II-COHERENT DIMENSIONS AND II-COHERENT RINGS

Lixin Mao

Abstract. R is called a right II-coherent ring in case every finitely generated torsionless right R-module is finitely presented. In this paper, we define a dimension for rings, called II-coherent dimension, which measures how far away a ring is from being II-coherent. This dimension has nice properties when the ring in question is coherent. In addition, we study some properties of II-coherent rings in terms of preenvelopes and precovers.

1. Introduction

Recall that R is called a right II-coherent ring [3] in case every finitely generated torsionless right R-module is finitely presented. This notion is also called strong coherence in [12]. R is called a right coherent ring in case every finitely generated right ideal is finitely presented. Clearly, we have the following implications:

right Noetherian rings ⇒ right II-coherent rings ⇒ right coherent rings.

But the converse does not hold generally (see [3]). II-coherent rings have been studied by many authors (see, for example, [3, 4, 6, 12, 21]).

The concept of FGT-injective dimensions for modules and rings were introduced in [4]. The FGT-injective dimension of a right R-module M, denoted by $\text{FGT} - \text{Id}(M)$, is defined as the smallest integer $n \geq 0$ such that $\text{Ext}_R^{n+1}(N, M) = 0$ for any finitely generated torsionless right R-module N. If no such n exists, set $\text{FGT} - \text{Id}(M) = \infty$. Put $r\text{FGT} = \text{I. dim}(R) = \sup\{\text{FGT} - \text{Id}(M) : M \text{ is any right R-module}\}$ and call $r\text{FGT} = \text{I. dim}(R)$ the right FGT-injective dimension of R. It is known that $r\text{FGT} = \text{I. dim}(R) = 0$ if and only if R is a right II-coherent left semihereditary ring (see [4]). So the right FGT-injective dimension of a ring R measures how far away R is from being a right II-coherent left semihereditary ring.

Recall that a right R-module M is called FP-injective (or absolutely pure) [19, 15] in case $\text{Ext}_R^1(N, M) = 0$ for all finitely presented right R-modules N. In [16], Mao and Ding introduced the concept of FP-projective dimensions.

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The $FP$-projective dimension of a right $R$-module $M$, denoted by $fpd(M_R)$, is defined as the smallest integer $n \geq 0$ such that $\text{Ext}_R^{n+1}(M, N) = 0$ for any $FP$-injective right $R$-module $N$. If no such $n$ exists, set $fpd(M_R) = \infty$. $M$ is called $FP$-projective if $fpd(M_R) = 0$, i.e., $\text{Ext}_R^1(M, N) = 0$ for any $FP$-injective right $R$-module $N$.

In Section 2 of the present paper, we define a dimension for rings, which measures how far away a ring is from being II-coherent. Put $r.\pi cD(R) = \sup\{fpd(M_R) : M_R$ is a finitely generated torsionless right $R$-module$\}$ and call $r.\pi cD(R)$ the right II-coherent dimension of $R$. This dimension has nice properties when the ring in question is coherent. For example, it is true that $rFGT - I. \dim(R) \leq wD(R) + r.\pi cD(R)$ for a right coherent ring $R$. Let $R$ and $S$ be right coherent rings, we show that $r.\pi cD(R \otimes S) = \sup\{r.\pi cD(R), r.\pi cD(S)\}$. We also consider the II-coherent dimensions under changes of rings, especially under excellent extensions of rings. It is proven that, if $R$ and $S$ is right coherent rings and $S$ is an excellent extension of $R$, then $r.\pi cD(S) = r.\pi cD(R)$.

In Section 3, we study some properties of II-coherent rings in terms of preenvelopes and precovers. Let $R$ be a right II-coherent ring. We first prove that every left $R$-module has an $FGT$-flat preenvelope, and every right $R$-module has an $FGT$-injective cover, where a right $R$-module $M$ (a left $R$-module $Q$) is called $FGT$-injective ($FGT$-flat) (see [4]) if $\text{Ext}_R^1(N, M) = 0$ ($\text{Tor}_R^1(N, Q) = 0$) for any finitely generated torsionless right $R$-module $N$. Next we show that the following are equivalent: (1) $R_R$ is $FGT$-injective. (2) Every left $R$-module has a monic $FGT$-flat preenvelope. (3) Every right $R$-module has an epic $FGT$-injective cover. It is also shown that the following are equivalent: (1) $rFGT - I. \dim(R) \leq 1$. (2) Every left $R$-module has an epic $FGT$-flat envelope. (3) Every right $R$-module has a monic $FGT$-injective cover. Finally we prove that the following are equivalent: (1) $rFGT - I. \dim(R) \leq 2$. (2) Every right $R$-module has an $FGT$-injective cover with the unique mapping property.

Throughout this paper, all rings are associative with identity and all modules are unitary. We write $M_R$ ($rM$) to indicate a right (left) $R$-module. $wD(R)$ stands for the weak global dimension of a ring $R$. For an $R$-module $M$, $pd(M)$ denotes the projective dimension of $M$, the dual module $\text{Hom}_R(M, R)$ is denoted by $M^*$, and the character module $M^+$ is defined by $M^+ = \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$. General background materials can be found in [10, 13, 18, 22].

