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CONTINUITY FOR MULTILINEAR INTEGRAL OPERATORS ON BESOV SPACES

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ABSTRACT. The continuity for the multilinear operators associated to some non-convolution type integral operators on Besov spaces are obtained. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

1. Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [1-7]). From [2], [7], [13], [15] and [18], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce some multilinear operators associated to certain non-convolution type integral operators and prove the continuity properties for the multilinear operators on the Besov spaces. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

2. Preliminaries and Theorem

In this paper, we will study a class of multilinear operators associated to some non-convolution type integral operators as following.

Let m_j be the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n $(j = 1, \ldots, l)$. Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\gamma| \le m_j} \frac{1}{\gamma!} D^\gamma A_j(y)(x-y)^\gamma.$$

Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

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and

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j; x, y)}{|x-y|^m} F_t(x,y) f(y) dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear operator associated to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$ and $0 \leq \delta < n-1$,

$$||F_t(x,y)|| \leq C|x-y|^{-n+\delta}$$

and

$$||F_t(y,x) - F_t(z,x)|| \leq C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \leq |x-z|$. We define $T(f)(x) = ||F_t(f)(x)||$.

Note that when $m = 0$, T^A is just the multilinear commutator of T and A (see [8-13], [21]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). The purpose of this paper is to study the boundedness properties for the multilinear operator T^A on Besov spaces. In Section 4, some applications of Theorem in this paper are given.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [14], [15])

$$f^{\#}(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $\beta \geq 0$, the Besov space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [18]).

Definition 1. Let $0 < p, q \leq \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha, p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\ p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\ p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,\ p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,\ p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \le q < \infty$, $\alpha \in R$. The central Campanato space is defined by

$$CL_{\alpha,\ q}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{CL_{\alpha,\ q}} < \infty\},$$

where

$$||f||_{CL_{\alpha,\ q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{1/q}.$$

Now we can state our theorem as following.

Theorem 1. *Let $0 < \beta < \min(1/l, \varepsilon/l)$ and $D^\gamma A_j \in \dot{\Lambda}_\beta$ for all γ with $|\gamma| = m_j$ and $j = 1, \ldots, l$. Suppose that T is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $0 \le \eta < n$, $1 < r < n/\eta$ and $1/r - 1/s = \eta/n$. Then T^A is bounded from $L^p(R^n)$ to $\dot{\Lambda}_{l\beta+\delta-n/p}$ for any $n/(l\beta + \delta) \le p \le n/\delta$.*

Theorem 2. *Let $0 < \beta < \min(1/l, \varepsilon/l)$, $1 < q_1 < n/(l\beta + \delta)$, $1/q_2 = 1/q_1 - (l\beta + \delta)/n$, $\max(-n/q_2 - 1, -n/q_2 - \varepsilon) < \alpha \le -n/q_2$ and $D^\gamma A_j \in \dot{\Lambda}_\beta$ for all γ with $|\gamma| = m_j$ and $j = 1, \ldots, l$. Suppose that T^A is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $0 \le \eta < n$, $1 < r < n/\eta$ and $1/r - 1/s = \eta/n$. Then T^A is bounded from $\dot{K}_{q_1}^{\alpha,\ \infty}(R^n)$ to $CL_{-\alpha/n-1/q_2,\ q_2}(R^n)$.*

Remark. Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1 (see [18]). *For $0 < \beta < 1, 1 \le p \le \infty$, we have*

$$||b||_{\dot{\Lambda}_\beta}$$

$$\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^p dx\right)^{1/p}$$

$$\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx \approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|}\int_Q |b(x) - c|^p dx\right)^{1/p}.$$

Lemma 2 (see [5]). *Let A be a function on R^n and $D^\gamma A \in L^q(R^n)$ for $|\gamma| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\gamma A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 3 (see [17]). *For $\alpha < 0, 0 < q < \infty$, we have*

$$||f||_{\dot{K}_q^{\alpha,\infty}} \approx \sup_{\mu \in Z} 2^{\mu\alpha} ||f\chi_{B_\mu}||_{L^q}.$$

Proof of Theorem 1. Without loss of generality, we may assume $l = 2$. By Lemma 1, it is only to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q |T^A(f)(x) - C_0| dx \le C||f||_{L^p}.$$

Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and

$$\tilde{A}_j(x) = A_j(x) - \sum_{|\gamma|=m} \frac{1}{\gamma!} (D^\gamma A_j)_{\tilde{Q}} x^\gamma.$$

