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THE VARIATIONAL THEORY OF A CIRCULAR ARCH WITH TORSIONAL SPRINGS AT BOTH EDGES

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Abstract. Arches are constrained with rotational resistance at both edges. An energy method is used to derive variational formulation which is used to prove the existence of equilibrium states of elastic circular arches for the torsional spring constants $\rho_- \geq 0$, $\rho_+ \geq 0$, and $\rho_- + \rho_+ > 0$. The boundary conditions are searched using the existence of minimum potential energy.

1. Introduction

Arches we consider are inextensible elastic circular subject to the action of normal pressure. The behavior of arches are nonlinear and sensitive to the buckling conditions, flexural rigidity, and opening angle. Classical authors, Dadeppo [2], Dickey and Broughton [3], and Tadjbakhsh and Odeh [6] have explored the energy principle of an elastic circular model with Drichlet and Neumann boundary conditions.

The arches are restrained by torsional spring at the bases (see Fig. 1). The resistance energy to conserve un-deformed arch at the both edges is taken into account for potential energy, which yields Robin boundary conditions. We address the existence of equilibria of the elastic circular arches and Robin boundary conditions are found using the result of the existence of minimization of energy. The variational formulation is set up using the energy method based on Hamilton's principal in section 2. This principal leads to a minimization of total energy. In section 3 the existence of solution, namely the existence of minimum of potential energy, is discussed using the variational formulation and boundary conditions are searched.

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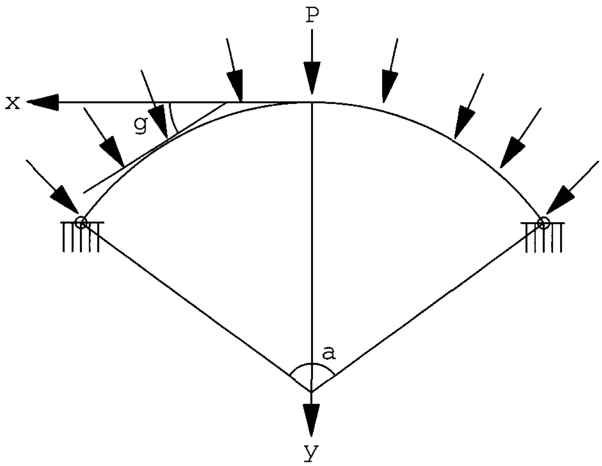

Fig. 1. Normal load uniformly distributed around arch

2. Variational formulation

Let $x(s)$ and $y(s)$ denote the coordinates of the point s on the cross section, and $g(s)$ the angle between the tangent to the cross section and x-axis, where s is arc-length along the cross section of a buckled arch. Then

(1a) $$x(s) = \int_{-a}^{s} \cos g(s) ds,$$

(1b) $$y(s) = \int_{-a}^{s} \sin g(s) ds,$$

and, for a fixed angle a,

(2) $$g : [-a, a] \longrightarrow R,$$

satisfying the constant base positions

(3a) $$\int_{-a}^{a} \sin g(s) ds = 0,$$

(3b) $$\int_{-a}^{a} \cos g(s) ds = 2 \sin a.$$

For spring constants $\rho_+ \geq 0$, $\rho_- \geq 0$, and pressure $p \in R$, the strain energy E is defined by

(4) $$E = \int_{-a}^{a} (1 - g_s(s))^2 ds,$$

and work done W by

$$
\begin{aligned}
W &= -p\left[\int_{-a}^{a}(xy_s - yx_s)ds - 2a\right] \\
&= -p\left[\int_{-a}^{a}(x\sin g(s) - y\cos g(s))ds - 2a\right],
\end{aligned} \tag{5}
$$

respectively. The work done is considered as the difference in area enclosed by the arch in its deformed and un-deformed states. The potential energy V then is

$$
\begin{aligned}
V(g) &= r\rho_-(g(-a)+a)^2 + \rho_+(g(a)-a)^2 \\
&\quad + \int_{-a}^{a}\left[(1-g_s(s))^2 + p\int_{-a}^{s}\sin(g(s)-g(\xi))d\xi\right]ds.
\end{aligned} \tag{6}
$$

The first and second terms in equation (6) imply the energy to keep un-deformed arch at the both edges. If the arch is buckled with clamped bases, then boundary conditions are $g(-a) = -a$ and $g(a) = a$, which yield first two terms in equation (6) vanish. Moreover, $\rho_- = \rho_+ = 0$ mean the buckling in hinged bases.