2. II-Coherent dimensions

We start with the following

**Definition 2.1.** Let $R$ be a ring. Put $r.\pi cD(R) = \sup\{fpd(M_R) : M_R$ is a finitely generated torsionless right $R$-module$\}$ and call $r.\pi cD(R)$ the right II-coherent dimension of $R$. Similarly, we have $l.\pi cD(R)$.

**Proposition 2.2.** The following are equivalent for a ring $R$:

1. $r.\pi cD(R) = 0$. 

(2) $R$ is a right II-coherent ring.
(3) Every $FP$-injective right $R$-module is $FGT$-injective.

Proof. (1) $\Rightarrow$ (2). Let $M$ be a finitely generated torsionless right $R$-module. Then $\text{Ext}_R^1(M, N) = 0$ for any $FP$-injective right $R$-module $N$ by (1). Therefore $M$ is finitely presented by [8], and so $R$ is a right II-coherent ring.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are clear by definition.

Remark 2.3. (1) By Proposition 2.2, the right II-coherent dimension $r\pi cD(R)$ measures how far away a ring $R$ is from being right II-coherent. It is known that right II-coherent rings need not be left II-coherent (see [13, Example 4.46 (e)]), so $r\pi cD(R) \neq l\pi cD(R)$ in general.

(2) If $R$ is a left $FP$-injective ring (i.e., $_RR$ is $FP$-injective), then every finitely presented right $R$-module is torsionless by [11, Theorem 2.3], and so every $FGT$-injective right $R$-module is $FP$-injective. In this case, $R$ is a right II-coherent ring if and only if $FGT$-injective right $R$-modules coincide with $FP$-injective right $R$-modules by Proposition 2.2.

The next lemma will be used frequently in the sequel.

Lemma 2.4. [16, Proposition 3.1] Let $R$ be a right coherent ring. For any right $R$-module $M$ and integer $n \geq 0$, the following are equivalent:

1. $fpd(M) \leq n$.
2. $\text{Ext}_R^{n+1}(M, N) = 0$ for any $FP$-injective right $R$-module $N$.
3. $\text{Ext}_R^{2n+j}(M, N) = 0$ for any $FP$-injective right $R$-module $N$ and $j \geq 1$.
4. There exists an exact sequence $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, where each $P_i$ is $FP$-projective.

Proposition 2.5. The following are equivalent for a right coherent ring $R$:

1. $r\pi cD(R) \leq 1$.
2. For any exact sequence of right $R$-modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A$ $FP$-injective and $B$ injective, $C$ is $FGT$-injective.
3. For any pure submodule $N$ of an injective right module $M$, the quotient $M/N$ is $FGT$-injective.

Proof. (1) $\Rightarrow$ (2). Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of right $R$-modules with $A$ $FP$-injective and $B$ injective. If $M$ is a finitely generated torsionless right $R$-module, then $fpd(M) \leq r\pi cD(R) \leq 1$ by (1). Thus $\text{Ext}_R^1(M, C) \cong \text{Ext}_R^2(M, A) = 0$ by Lemma 2.4, and so $C$ is $FGT$-injective.

(2) $\Rightarrow$ (3) is obvious since $N$ is $FP$-injective.

(3) $\Rightarrow$ (1). Let $M$ be a finitely generated torsionless right $R$-module and $N$ an $FP$-injective right $R$-module. Then there exists an exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ with $E$ injective. So $L$ is $FGT$-injective by (3) since $N$ is a pure submodule of $M$. Thus $\text{Ext}_R^2(M, N) \cong \text{Ext}_R^1(M, L) = 0$, and hence $fpd(M) \leq 1$. It follows that $r\pi cD(R) \leq 1$. □

Proposition 2.6. Let $R$ be a right coherent ring. Then
(1) \( \text{sup}\{\text{FGT} - \text{Id}(M) : M \text{ is an FP-injective right } R\text{-module}\} \leq r.\pi c D(R) \);
(2) \( r.\text{FGT} - I. \dim(R) \leq \text{sup}\{\text{pd}(F) : F \text{ is a finitely generated torsionless } R\text{-module}\} \leq wD(R) + r.\pi c D(R) \).

Proof. (1). We may assume that \( r.\pi c D(R) = n < \infty \). Let \( M \) be an FP-injective right \( R\)-module and \( N \) any finitely generated torsionless right \( R\)-module. Then \( \text{fpd}(N) \leq n \). Thus \( \text{Ext}^{n+1}_R(N, M) = 0 \) by Lemma 2.4, and so \( \text{FGT} - \text{Id}(M) \leq n \). Consequently \( \text{sup}\{\text{FGT} - \text{Id}(M) : M \text{ is any FP-injective right } R\text{-module}\} \leq r.\pi c D(R) \).

(2). It is easy to check that \( r.\text{FGT} - I. \dim(R) \leq \text{sup}\{\text{pd}(F) : F \text{ is a finitely generated torsionless } R\text{-module}\} \).