Then $R_m(A_j;x,y) = R_m(\tilde{A}_j;x,y)$ and $D^\gamma \tilde{A}_j = D^\gamma A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\gamma| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
&F_t^A(f)(x) \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f(y) dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_2(y) dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_1(y) dy \\
&\quad - \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\gamma_1}}{|x-y|^m} D^{\gamma_1} \tilde{A}_1(y) F_t(x,y) f_1(y) dy \\
&\quad - \sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\gamma_2}}{|x-y|^m} D^{\gamma_2} \tilde{A}_2(y) F_t(x,y) f_1(y) dy \\
&\quad + \sum_{\substack{|\gamma_1|=m_1 \\ |\gamma_2|=m_2}} \frac{1}{\gamma_1!\gamma_2!} \int_{R^n} \frac{(x-y)^{\gamma_1+\gamma_2} D^{\gamma_1}\tilde{A}_1(y) D^{\gamma_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) f_1(y) dy,
\end{aligned}$$

then

$$
\begin{aligned}
&\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| = \left|||F_t^A(f)(x)|| - ||F_t^{\tilde{A}}(f_2)(x_0)||\right| \\
&\le ||F_t^A(f)(x) - F_t^{\tilde{A}}(f_2)(x_0)|| \\
&\le \left\|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) f_1(y) dy\right\| \\
&\quad + \left\|\sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\gamma_1}}{|x-y|^m} D^{\gamma_1}\tilde{A}_1(y) F_t(x,y) f_1(y) dy\right\| \\
&\quad + \left\|\sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\gamma_2}}{|x-y|^m} D^{\gamma_2}\tilde{A}_2(y) F_t(x,y) f_1(y) dy\right\| \\
&\quad + \left\|\sum_{\substack{|\gamma_1| = m_1 \\ |\gamma_2| = m_2}} \frac{1}{\gamma_1!\gamma_2!} \int_{R^n} \frac{(x-y)^{\gamma_1+\gamma_2} D^{\gamma_1}\tilde{A}_1(y) D^{\gamma_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) f_1(y) dy\right\| \\
&\quad + \left|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)\right|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| dx \\
&\le \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left\|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x,y) f_1(y) dy\right\| dx \\
&\quad + \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \\
&\quad \times \int_Q \left\|\sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\gamma_1}}{|x-y|^m} D^{\gamma_1}\tilde{A}_1(y) F_t(x,y) f_1(y) dy\right\| dx \\
&\quad + \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \\
&\quad \times \int_Q \left\|\sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\gamma_2}}{|x-y|^m} D^{\gamma_2}\tilde{A}_2(y) F_t(x,y) f_1(y) dy\right\| dx \\
&\quad + \frac{C}{|Q|^{1+(2\beta+\delta)/n-1/p}}
\end{aligned}
$$

$$\times \int_Q \left\| \sum_{\substack{|\gamma_1| = m_1 \\ |\gamma_2| = m_2}} \int_{R^n} \frac{(x-y)^{\gamma_1+\gamma_2} D^{\gamma_1}\tilde{A}_1(y) D^{\gamma_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) f_1(y)dy \right\| dx$$

$$+ \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left| T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0) \right| dx$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, by Lemma 5 and Lemma 2, we get, for $x \in Q$ and $y \in \tilde{Q}$,

$$\begin{aligned} &|R_m(\tilde{A}_j; x, y)| \\ &\le C|x-y|^m \sum_{|\gamma|=m} \sup_{x \in \tilde{Q}} |D^\gamma A_j(x) - (D^\gamma A_j)_{\tilde{Q}}| \\ &\le C|x-y|^m |Q|^{\beta/n} \sum_{|\gamma|=m} ||D^\gamma A_j||_{\dot{\Lambda}_\beta}, \end{aligned}$$

thus, by the (L^r, L^s)-boundedness of T with $1 < r < p \le n/\delta$ and $1/s = 1/r - \delta/n$, we obtain, using Hölder's inequality,

$$\begin{aligned} I_1 &\le C\prod_{j=1}^2 \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{1}{|Q|^{1+\delta/n-1/p}} \int_Q |T(f_1)(x)|dx \\ &\le C\prod_{j=1}^2 \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left(\int_Q |T(f_1)(x)|^s dx \right)^{1/s} \\ &\le C\prod_{j=1}^2 \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left(\int_{\tilde{Q}} |f(x)|^r dx \right)^{1/r} \\ &\le C\prod_{j=1}^2 \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} ||f||_{L^p} |Q|^{1/r-1/p} \\ &\le C\prod_{j=1}^2 \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}. \end{aligned}$$