The transform $z(s) = g(s) - s$ yields

$$\int_{-a}^{a}\sin(z(s)+s)ds = 0, \tag{7a}$$

$$\int_{-a}^{a}\cos(z(s)+s)ds = 2\sin a, \tag{7b}$$

and

$$
\begin{aligned}
V(z) &= \rho_- z(-a)^2 + \rho_+ z(a)^2 \\
&\quad + \int_{-a}^{a}\left[z_s(s)^2 + p\int_{-a}^{s}\sin[(z(s)-z(\xi)) + (s-\xi)]d\xi\right]ds.
\end{aligned} \tag{8}
$$

3. Existence of solution

Assume that $0 \le \rho_-$, $0 \le \rho_+$, and $\rho_+ + \rho_- > 0$. Let us fix pressure p and angle a. We want to minimize the E

$$
\begin{aligned}
E(z) &= \rho_- z(-a)^2 + \rho_+ z(a)^2 \\
&\quad + \int_{-a}^{a}\left[z_s(s)^2 + p\int_{-a}^{s}\sin[(z(s)-z(\xi) + (s-\xi)]d\xi\right]ds
\end{aligned} \tag{9}
$$

subject to the constraints

$$\int_{-a}^{a}\sin(z(s)+s)ds = 0, \tag{10a}$$

$$\int_{-a}^{a}\cos(z(s)+s)ds = 2\sin a. \tag{10b}$$

Let $H \equiv W^{1,2}$ and define

$$f_1(u) = \int_{-a}^{a} \cos(u(s) + s)ds, \tag{11}$$

$$f_2(u) = \int_{-a}^{a} \sin(u(s) + s)ds, \tag{12}$$

for $u \in H$. Define j and Ψ by

$$j(v) = \int_{-a}^{a}\int_{-a}^{s} \sin[(v(s) - v(\xi)) + (s - \xi)]d\xi ds, \tag{13}$$

$$\Psi(u,v) = \rho_- u(-a)^2 + \rho_+ u(a)^2 + \int_{-a}^{a} u'^2 ds + pj(v), \tag{14}$$

for $u, v \in H$, and note

$$E(u) = \Psi(u,u). \tag{15}$$

Lemma 3.1. *There exists a constant c such that*

$$||u||_\infty \leq c|u|_{1,2} \tag{16}$$

for every $u \in H$.

Proof. Since $u(x_0) = u(s) - \int_{x_0}^{s} u'$ for any $x_0 \in [-a,a]$, $|u(x_0)| \leq |u(s)| + \int_{-a}^{a} |u'|$. The integration and Hölder inequality yield

$$\begin{aligned} 2a|u(x_0)| &\leq \int_{-a}^{a} |u(s)| + 2a\int_{-a}^{a} |u'| \\ &\leq \sqrt{2a}||u||_{L^2} + (2a)^{\frac{3}{2}}||u'||_{L^2} \\ &\leq \sqrt{8a^3 + 2a}|u|_{1,2}. \end{aligned}$$

The last inequality is from the fact $|\beta_1 A + \beta_2 B| \leq \sqrt{\beta_1^2 + \beta_1^2}\sqrt{A^2 + B^2}$. □

Lemma 3.2. *The set $S = \{u \in H | f_1(u) = 2\sin a, f_2(u) = 0\}$ is nonempty and weakly closed in H.*

Proof. The set is not empty since it contains $u(s) = -2s$. Let u_n be a sequence in S such that $u_n \rightharpoonup u$ in H. Then, by the imbedding theorem, u_n converges to u in L^2-norm. On the other hand,

$$\begin{aligned} |f_1(u_n) - f_1(u)| &= \left|\int_{-a}^{a} [\cos(u_n(s) + s) - \cos(u(s) + s)]ds\right| \\ &\leqslant 2\int_{-a}^{a} |\sin\frac{1}{2}(u_n(s) - u(s))\sin\frac{1}{2}(u_n(s) + u(s) + 2s)|ds \\ &\leqslant \int_{-a}^{a} |u_n - u|ds \\ &\leqslant c||u_n - u||_{L^2} \to 0, \end{aligned} \tag{17}$$

where c is a constant. Thus $f_1(u) = \int_{-a}^{a} \cos(u(s)+s)ds = 2\sin a$. Similarly, the proof of $f_2(u) = \int_{-a}^{a} \sin(u(s)+s)ds = 0$ can be completed. Hence S is weakly closed. □