Now we assume that, without loss of the generality, both \( r.\pi c D(R) \) and \( wD(R) \) are finite. Let \( r.\pi c D(R) = m < \infty \) and \( wD(R) = n < \infty \). Suppose \( M \) is a finitely generated torsionless right \( R\)-module, then \( \text{fpd}(M) \leq m \). So by Lemma 2.4, \( M \) admits an FP-projective resolution

\[
0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,
\]

where each \( P_i \) is FP-projective. Note that \( \text{pd}(P_i) \leq wD(R) = n \) by [16, Theorem 4.2], \( i = 0, 1, 2, \ldots , m \). It follows that \( \text{pd}(M) \leq m + n \) by dimension shifting, and hence \( r.\text{FGT} - I. \dim(R) \leq m + n \), as desired. \( \square \)

**Theorem 2.7.** Let \( R \) and \( S \) be right coherent rings. Then

\[
r.\pi c D(R \oplus S) = \text{sup}\{r.\pi c D(R), r.\pi c D(S)\}.
\]

Proof. We first show that \( r.\pi c D(R \oplus S) \leq \text{sup}\{r.\pi c D(R), r.\pi c D(S)\} \). We may assume that \( r.\pi c D(R) = m < \infty, r.\pi c D(S) = n < \infty \), and \( m \geq n \). Let \( M \) be a finitely generated torsionless right \((R \oplus S)\)-module. Then \( M \) has a unique decomposition that \( M = A \oplus B \), where \( A = M(R, 0) \) is a right \( R\)-module and \( B = M(0, S) \) is a right \( S\)-module via \( x r = x(r, 0) \) for \( x \in A, r \in R \), and \( y s = y(0, s) \) for \( y \in B, s \in S \). It is easy to verify that \( A \) is a finitely generated torsionless right \( R\)-module and \( B \) is a finitely generated torsionless right \( S\)-module. Thus \( \text{fpd}(A_R) \leq m \) and \( \text{fpd}(B_S) \leq n \leq m \). By Lemma 2.4, there exist two exact sequences

\[
0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0
\]

and

\[
0 \rightarrow Q_m \rightarrow Q_{m-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow B \rightarrow 0
\]

of right \( R\)-modules and right \( S\)-modules respectively, where each \( P_i \) is an FP-projective right \( R\)-module, and each \( Q_i \) is an FP-projective right \( S\)-module. Regarding these as exact sequences of right \((R \oplus S)\)-modules, we have an exact sequence of right \((R \oplus S)\)-modules

\[
0 \rightarrow P_m \oplus Q_m \rightarrow P_{m-1} \oplus Q_{m-1} \rightarrow \cdots \rightarrow P_1 \oplus Q_1 \rightarrow P_0 \oplus Q_0 \rightarrow A \oplus B \rightarrow 0.
\]

Note that each \( P_i \oplus Q_i \) is an FP-projective right \((R \oplus S)\)-module by [16, Lemma 3.15]. Thus \( \text{fpd}(M_{R \oplus S}) \leq m \), and hence \( r.\pi c D(R \oplus S) \leq \text{sup}\{r.\pi c D(R), r.\pi c D(S)\} \).

Next we prove that \( \text{sup}\{r.\pi c D(R), r.\pi c D(S)\} \leq r.\pi c D(R \oplus S) \). We may assume that \( r.\pi c D(R \oplus S) = k < \infty \). Let \( M \) be a finitely generated torsionless right \( R\)-module. Note that \( M \) may be regarded as a finitely generated torsionless right \((R \oplus S)\)-module, so \( \text{fpd}(M_{R \oplus S}) \leq r.\pi c D(R \oplus S) = k \). Thus there
exists an exact sequence 0 → P_k → P_{k-1} → · · · → P_1 → P_0 → M → 0 of right (R ⊕ S)-modules, where each P_i is an FP-projective right (R ⊕ S)-module. Let P_i = A_i ⊕ B_i, where A_i is a right R-module and B_i is a right S-module, i = 0, 1, . . . , k. Since M is a right R-module, we have the exact sequence 0 → A_k → A_{k-1} → · · · → A_1 → A_0 → M → 0 of right R-modules. Note that each A_i is an FP-projective right (R ⊕ S)-module, and so an FP-projective right R-module by [16, Lemma 3.15], whence fpd(M_R) ≤ k, and so r.πcD(R) ≤ k. Similarly r.πcD(S) ≤ k. Thus sup{r.πcD(R), r.πcD(S)} ≤ r.πcD(R ⊕ S). The proof is complete.

□

Remark 2.8. Theorem 2.7 shows that r.πcD( ⊕ \limits_{i=1}^{n} R_i) = sup \{r.πcD(R_i)\} if each R_i is right coherent. In particular, we get that ⊕ R_i is a right II-coherent ring if and only if each R_i is right II-coherent, i = 1, 2, . . . , n.

Next we investigate the II-coherent dimensions under changes of rings.

Proposition 2.9. Let R and S be right coherent rings. If ϕ : R → S is a surjective ring homomorphism with S flat as a left R-module and projective as a right R-module, then r.πcD(S) ≤ r.πcD(R).