For I_2, using Hölder's inequality, we get, for $1 < r < p \le n/\delta$ and $1/s = 1/r - \delta/n$,

$$\begin{aligned} I_2 &\le C \sum_{|\gamma_2|=m_2} ||D^{\gamma_2} A_2||_{\dot{\Lambda}_\beta} \sum_{|\gamma_1|=m_1} \frac{1}{|Q|^{1+(\beta+\delta)/n-1/p}} \int_Q |T(D^{\gamma_1}\tilde{A}_1 f_1)(x)|dx \\ &\le C \sum_{|\gamma_2|=m_2} ||D^{\gamma_2} A_2||_{\dot{\Lambda}_\beta} \end{aligned}$$

$$\times \sum_{|\gamma_1|=m_1} \frac{|Q|^{1-1/s}}{|Q|^{1+(\beta+\delta)/n-1/p}} \left(\int_Q |T(D^{\gamma_1} A - (D^{\gamma_1} A)_{\tilde{Q}}) f_1)(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{|\gamma_2|=m_2} ||D^{\gamma_2} A_2||_{\dot{\Lambda}_\beta}$$

$$\times \sum_{|\gamma_1|=m_1} \frac{|Q|^{1-1/s}}{|Q|^{1+(\beta+\delta)/n-1/p}} \left(\int_{R^n} |(D^{\gamma_1} A(x) - (D^{\gamma_1} A)_{\tilde{Q}}) f_1(x)|^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left(\int_{\tilde{Q}} |f(x)|^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} ||f||_{L^p} |Q|^{1/r-1/p}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}.$$

For I_3, similar to the proof of I_2, we get

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}.$$

Similarly, for I_4, we obtain, for $1 < r < p \leq n/\delta$ and $1/s = 1/r - \delta/n$,

$$I_4$$

$$\leq C \sum_{\substack{|\gamma_1| = m_1 \\ |\gamma_2| = m_2}} \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q |T(D^{\gamma_1} \tilde{A}_1 D^{\gamma_2} \tilde{A}_2 f_1)(x)| dx$$

$$\leq C \sum_{\substack{|\gamma_1| = m_1 \\ |\gamma_2| = m_2}} \frac{|Q|^{1-1/s}}{|Q|^{1+(2\beta+\delta)/n-1/p}} \left(\int_{R^n} |T(D^{\gamma_1} \tilde{A}_1 D^{\gamma_2} \tilde{A}_2 f_1)(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{\substack{|\gamma_1| = m_1 \\ |\gamma_2| = m_2}} \frac{|Q|^{1-1/s}}{|Q|^{1+(2\beta+\delta)/n-1/p}} \left(\int_{R^n} |D^{\gamma_1} \tilde{A}_1(x) D^{\gamma_2} \tilde{A}_2(x) f_1(x)|^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left(\int_{\tilde{Q}} |f(x)|^r dx \right)^{1/r}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)\frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}}||f||_{L^p}|Q|^{1/r-1/p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)||f||_{L^p}.$$

For I_5, We write

$$\begin{aligned}
&F_t^{\tilde{A}}(f_2)(x)-F_t^{\tilde{A}}(f_2)(x_0)\\
=&\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)f_2(y)dy\\
&+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)dy\\
&+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)dy\\
&-\sum_{|\gamma_1|=m_1}\frac{1}{\gamma_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\gamma_1}}{|x-y|^m}F_t(x,y)\right.\\
&\qquad\left.-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\gamma_1}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\times D^{\gamma_1}\tilde{A}_1(y)f_2(y)dy\\
&-\sum_{|\gamma_2|=m_2}\frac{1}{\gamma_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\gamma_2}}{|x-y|^m}F_t(x,y)\right.\\
&\qquad\left.-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\gamma_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\times D^{\gamma_2}\tilde{A}_2(y)f_2(y)dy\\
&+\sum_{\substack{|\gamma_1|=m_1\\|\gamma_2|=m_2}}\frac{1}{\gamma_1!\gamma_2!}\int_{R^n}\left[\frac{(x-y)^{\gamma_1+\gamma_2}}{|x-y|^m}F_t(x,y)-\frac{(x_0-y)^{\gamma_1+\gamma_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\
&\times D^{\gamma_1}\tilde{A}_1(y)D^{\gamma_2}\tilde{A}_2(y)f_2(y)dy\\
=&I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}$$