Lemma 3.3. *The function $\Psi(\cdot,\cdot)$ is a semi-convex function on $H \times H$.*

Proof. Pick v in H and $c \in R$. Let us take u_1 and u_2 in $S_{c,v} = \{u \in H | \Psi(u,v) \leq c\}$. For $t \in [0,1]$, define

$$g(t) = \Psi(tu_1 + (1-t)u_2, v) - t\Psi(u_1, v) - (1-t)\Psi(u_2, v). \tag{18}$$

Then $g(t)$ is a parabola with $g(0) = g(1) = 0$, and the second derivative with respect to t is

$$\begin{aligned} g''(t) &= 2\rho_-(u_1(-a) - u_2(-a))^2 + 2\rho_+(u_1(a) - u_2(a))^2 \\ &\quad + 2\int_{-a}^{a} [(u_1' - u_2')]^2 \geq 0. \end{aligned} \tag{19}$$

Thus $g(t) \leq 0$ for $0 \leq t \leq 1$ and

$$\Psi(tu_1 + (1-t)u_2, v) \leq t\Psi(u_1, v) + (1-t)\Psi(u_2, v) \leq c. \tag{20}$$

Therefore, $tu_1 + (1-t)u_2 \in S_{c,v}$, which implies $S_{c,v}$ is convex. □

Let v_n be a sequence that converges weakly to v in H. Then we can prove $\Psi(u, v_n) \to \Psi(u, v)$ uniformly similarly to Lemma 3.2.

Now, for any sequence u_n that converges to u in H and c as in Lemma 3.1 we have

$$\begin{aligned} &|\Psi(u_n, v) - \Psi(u, v)| \\ &\leqslant \rho_- ||u_n + u||_\infty ||u_n - u||_\infty + \rho_+ ||u_n + u||_\infty ||u_n - u||_\infty \\ &\quad + ||u_n' + u'||_{L^2} ||u_n' - u'||_{L^2} \\ &\leqslant (\rho_- c^2 + \rho_+ c^2 + 1)|u_n + u|_{1,2} |u_n - u|_{1,2}. \end{aligned} \tag{21}$$

Hence Ψ is semi-convex on $H \times H$.

Lemma 3.4. *The $E(u) \to \infty$ as $|u|_{1,2} \to \infty$ on H.*

Proof. Note that

$$E(u) \geq \rho_- u^2(-a) + \rho_+ u^2(a) + \int_{-a}^{a} u'^2 - 2|p|a^2. \tag{22}$$

If $\int_{-a}^{a} u'^2 \to \infty$ then $E(u) \to \infty$ clearly. If on the other hand $\int_{-a}^{a} u'^2$ remain bounded then $E(u) \to \infty$, because $|u(\pm a)| \to \infty$ by the following argument;

using $u(s) = u(-a) + \int_{-a}^{s} u'$,

$$\begin{aligned}
&\int_{-a}^{a} u^2 \\
\leq\ & \int_{-a}^{a} u^2(-a) + 2|u(-a)| \int_{-a}^{a}\int_{-a}^{a} |u'| + \int_{-a}^{a} \left[\int_{-a}^{a} |u'|\right]^2 \\
\leq\ & 2au^2(-a) + 4\sqrt{2a^3}|u(-a)| \left[\int_{-a}^{a} |u'|^2\right]^{\frac{1}{2}} + 4a^2 \int_{-a}^{a} |u'|^2.
\end{aligned}$$

□

Lemma 3.5. *The $E : H \to R^1$ and $f_i : H \to R^1$, $i = 1, 2$, are C^1 on H. Moreover,*

$$\begin{aligned}
E'(u)h = {}& 2\rho_- u(-a)h(-a) + 2\rho_+ u(a)h(a) + \int_{-a}^{a} 2u'h' \\
& + p \int_{-a}^{a}\int_{-a}^{s} \cos[u(s) - u(\xi) + (s - \xi)](h(s) - h(\xi)),
\end{aligned} \tag{23}$$

$$f_1'(u)h = \int_{-a}^{a} -\sin(u + s)h(s), \tag{24}$$

$$f_2'(u)h = \int_{-a}^{a} \cos(u + s)h(s), \tag{25}$$

for all $u, h \in H$.