Proof. We may assume that r.πcD(R) = n < ∞. Let M_S be a finitely generated torsionless right S-module and F_S an FP-injective right S-module. We claim that F_R is an FP-injective right R-module. In fact, if N is a finitely presented right R-module, then there is an exact sequence 0 → K → P → N → 0 of right R-modules with K finitely generated and P finitely generated projective. Since R_S is flat, we have the right S-module exact sequence 0 → K ⊗_R S → P ⊗_R S → N ⊗_R S → 0. Note that K ⊗_R S is a finitely generated right S-module, P ⊗_R S is a finitely generated projective right S-module, and so N ⊗_R S is a finitely presented right S-module. Therefore Ext^1_R(N_R, F_R) ≅ Ext^1_S(N ⊗_R S, F_S) = 0 by [18, Theorem 11.65] since F_S is FP-injective. So F_R is FP-injective. On the other hand, M_R is a finitely generated torsionless right R-module since S_R is projective. Thus fpd(M_R) ≤ r.πcD(R) ≤ n, and hence Ext^{n+1}_S(M_S, Hom_R(S, F_R)) ≅ Ext^{n+1}_R(M_R, F_R) = 0 by [18, Theorem 11.66]. Note that F_S ≅ Hom_R(S, F_R) (for ϕ is surjective), so Ext^{n+1}_S(M_S, F_S) = 0. Thus fpd(M_S) ≤ n. It follows that r.πcD(S) ≤ r.πcD(R).

□

Recall that a ring S is said to be an excellent extension of a subring R [2] if the following conditions are satisfied:

(1) R and S have the same identity and S is free with basis s_1, . . . , s_n as both a right and a left R-module, s_1 = 1_R, and Rs_i = s_iR for all i = 1, . . . , n;

(2) If M_S is a submodule of N_S and M_R is a direct summand of N_R, then M_S is a direct summand of N_S.
Theorem 2.10. Let $R$ and $S$ be right coherent rings. If $S$ is an excellent extension of $R$, then $r.piD(S) = r.piD(R)$.

Proof. Note that $R$ is an $R$-bimodule direct summand of $S$ since $S$ is an excellent extension of $R$. So we may let $RS = R \oplus T$.

We first prove that $r.piD(S) \leq r.piD(R)$. We may assume that $r.piD(R) = m < \infty$. Let $N_S$ be a finitely generated torsionless right $S$-module. It is easy to see that $N_R$ is a finitely generated torsionless right $R$-module. Thus $fpd(N_R) \leq m$, and so there exists an exact sequence $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ of right $R$-modules, where each $P_i$ is an FP-projective right $R$-module. Since $RS$ is free, we have the exact sequence $0 \to P_m \otimes_R S \to P_{m-1} \otimes_R S \to \cdots \to P_1 \otimes_R S \to P_0 \otimes_R S \to N \otimes_R S \to 0$ of right $S$-modules. Note that each $P_i \otimes_R S$ is an FP-projective right $S$-module by [16, Lemma 3.18], and so $fpd(N \otimes_R S)_S \leq m$. Note that $(N \otimes_R S)_R \cong N_R \oplus (N \otimes_R T)$, and so we have $N_S$ is isomorphic to a direct summand of $(N \otimes_R S)_S$ since $S$ is an excellent extension of $R$. Thus $fpd(N_S) \leq fpd(N \otimes_R S)_S$, and hence $fpd(N_S) \leq m$. It follows that $r.piD(S) \leq r.piD(R)$.

Next we prove that $r.piD(R) \leq r.piD(S)$. We may assume that $r.piD(S) = n < \infty$. Let $M_R$ be a finitely generated torsionless right $R$-module. It is easy to check that $M \otimes_R S$ is a finitely generated torsionless right $S$-module and hence $fpd(M \otimes_R S)_S \leq n$. By Lemma 2.4, there exists an exact sequence of right $S$-modules $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \otimes_R S \to 0$, where each $Q_i$ is an FP-projective right $S$-module. Thus each $Q_i$ is a direct summand in a right $S$-module $U_i$ such that $U_i$ is a union of a continuous chain, $(U_{\alpha,i} : \alpha < \lambda)$, for a cardinal $\lambda$, $U_{0,i} = 0$, and $U_{\alpha+1,i}/U_{\alpha,i}$ is a finitely presented right $S$-module for all $\alpha < \lambda$ (see [20, Definition 3.3]). It is easy to verify that $U_{\alpha+1,i}/U_{\alpha,i}$ is a finitely presented right $R$-module for all $\alpha < \lambda$ since $S$ is an excellent extension of $R$. Thus each $Q_i$ is an FP-projective right $R$-module, and so $fpd(M \otimes_R S)_R \leq n$. Since $(M \otimes_R S)_R \cong M_R \oplus (M \otimes_R T)$, we have $fpd(M_R) \leq fpd(M \otimes_R S)_R \leq n$ and hence $r.piD(R) \leq r.piD(S)$. So we have the desired equality. \hfill \Box

It has been proven that, if $S$ is an excellent extension of $R$, then $R$ is right coherent if and only if $S$ is right coherent (see [14, Lemma 8]). As an immediate consequence of Theorem 2.10, we have

Corollary 2.11. Let $S$ be an excellent extension of $R$. Then $R$ is a right $\Pi$-coherent ring if and only if $S$ is right $\Pi$-coherent.

