π is not discrete.*

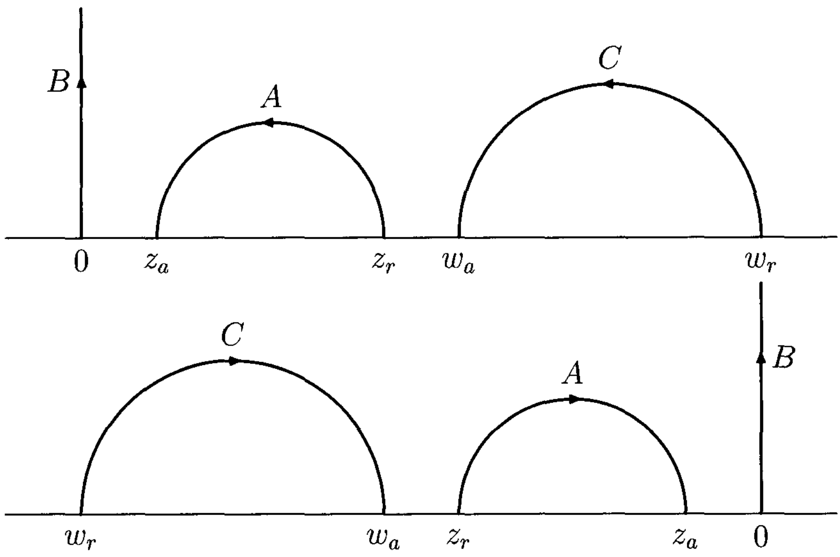

FIGURE 5. The principal lines of a discrete group π

Proof. If π is discrete, then the locations of principal lines of A and B should one of Figure 5. Thus $\mathsf{CR}(A, B) = z_r/z_a > 1$. Since A is hyperbolic, we have $z_r \neq z_a$. This implies $\mathsf{CR}(A, B) = z_r/z_a$ can not be 1. If $\mathsf{CR}(A, B) = 0$ or ∞, then the hyperbolic matrices A and B have common fixed point. Thus π is not discrete. If $0 < \mathsf{CR}(A, B) < 1$ or $\mathsf{CR}(A, B) < 0$, then from the above mentioned Keen's equivalent relation of discreteness, π is not discrete. □

Proposition 3.5. *Suppose that C is the hyperbolic matrix in* (3.4). *Then the fixed points w_a and w_r of C are*

$$w_a,\ w_r = \frac{E \pm \sqrt{D}}{-2\,c},$$

where $E = (\lambda + \lambda^{-1} - a - \mu^2 a)$ *and* $D = \left(\lambda + \lambda^{-1} - a + \mu^2 a\right)^2 - 4\mu^2$.

Proof. By the Equation (2.4), the fixed points w_a, w_r of C is

$$\frac{[\mu^{-1}(-a + \lambda + \lambda^{-1}) - \mu\, a] \pm \sqrt{[\mu^{-1}(-a + \lambda + \lambda^{-1}) + \mu\, a]^2 - 4}}{-2\,\mu^{-1} c}$$
$$= \frac{[(-a + \lambda + \lambda^{-1}) - \mu^2 a] \pm \sqrt{[(-a + \lambda + \lambda^{-1}) + \mu^2 a]^2 - 4\mu^2}}{-2\,c}.$$

□

Theorem 3.6 (Main Theorem). *Let A, B be hyperbolic matrices in* $\mathbf{SL}(2, \mathbb{R})$ *with* $\mathrm{tr}(A) > 2$ *and* $\mathrm{tr}(B) > 2$ *and* $\pi = \langle A, B, C \mid CBA = I \rangle$. *Then π is discrete if and only if* $\mathsf{CR}(A, B) > 1$ *and* $\mathrm{tr}(C) < -2$.

Proof. ($\Rightarrow$) Without loss of generality we may assume A, B, C are the matrices in (3.3) and (3.4). Suppose π is discrete. Then $\mathsf{CR}(A, B) > 1$ by Theorem 3.4. If $0 < z_a < z_r < w_a < w_r$, then we have $c < 0$ and $a < \lambda^{-1} < 1 < \lambda$ by Proposition 3.1. Let C_{ij} stand for the (i, j)-th entry of the matrix C. Then the condition

$$C_{12}C_{22} = \mu^2 a(\lambda - a)(\lambda^{-1} - a)c^{-1} > 0$$

implies $a < 0$. If $w_r < w_a < z_r < z_a < 0$, then we have $c > 0$ and $a < \lambda^{-1}$. Then the condition $C_{12}C_{22} < 0$ also implies $a < 0$. Thus both cases we have $a < 0$. The condition $\mathrm{tr}(B) > 2$ implies $\mu > 1$. Since

$$|C_{11}| = |\mu^{-1}|\,| - a + \lambda + \lambda^{-1}| < |C_{22}| = |\mu|\,|a|,$$

we have $\mu^{-1}(-a + \lambda + \lambda^{-1}) < \mu(-a)$. Thus

$$\mathrm{tr}(C) = \mu^{-1}(-a + \lambda + \lambda^{-1}) + \mu\, a < 0.$$

Since C is hyperbolic, the trace of C should be less than -2.