Proof. Using the Lemma 3.1,

$$\begin{aligned}
&\big|\rho_-(u(-a) + h(-a))^2 + \rho_+(u(a) + h(a))^2 \\
&+ \int_{-a}^{a} (u' + h')^2 - [\rho_- u(-a)^2 + \rho_+ u(a)^2 + \int_{-a}^{a} u'^2] \\
&- [2\rho_- u(-a)h(-a) + 2\rho_+ u(a)h(a) + 2\int_{-a}^{a} u'h']\big| \\
&\leq \rho_-|h(-a)|^2 + \rho_+|h(a)|^2 + \int_{-a}^{a} h'^2 \\
&\leq |h|_{1,2}^2 \left[c^2(\rho_- + \rho_+) + 1\right].
\end{aligned}$$

Moreover, for some constant c_1,

$$\begin{aligned}
&\Big| \int_{-a}^{a}\int_{-a}^{s} \sin[(u(s) + h(s)) - (u(\xi) + h(\xi)) + (s - \xi)] \\
&- \int_{-a}^{a}\int_{-a}^{s} \sin[u(s) - u(\xi) + (s - \xi)] \\
&- \int_{-a}^{a}\int_{-a}^{s} \cos[u(s) - u(\xi) + (s - \xi)](h(s) - h(\xi))\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{-a}^{a}\int_{-a}^{a} |\cos(h(s)-h(\xi))-1| + \int_{-a}^{a}\int_{-a}^{a} |\sin(h(s)-h(\xi))-(h(s)-h(\xi))| \\
&\leq \int_{-a}^{a}\int_{-a}^{a} |h(s)-h(\xi)|^2 \\
&\leq c_1|h|_{1,2}^2.
\end{aligned}
$$

Thus, using (23),

$$|E(u+h)-E(u)-E'(u)h| \leq c_2|h|_{1,2}^2,$$

where c_2 is a constant. Hence $E(u)$ is differentiable in H and the derivative is given by (23).
To show that E is C^1, pick u, u_1, and h in H and note

$$
\begin{aligned}
\left|\int_{-a}^{a}(u_1'-u')h'\right| &\leq \left[\int_{-a}^{a}|u_1'-u'|^2\right]^{\frac{1}{2}}\left[\int_{-a}^{a}|h'|^2\right]^{\frac{1}{2}} \\
&\leq |u_1-u|_{1,2}|h|_{1,2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{-a}^{a}\int_{-a}^{s}\{\cos[u_1(s)-u_1(\xi)+(s-\xi)] \\
&\qquad\qquad -\cos[u(s)-u(\xi)+(s-\xi)]\}\,(h(s)-h(\xi))\Big| \\
&\leq 2\int_{-a}^{a}\int_{-a}^{a}|\sin\frac{1}{2}[(u_1(s)-u(s))-(u_1(\xi)-u(\xi))]||(h(s)-h(\xi)| \\
&\leq c_3|u_1-u|_{1,2}|h|_{1,2},
\end{aligned}
$$

where c_3 is a constant. Hence, using the Lemma 3.1, for some constants k_1, k_2, and k_3,

$$
\begin{aligned}
&|E'(u_1)h-E'(u)h| \\
&\leq \rho_-|u_1(-a)-u(-a)||h(-a)|+\rho_+|u_1(a)-u(a)||h(a)|+k_1|u_1-u|_{1,2}|h|_{1,2} \\
&\leq \rho_-k_2|u_1-u|_{1,2}|h|_{1,2}+\rho_+k_3|u_1-u|_{1,2}|h|_{1,2}+k_1|u_1-u|_{1,2}|h|_{1,2} \\
&\leq (k_1+\rho_-k_2+\rho_+k_3)|u_1-u|_{1,2}|h|_{1,2},
\end{aligned}
$$

and so $u \to E'(u)$ is continuous. Similar argument can be applied to the differentiability of the functions f_i, $i=1,2$. □