3. Some properties of $\Pi$-coherent rings

Let $C$ be a class of $R$-modules and $M$ an $R$-module. Following [9], we say that a homomorphism $\phi : M \to C$ is a $C$-preenvelope of $M$ if $C \in C$ and the abelian group homomorphism $\text{Hom}_R(\phi, C') : \text{Hom}_R(C, C') \to \text{Hom}_R(M, C')$ is surjective for each $C' \in C$. A $C$-preenvelope $\phi : M \to C$ is said to be a $C$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism.
Dually we have the definitions of a $C$-	extit{precover} and a $C$-	extit{cover}. $C$-envelopes
($C$-covers) may not exist in general, but if they exist, they are unique up to
isomorphism.

Let $R$ be a right coherent ring. It is known that every left $R$-module has a
flat preenvelope (see [9]) and every right $R$-module has an $FP$-injective cover
(see [17]). In this section, we will investigate the analogous properties of II-
coherent rings in terms of preenvelopes and precovers. The following lemmas
will be needed.

**Lemma 3.1.** Let $R$ be a ring. Then

1. A left $R$-module $M$ is FGT-flat if and only if $M^+$ is FGT-injective.
2. Pure submodules of FGT-flat left $R$-modules are FGT-flat.
3. For a short exact sequence of left $R$-modules $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$,
   if $N$ and $L$ are FGT-flat, then $M$ is FGT-flat.

**Proof.** (1) holds by the standard isomorphism $\text{Ext}^1_R(N, M^+) \cong \text{Tor}^R_1(N, M)^+$
for any right $R$-module $N$.

(2). Let $A$ be a pure submodule of an FGT-flat left $R$-module $B$, then
the pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$ induces the split exact
sequence $0 \rightarrow (B/A)^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$. Thus $A^+$ is FGT-injective since $B^+$
is FGT-injective by (1). So $A$ is FGT-flat.

(3). Let $H$ be a finitely generated torsionless right $R$-module. Then there is
an exact sequence $0 \rightarrow K \rightarrow F \rightarrow H \rightarrow 0$ with $F$ finitely generated free. Note
that $K = \lim \rightarrow K_i$, where each $K_i$ is a finitely generated submodule of $K$ (see
[10, Example 1.55 (2), p. 32]). Thus each $K_i$ is a finitely generated torsion-
less right $R$-module, and so $\text{Tor}^R_1(H, L) \cong \text{Tor}^R_1(K, L) = \text{Tor}^R_1(\lim \rightarrow K_i, L) \cong
\lim \rightarrow \text{Tor}^R_1(K_i, L) = 0$ since $L$ is FGT-flat. On the other hand, we have the
exact sequence $0 = \text{Tor}^R_2(H, L) \rightarrow \text{Tor}^R_1(H, M) \rightarrow \text{Tor}^R_1(H, N) = 0$ since $N$ is
FGT-flat. Thus $\text{Tor}^R_1(H, M) = 0$, and so $M$ is FGT-flat. \hfill \square

**Lemma 3.2.** Let $R$ be a right II-coherent ring. Then

1. A right $R$-module $M$ is FGT-injective if and only if $M^+$ is FGT-flat.
2. Any direct limit (direct sum) of FGT-injective right $R$-modules is FGT-
injective.
3. Any direct product of FGT-flat left $R$-modules is FGT-flat.
4. Pure submodules of FGT-injective right $R$-modules are FGT-injective.
5. For a short exact sequence of right $R$-modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$,
   if $A$ and $B$ are FGT-injective, then $C$ is FGT-injective.
6. $\text{FGT} - \text{Id}(M) \leq n$ if and only if $\text{Ext}^n_R(N, M) = 0$ for any finitely
generated torsionless right $R$-module $N$.

**Proof.** (1) through (3) follow from [4].

(4). Let $N$ be a pure submodule of an FGT-injective right $R$-module $M$.
For any finitely generated torsionless right $R$-module $L$, we have the exact
sequence
\[ \text{Hom}_R(L, M) \to \text{Hom}_R(L, M/N) \to \text{Ext}_R^1(L, N) \to \text{Ext}_R^1(L, M) = 0. \]

But the sequence \( \text{Hom}_R(L, M) \to \text{Hom}_R(L, M/N) \to 0 \) is exact since \( L \) is finitely presented and \( N \) is a pure submodule of \( M \). Therefore \( \text{Ext}_R^1(L, N) = 0 \) and so \( N \) is \( \text{FGT} \)-injective.

(5). The exact sequence \( 0 \to A \to B \to C \to 0 \) induces an exact sequence \( 0 \to C^+ \to B^+ \to A^+ \to 0 \). Note that \( B^+ \) and \( A^+ \) are \( \text{FGT} \)-flat by (1), so \( C^+ \) is \( \text{FGT} \)-flat by Lemma 3.1 (3), and hence \( C \) is \( \text{FGT} \)-injective.

(6) holds by [4]. \( \square \)

The next lemma is a special case of [1, Theorem 5].

**Lemma 3.3.** Let \( R \) be a ring. Then for each cardinal \( \lambda \), there is a cardinal \( \kappa \) such that for any \( R \)-module \( M \) and for any \( L \leq M \) such that \( \text{Card}(M) \geq \kappa \) and \( \text{Card}(M/L) \leq \lambda \), the submodule \( L \) contains a non-zero submodule that is pure in \( M \).