($\Leftarrow$) Since $\mathsf{CR}(A, B) > 1$, the fixed points of A are $0 < z_a < z_r$ or $z_r < z_a < 0$ by Proposition 3.3. If $0 < z_a < z_r$, then we have $c < 0$ and $a < \lambda^{-1}$. The condition $\mathrm{tr}(C) = \mu^{-1}(-a + \lambda + \lambda^{-1}) + \mu\, a < -2$ and $\mu > 1$ implies $a < 0$. Therefore we have the followings ;

$$\begin{aligned} C_{12}C_{21} &= -(\lambda - a)(\lambda^{-1} - a) < 0 \\ C_{12}C_{22} &= \mu^2 a(\lambda - a)(\lambda^{-1} - a)c^{-1} > 0 \\ |C_{11}| - |C_{22}| &= \mu^{-1}(-a + \lambda + \lambda^{-1}) - \mu(-a) = \mathrm{tr}(C) < -2. \end{aligned}$$

From Proposition 2.3, we get $0 < w_a < w_r$. Since c is negative, from Proposition 3.5, the fixed point w_a and w_r of C should be

$$w_a = \frac{E - \sqrt{D}}{-2\,c} \quad \text{and} \quad w_r = \frac{E + \sqrt{D}}{-2\,c}.$$

To show π is discrete, it is enough to show that $z_r < w_a$, i.e.,

$$z_r = \frac{(\lambda - a)}{(-c)} < w_a = \frac{E - \sqrt{D}}{(-2\,c)}$$

that is equivalent to $2(\lambda - a) < E - \sqrt{D}$ since $c < 0$. Since

$$\sqrt{D} < E - 2(\lambda - a) = -\lambda + \lambda^{-1} + a - \mu^2 a, \tag{3.6}$$

and $-\lambda + \lambda^{-1} + a - \mu^2 a > -\lambda - \lambda^{-1} + a - \mu^2 a = -\mathrm{tr}(C)\mu > 0$, it is equivalent to show that

$$D = \left(\lambda + \lambda^{-1} - a + \mu^2 a\right)^2 - 4\mu^2 < \left(-\lambda + \lambda^{-1} + a - \mu^2 a\right)^2.$$

After some calculations we can get the equivalent condition

$$0 < (\mu^2 - 1)(\lambda - a).$$

This is true since $\mu > 1$ and $\lambda > a$, it proves the discreteness of π.

If $z_r < z_a < 0$, then we have $c > 0$ and $a < \lambda^{-1}$. The condition $\text{tr}(C) = \mu^{-1}(-a+\lambda+\lambda^{-1}) + \mu a < -2$ and $\mu > 1$ implies $a < 0$. Therefore we have the followings ;

$$\begin{aligned} C_{12}C_{21} &= -(\lambda - a)(\lambda - a) < 0 \\ C_{12}C_{22} &= \mu^2 a(\lambda - a)(\lambda^{-1} - a)c^{-1} < 0 \\ |C_{11}| - |C_{22}| &= \mu^{-1}(-a+\lambda+\lambda^{-1}) - \mu(-a) = \text{tr}(C) < -2. \end{aligned}$$

From Proposition 2.3, we get $w_r < w_a < 0$. Since c is positive, from Proposition 3.5, the fixed point w_r and w_a of C should be

$$w_r = \frac{E+\sqrt{D}}{-2\,c} = \frac{-E-\sqrt{D}}{2\,c} \quad \text{and} \quad w_a = \frac{E-\sqrt{D}}{-2\,c} = \frac{-E+\sqrt{D}}{2\,c}.$$

To show π is discrete, it is enough to show that $w_a < z_r$, i.e.,

$$w_a = \frac{-E+\sqrt{D}}{(2\,c)} < z_r = \frac{-(\lambda - a)}{c}$$

that is equivalent to $-E+\sqrt{D} < -2(\lambda - a)$ since $c > 0$. Then we have

$$\sqrt{D} < E - 2(\lambda - a) = -\lambda + \lambda^{-1} + a - \mu^2 a.$$

It is the same inequality in (3.6), thus it holds since $\mu > 1$ and $\lambda > a$, it proves the main theorem. □

4. Application : algorithm for deciding the discreteness

Finally we can present an algorithm for deciding the discreteness of a holonomy group of a pair of pants $\Sigma(0,3)$. For given two hyperbolic elements A, B in $\mathbf{SL}(2,\mathbb{R})$,

Step 1: Compute $\text{tr}(A)$ and $\text{tr}(B)$. If $\text{tr}(A) < -2$, then replace A by $-A$. Similarly if $\text{tr}(B) < -2$, then replace B by $-B$.

Step 2: By step 1, without loss of generality, we may assume that $\text{tr}(A) > 2$ and $\text{tr}(B) > 2$. Compute the attracting and repelling fixed points A_a, A_r of A and B_a, B_r of B.

Step 3: Compute $\mathsf{CR}(A,B) = [B_a, A_r, A_a, B_r]$. If $\mathsf{CR}(A,B) > 1$, then go to step 4. If $0 < \mathsf{CR}(A,B) < 1$, replace A by A^{-1}. Then we have $\mathsf{CR}(A,B) > 1$. If $\mathsf{CR}(A,B) \leq 0$ or $\mathsf{CR}(A,B) = 1$ or $\mathsf{CR}(A,B) = \infty$, then hyperbolic elements A, B can not generate a discrete holonomy group.

Step 4: Compute $C = A^{-1}B^{-1}$. If $\text{tr}(C) < -2$, then

$$\pi = \langle\ A, B, C \in \mathbf{SL}(2,\mathbb{R}) \mid R = CBA = I\ \rangle$$

is a discrete group. If $\text{tr}(C) \geq -2$, then π is not discrete.

Using above algorithm we can make a computer program determine the discreteness of a holonomy group.

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