Theorem 3.1. *There exists a minimizer $u_0 \in H$ of*

$$
\begin{aligned}
E(u) \;=\;& \rho_-u(-a)^2+\rho_+u(a)^2 \\
&+\int_{-a}^{a}u'^2ds+p\int_{-a}^{a}\int_{-a}^{s}\sin[(u(s)-u(\xi))+(s-\xi)]d\xi ds, \qquad (26)
\end{aligned}
$$

with side conditions

$$\int_{-a}^{a} \sin(u(s)+s)ds = 0, \tag{27a}$$

$$\int_{-a}^{a} \cos(u(s)+s)ds = 2\sin a. \tag{27b}$$

Moreover, for some (μ_1, μ_2) *in* R^2,

$$E'(u_0) = \mu_1 f_1'(u_0) + \mu_2 f_2'(u_0).$$

Proof. There exists a closed ball B that contains an element u_1 in S and, using the Lemma 3.4, has a large enough radius R such that $E(u_1) < E(u)$ for any u not in B. Lemma 3.2 implies that the set $A \equiv B \cap S$ is weakly closed and bounded. It follows from the theorem [1] that E is bounded below and assumes its minimum at some u_0 in A. If $u \in S \setminus B$ then $E(u_0) \leq E(u_1) < E(u)$ and therefore the minimum of E on S is attained at u_0.

Now, it is needed to show that, for each $(w_1, w_2) \in R^2$, there exist $h \in H$ such that

$$f_i'(u_0)h = w_i, \qquad i = 1, 2 \tag{28}$$

to apply to the surjective implicit function theorem to Lagrange Multiplier Rule Theorem [8, p.270]. Let $h(s) = c_1 \cos(u_0(s)+s) + c_2 \sin(u_0(s)+s)$, where c_1 and c_2 are constants. The (28) implies that we need to find c_1 and c_2 such that

$$\begin{aligned} &- c_1 \int_{-a}^{a} \sin(u_0(s)+s)\cos(u_0(s)+s)ds \\ &- c_2 \int_{-a}^{a} \sin(u_0(s)+s)\sin(u_0(s)+s)ds = w_1, \\ &c_1 \int_{-a}^{a} \cos(u_0(s)+s)\cos(u_0(s)+s)ds \\ &+ c_2 \int_{-a}^{a} \sin(u_0(s)+s)\cos(u_0(s)+s)ds = w_2. \end{aligned}$$

This can be done if the determinant

$$\begin{aligned} \det = &\int_{-a}^{a} \sin^2(u_0(s)+s) \int_{-a}^{a} \cos^2(u_0(s)+s) \\ &- \left[\int_{-a}^{a} \sin(u_0(s)+s)\cos(u_0(s)+s)\right]^2 \\ = &\, a^2 - \frac{1}{4}\left(\left[\int_{-a}^{a} \cos 2(u_0(s)+s)\right]^2 + \left[\int_{-a}^{a} \sin 2(u_0(s)+s)\right]^2\right) \end{aligned}$$

is nonzero. Hölder inequality yields

$$\begin{aligned}\det &\geq a^2 - \frac{1}{4}\left[2a\int_{-a}^{a}\cos^2 2(u_0(s)+s) + 2a\int_{-a}^{a}\sin^2 2(u_0(s)+s)\right]^2 \\ &\geq 0,\end{aligned}$$

however, the equality in above inequality holds when $\sin 2(u_0(s)+s) = constant$ and $\cos 2(u_0(s)+s) = constant$. From the side condition $\int_{-a}^{a}\sin(u_0(s)+s)ds = 0$, $u_0(s)+s = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ which contradicts the constraint $\int_{-a}^{a}\cos(u_0(s)+s)ds = 2\sin a$. Thus $u_0(s)+s$ cannot be constant and $\det(M) > 0$. Hence, c_i, $i = 1,2$ will be determined uniquely and the theorem [8, p.270] can be applied. □