Let \( R \) be an arbitrary ring, it is easy to see that every left \( R \)-module has an \( \text{FGT} \)-flat cover by [20, Lemma 1.11 and Theorem 2.8], and every right \( R \)-module has an \( \text{FGT} \)-injective preenvelope by [7, Theorem 10]. If \( R \) is a right \( \Pi \)-coherent ring, then we can obtain additional results as follows.

**Theorem 3.4.** The following are true for a right \( \Pi \)-coherent ring \( R \):

1. Every left \( R \)-module has an \( \text{FGT} \)-flat preenvelope.
2. Every right \( R \)-module has an \( \text{FGT} \)-injective cover.

**Proof.** (1). Let \( M \) be any left \( R \)-module. By [10, Lemma 5.3.12], there is a cardinal number \( \aleph_\alpha \) such that for any homomorphism \( g : M \to L \) with \( L \) \( \text{FGT} \)-flat, there is a pure submodule \( Q \) of \( L \) such that \( \text{Card}(Q) \leq \aleph_\alpha \) and \( g(M) \subseteq Q \). Note that \( Q \) is \( \text{FGT} \)-flat by Lemma 3.1 (2), and so \( M \) has an \( \text{FGT} \)-flat preenvelope by [10, Proposition 6.2.1] since any direct product of \( \text{FGT} \)-flat left \( R \)-modules is \( \text{FGT} \)-flat by Lemma 3.2 (3).

(2). The proof is motivated by that of [17, Lemma 4.8]. Suppose that \( N \) is a right \( R \)-module with \( \text{Card}(N) = \lambda \). Let \( \kappa \) be a cardinal as in Lemma 3.3. By [10, Proposition 5.2.2], we only need to show that any homomorphism \( f : D \to N \) with \( D \) \( \text{FGT} \)-injective has a factorization \( D \to C \to N \) with \( C \) \( \text{FGT} \)-injective and \( \text{Card}(C) \leq \kappa \).

If \( \text{Card}(D) \leq \kappa \), then we have done. So we may assume that \( \text{Card}(D) > \kappa \).

Let \( K = \ker(f) \). Note that \( \text{Card}(D/K) \leq \lambda \) since \( D/K \) embeds in \( N \). Thus \( K \) contains a non-zero submodule \( D_0 \) which is pure in \( D \) by Lemma 3.3. Since \( D_0 \) is \( \text{FGT} \)-injective by Lemma 3.2 (4), \( D/D_0 \) is \( \text{FGT} \)-injective by Lemma 3.2 (5).

If \( \text{Card}(D/D_0) \leq \kappa \), then we have done since \( f \) factors through \( D/D_0 \).

Suppose that \( \text{Card}(D/D_0) > \kappa \). Note that there exists \( g : D/D_0 \to N \) with \( \ker(g) = K_1/D_0 \). So \( K_1/D_0 \) contains a non-zero submodule \( D_1/D_0 \) which is
pure in \( D/D_0 \) by Lemma 3.3. Therefore \( D/D_1 \cong (D/D_0)/(D_1/D_0) \) is FGT-injective by Lemma 3.2 (5).

If Card\((D/D_1) \leq \kappa\), then we have done since \( f \) factors through \( D/D_1 \).

If Card\((D/D_1) > \kappa\), then we can continue the process above. Ultimately we arrive at a direct limit \( \lim(D/D_i) = D/\lim D_i \) such that Card\((\lim(D/D_i)) \leq \kappa\). Note that \( \lim(D/D_i) \) is FGT-injective by Lemma 3.2 (2), and \( f \) factors through \( \lim(D/D_i) \). So \( N \) has an FGT-injective precover, and hence has an FGT-injective cover by [10, Corollary 5.2.7].

In general, an FGT-flat preenvelope or an FGT-injective cover need not be a monomorphism or epimorphism. Next, we shall consider when every left \( R \)-module has a monic or epic FGT-flat preenvelope and when every right \( R \)-module has a monic or epic FGT-injective cover in case \( R \) is a right \( II \)-coherent ring.

**Theorem 3.5.** The following are equivalent for a right \( II \)-coherent ring \( R \):

1. \( R_R \) is FGT-injective.
2. Every left \( R \)-module has a monic FGT-flat preenvelope.
3. Every right \( R \)-module has an epic FGT-injective cover.

If any of the above conditions holds, then \( rFGT - I. \dim(R) = 0 \) or \( \infty \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( M \) be any left \( R \)-module. Since \((R_R)^+\) is a cogenerator in the category of left \( R \)-modules, there is an exact sequence \( 0 \to M \to II(R_R)^+ \). Note that \((R_R)^+\) is FGT-flat by (1) and Lemma 3.2 (1), and so \( II(R_R)^+ \) is FGT-flat by Lemma 3.2 (3). Thus \( M \) embeds in an FGT-flat left \( R \)-module. But \( M \) has an FGT-flat preenvelope \( f : M \to F \) by Theorem 3.4 (1). So \( f \) is a monomorphism.

(2) \( \Rightarrow \) (1). Note that the injective left \( R \)-module \((R_R)^+\) embeds in an FGT-flat left \( R \)-module by (2). Thus \((R_R)^+\) is FGT-flat, and so \( R_R \) is FGT-injective.

(1) \( \Rightarrow \) (3). Let \( M \) be a right \( R \)-module. Then there is an exact sequence \( F \to M \to 0 \) with \( F \) free. Note that \( F \) is FGT-injective by (1) and Lemma 3.2 (2). Since \( M \) has an FGT-injective cover \( f \) by Theorem 3.4 (2), \( f \) is an epimorphism.

(3) \( \Rightarrow \) (1) is clear since \( R_R \) has an epic FGT-injective cover.

Now suppose that any of the equivalent conditions above holds, and \( rFGT - I. \dim(R) = n < \infty \). For any right \( R \)-module \( M \), there exists an exact sequence \( 0 \to L \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0 \), where each \( F_i \) is free. Thus each \( F_i \) is FGT-injective by (1). So \( M \) is FGT-injective by Lemma 3.2 (6) since \( FGT - Id(L) \leq rFGT - I. \dim(R) \leq n \). It follows that \( rFGT - I. \dim(R) = 0 \).  

**Theorem 3.6.** The following are equivalent for a right \( II \)-coherent ring \( R \):

1. \( rFGT - I. \dim(R) \leq 1 \).
2. Every left \( R \)-module has an epic FGT-flat envelope.
(3) Every right $R$-module has a monic $FGT$-injective cover.

Proof. (1) $\Rightarrow$ (2). For any left $R$-module $M$, there is an $FGT$-flat preenvelope $f : M \to F$ by Theorem 3.4 (1). The exact sequence $0 \to \ker(f) \to F \to L \to 0$ induces the exactness of the sequence $0 \to L^+ \to F^+ \to (\ker(f))^+ \to 0$. Note that $F^+$ is $FGT$-injective, so $(\ker(f))^+$ is $FGT$-injective by Lemma 3.2 (6) since $FGT - \text{Id} \langle L \rangle \leq r F G T - I. \dim(R) \leq 1$. Thus $(\ker(f))^+$ is $FGT$-flat, and hence $M \to \ker(f)$ is an epic $FGT$-flat preenvelope, equivalently, an epic $FGT$-flat envelope.

(2) $\Rightarrow$ (3). Let $M$ be a right $R$-module. Write $F = \bigoplus \{N \leq M : N \text{ is } F G T \text{-injective}\}$ and $G = \bigoplus \{N \leq M : N \text{ is } F G T \text{-injective}\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$, which induces an exact sequence $0 \to F^+ \to G^+ \to \ker(f) \to 0$. Since $F^+$ has an epic $FGT$-flat envelope by (2) and $G^+$ is $FGT$-flat, we have $F^+$ is $FGT$-flat, and so $F$ is $FGT$-injective. Next we prove that the inclusion $i : F \to M$ is an $FGT$-injective precover of $M$. Let $\psi : F \to M$ with $F$ $FGT$-injective be an arbitrary homomorphism. Note that $\psi(F) \subseteq F$ by the proof above. Define $\zeta : F \to F$ via $\zeta(x) = \psi(x)$ for $x \in F$. Then $i \zeta = \psi$, and so $i : F \to M$ is an $FGT$-injective precover of $M$. In addition, it is clear that the identity map $1_F$ of $F$ is the only homomorphism $g : F \to F$ such that $ig = i$, and hence $i$ is an $FGT$-injective cover of $M$.

(3) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with $E$ injective. Since $L$ has a monic $FGT$-injective cover by (3), we have $L$ is $FGT$-injective. Thus $FGT - \text{Id}(M) \leq 1$ and so $r F G T - I. \dim(R) \leq 1$. □

Recall that $\phi : M \to C$ is said to be a $C$-envelope with the unique mapping property [5] if for any homomorphism $f : M \to C'$ with $C' \subseteq C$, there is a unique homomorphism $g : C \to C'$ such that $g \phi = f$. Dually we have the definition of a $C$-cover with the unique mapping property.

**Theorem 3.7.** The following are equivalent for a right $\Pi$-coherent ring $R$:

1. $r F G T - I. \dim(R) \leq 2$.
2. Every right $R$-module has an $FGT$-injective cover with the unique mapping property.
   Moreover, the above conditions hold if $R$ satisfies that
3. Every left $R$-module has an $FGT$-flat envelope with the unique mapping property.

Proof. (1) $\Rightarrow$ (2). Let $M$ be a right $R$-module. Then $M$ has an $FGT$-injective cover $f : F \to M$ by Theorem 3.4 (2). It is enough to show that, for any $FGT$-injective right $R$-module $G$ and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\ker(g) \to M$ such that $\beta \pi = f$ since $\ker(g) \subseteq \ker(f)$, where $\pi : F/\ker(g) \to F/\ker(g)$ is the natural map. Note that $F/\ker(g)$ is $FGT$-injective by Lemma 3.2 (6) since $FGT - \text{Id}(\ker(g)) \leq 2$. Thus there exists $\alpha : F/\ker(g) \to F$ such that $\beta = f \alpha$, and so $f \alpha \pi = f$. Hence $\alpha \pi$ is an isomorphism since $f$ is a cover. Therefore $\pi$ is monic, and so $g = 0$. 