Theorem 3.2. *If* $u_0 \in W^{1,2}(-a,a) \cap S$ *satisfies*

$$E'(u_0)h = \mu_1 f_1'(u_0)h + \mu_2 f_2'(u_0)h \quad \textit{for all} \quad h \in W^{1,2}(-a,a) \tag{29}$$

for some constants μ_1 *and* μ_2*, then* $u_0 \in C^2[-a,a]$*, and*

$$\begin{aligned} &u_0'' - p\int_{-a}^{s}\cos[u_0(s)-u_0(\xi)+(s-\xi)]d\xi + p\sin a\cos(u_0(s)+s) \\ &= \frac{\mu_1}{2}\sin(u_0(s)+s) - \frac{\mu_2}{2}\cos(u_0(s)+s), \end{aligned} \tag{30}$$

$$u_0'(-a) - \rho_- u_0(-a) = 0, \tag{31a}$$

$$u_0'(a) + \rho_+ u_0(a) = 0. \tag{31b}$$

Proof. Let us bring back the equation

$$\begin{aligned} E'(u_0)h = \quad & 2\rho_- u_0(-a)h(-a) + 2\rho_+ u_0(a)h(a) + \int_{-a}^{a} 2u_0'h'ds \\ & + p\int_{-a}^{a} h(s)\int_{-a}^{s}\cos[u_0(s)-u_0(\xi)+(s-\xi)]d\xi ds \\ & - p\int_{-a}^{a}\int_{-a}^{s} h(\xi)\cos[u_0(s)-u_0(\xi)+(s-\xi)]d\xi ds. \end{aligned} \tag{32}$$

The change of the order of integration of the last term in the equation (32) on the right hand side and the constraints furnish

$$\begin{aligned} &\int_{-a}^{a}\int_{-a}^{s} h(\xi)\cos(u_0(s)-u_0(\xi)+(s-\xi))d\xi ds \\ &= \int_{-a}^{a} -h(s)\left[\int_{-a}^{s}\cos(u_0(s)-u_0(\xi)+(s-\xi))d\xi - 2\sin a\cos(u_0(s)+s)\right] ds. \end{aligned}$$

Thus the equation (32) is changed to

$$\begin{aligned} &E'(u_0)h \\ &= 2\rho_- u_0(-a)h(-a) + 2\rho_+ u_0(a)h(a) + \int_{-a}^{a} 2u_0'h' \\ &\quad + \int_{-a}^{a} h(s)2p\left[\int_{-a}^{s} \cos[(u_0(s) - u_0(\xi) + (s-\xi)]d\xi - \sin a\cos(u_0(s)+s)\right] ds. \end{aligned}$$

Now, define

$$\begin{aligned} k(s) = 2p\int_{-a}^{s} \cos[u_0(s) - u_0(\xi) + (s-\xi)]d\xi - 2p\sin a\cos(u_0(s)+s) \\ + \mu_1 \sin(u_0(s)+s) - \mu_2\cos(u_0(s)+s). \end{aligned}$$

Then the above equation (29) is reduced to

$$2\rho_- u_0(-a)h(-a) + 2\rho_+ u_0(a)h(a) + 2\int_{-a}^{a} u_0'h'(s)ds = -\int_{-a}^{a} k(s)h(s)ds$$

for all $h \in W_0^{1,2}(-a,a)$. Since $k(s)$ is continuous, regularity [4, p65] implies that u_0' is absolutely continuous, and thus $u_0 \in C^2[-a,a]$. Integration by part yields

$$\begin{aligned} 2[\rho_- u_0(-a) - u_0'(-a)]h(-a) + 2[\rho_+ u_0(a) + u_0'(a)]h(a) \\ + \int_{-a}^{a} [-2u_0''(s) + k(s)]h(s)ds = 0. \end{aligned}$$

[4, p.65] also implies that $2u_0''(s) = k(s)$, hence

(33a) $$u_0'(-a) - \rho_- u_0(-a) = 0,$$

(33b) $$u_0'(a) + \rho_+ u_0(a) = 0.$$

□

4. Conclusion

We discuss the equilibrium behaviors of circular arches constrained by torsional elastic spring at both edges. In the formulation of potential energy we take into account the resistance energy generated by torsional springs at both ends. The consideration of the energy to keep initial circular form is a crucial part to study the existence of equilibrium states, which yields the Robin boundaries.

There is a difficulty to follow earlier discussion style. Usually boundary conditions are given and then argument is developed under the boundary conditions. However, we first investigate the minimum principal of potential energy for each given pressure and opening angle, and next the boundary conditions is searched. The work on variational methods for nonlinear elliptic eigenvalue problem by F. E. Browder [1] serves the proof of the existence of solution.

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