(2) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then we have the exact sequence $0 \to M \to E^0 \xrightarrow{\varphi} E^1 \xrightarrow{\psi} N \to 0$, where $E^0$ and $E^1$ are injective. Let $\theta : H \to N$ be an $FGT$-injective cover with the unique mapping property. Then there exists $\delta : E^1 \to H$ such that $\psi = \theta \delta$. Thus $\theta \delta \varphi = \psi \varphi = 0 = \theta \theta$, and hence $\delta \varphi = 0$, which implies that $\ker(\psi) = \im(\varphi) \subseteq \ker(\delta)$. Therefore there exists $\gamma : N \to H$ such that $\gamma \psi = \delta$, and so $\theta \gamma \psi = \psi$. Thus $\theta \gamma = 1_N$ since $\psi$ is epic. It follows that $N$ is isomorphic to a direct summand of $H$, and hence $N$ is $FGT$-injective. So $FGT - Id(M) \leq 2$, and hence $rFGT - I.\dim(R) \leq 2$.

(3) $\Rightarrow$ (1). Let $M$ be a right $R$-module. Then we have the exact sequence $0 \to M \to E^0 \to E^1 \to L \to 0$ with $E^0$ and $E^1$ injective. It gives rise to the exactness of the sequence

$$0 \to L^+ \xrightarrow{\psi} (E^1)^+ \xrightarrow{\varphi} (E^0)^+ \to M^+ \to 0.$$ 

Note that $(E^0)^+$ and $(E^1)^+$ are flat. Let $\theta : L^+ \to H$ be the $FGT$-flat envelope with the unique mapping property. Then there exists $\delta : H \to (E^1)^+$ such that $\psi = \theta \delta$. Thus $\varphi \theta \delta = \varphi \psi = 0$, and hence $\varphi \delta = 0$, which implies that $\im(\delta) \subseteq \ker(\varphi) = \im(\psi)$. So there exists $\gamma : H \to L^+$ such that $\psi \gamma = \delta$, and hence $\psi \gamma \theta = \psi$. Thus $\gamma \theta = 1_{L^+}$ since $\psi$ is monic. Consequently $L^+$ is isomorphic to a direct summand of $H$, and hence $L^+$ is $FGT$-flat. Thus $L$ is $FGT$-injective. It follows that $FGT - Id(M) \leq 2$, and so $rFGT - I.\dim(R) \leq 2$. $\square$

Finally, we consider the special case that $R$ is a commutative $\Pi$-coherent ring.

**Proposition 3.8.** Let $R$ be a commutative $\Pi$-coherent ring and $n$ a nonnegative integer. Then $wD(R) \leq n + 1$ if and only if $FGT - I.\dim(R) \leq n$.

**Proof.** It is easy to verify that $FGT - I.\dim(R) = \sup\{pd(F) : F$ is a finitely generated torsionless $R$-module\} by Lemma 3.2 (6) since $R$ is a $\Pi$-coherent ring.

"$\Rightarrow\"$. Let $M$ be a finitely generated torsionless $R$-module. Then $M^*$ is finitely generated by [3, Theorem 1]. So there exists an exact sequence $R^m \to M^* \to 0$ with $m$ a nonnegative integer, which induces an exact sequence $0 \to M^{**} \to R^m$. Thus there exists an exact sequence $0 \to M \to R^m \to L \to 0$ since $M$ is torsionless. Note that $pd(L) \leq wD(R) \leq n + 1$ by [19, Theorem 3.3] since $L$ is finitely presented. Hence $pd(M) \leq n$, and so $FGT - I.\dim(R) \leq n$.

"$\Leftarrow\"$. Let $N$ be a finitely presented $R$-module. Then there is an exact sequence $0 \to K \to F \to N \to 0$ with $F$ finitely generated free and $K$ finitely generated torsionless. Note that $pd(K) \leq FGT - I.\dim(R) \leq n$, and so $pd(N) \leq n + 1$. Thus $wD(R) \leq n + 1$ by [19, Theorem 3.3]. $\square$

**Remark 3.9.** (1) Let $R$ be a commutative $\Pi$-coherent ring. Then Theorem 3.6 and 3.7 characterize respectively those rings such that $wD(R) \leq 2$ and $wD(R) \leq 3$ by Proposition 3.8.
(2) Let \( R = F[x_1, x_2, \ldots, x_n] \), the ring of polynomials in \( n \) indeterminates over a field \( F \). Then \( R \) is a commutative Noetherian ring, and hence a \( \Pi \)-coherent ring. Note that \( wD(R) = n \), and so \( FGT - I. \dim(R) = n - 1 \) by Proposition 3.8. This fact also shows that the inequality \( rFGT - I. \dim(R) \leq wD(R) + r.\pi cD(R) \) in Proposition 2.6 may be strict.

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References


Department of Basic Courses  
Nanjing Institute of Technology  
Hongjing Road, No. 1, Jiangning District  
Nanjing, Jiangsu Province, 211167, P. R. China  
E-mail address: maolx2@hotmail.com