By the following inequality, for $b\in\dot{\Lambda}_\beta$,

$$|b(x)-b_Q|\leq\frac{1}{|Q|}\int_Q||b||_{\dot{\Lambda}_\beta}|x-y|^\beta dy\leq||b||_{\dot{\Lambda}_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_j}(\tilde{A}_j;x,y)| \le \sum_{|\gamma|=m_j} ||D^\gamma A_j||_{\dot{\Lambda}_\beta}(|x-y|+d)^{m_j+\beta}.$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of F_t,

$$\begin{aligned}
&||I_5^{(1)}|| \\
&\le C\int_{R^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}}\right)\prod_{j=1}^2|R_{m_j}(\tilde{A}_j;x,y)||f(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right) \\
&\quad\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta-2\beta}}\right)|f(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right) \\
&\quad\times\sum_{k=0}^\infty\left(\frac{d}{(2^kd)^{n+1-\delta-2\beta}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon-\delta-2\beta}}\right)\int_{2^k\tilde{Q}}|f(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{\delta/n+2\beta/n-1/p} \\
&\quad\times\sum_{k=0}^\infty\left(2^{k(\delta+2\beta-1-n/p)}+2^{k(\delta+2\beta-\varepsilon-n/p)}\right)||f||_{L^p} \\
&\le C\prod_{j=1}^2\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{\delta/n+2\beta/n-1/p}||f||_{L^p}.
\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)=\sum_{|\eta|<m_j}\frac{1}{\eta!}R_{m_j-|\eta|}(D^\eta\tilde{A}_j;x,x_0)(x-y)^\eta$$

and Lemma 5, we get

$$\begin{aligned}
&||I_5^{(2)}|| \\
&\le C\prod_{j=1}^2\left(\sum_{|\gamma|=m_j}||D^\gamma A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}}|f(y)|dy
\end{aligned}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma|=m_j} ||D^\gamma A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}.$$

Similarly,

$$||I_5^{(3)}|| \leq C \prod_{j=1}^{2} \left(\sum_{|\gamma|=m_j} ||D^\gamma A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}.$$

For $I_5^{(4)}$, similar to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, we obtain,

$$\begin{aligned} ||I_5^{(4)}|| \leq & C \sum_{|\gamma_1|=m_1} \int_{R^n \setminus \tilde{Q}} \left|\left| \frac{(x-y)^{\gamma_1} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\gamma_1} F_t(x_0,y)}{|x_0-y|^m} \right|\right| \\ & \times |R_{m_2}(\tilde{A}_2; x, y)| |D^{\gamma_1} \tilde{A}_1(y)| |f(y)| dy \\ & + C \sum_{|\gamma_1|=m_1} \int_{R^n \setminus \tilde{Q}} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)| \\ & \times \frac{||(x_0-y)^{\gamma_1} F_t(x_0,y)||}{|x_0-y|^m} |D^{\gamma_1} \tilde{A}_1(y)| |f(y)| dy \\ \leq & C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p} \\ & \times \sum_{k=0}^{\infty} \left(2^{k(\delta+2\beta-1-n/p)} + 2^{k(\delta+2\beta-\varepsilon-n/p)} \right) ||f||_{L^p} \\ \leq & C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}. \end{aligned}$$

Similarly,

$$||I_5^{(5)}|| \leq C \prod_{j=1}^{2} \left(\sum_{|\gamma|=m_j} ||D^\gamma A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}.$$

For $I_5^{(6)}$, we get

$$\begin{aligned} & ||I_5^{(6)}|| \\ & \leq C \sum_{|\gamma_1|=m_1, |\gamma_2|=m_2} \int_{R^n \setminus \tilde{Q}} \left|\left| \frac{(x-y)^{\gamma_1+\gamma_2} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\gamma_1+\gamma_2} F_t(x_0,y)}{|x_0-y|^m} \right|\right| \\ & \times |D^{\gamma_1} \tilde{A}_1(y)| |D^{\gamma_2} \tilde{A}_2(y)| |f(y)| dy \end{aligned}$$

$$\leq C \sum_{|\gamma_1|=m_1,|\gamma_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}} \right)$$
$$\times |D^{\gamma_1}\tilde{A}_1(y)||D^{\gamma_2}\tilde{A}_2(y)||f(y)|dy$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta} \right)$$
$$\times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta-2\beta}} \right) |f(y)|dy$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p}||f||_{L^p}.$$

Thus

$$|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)|$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta} \right) |Q|^{\delta/n+2\beta/n-1/p}||f||_{L^p}.$$

and

$$I_5 \leq C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}.$$

This completes the proof of the theorem. □

Proof of Theorem 2. Without loss of generality, we may assume $l = 2$. Fix a ball $B = B(0,d)$, there exists $\mu_0 \in Z$ such that $2^{\mu_0-1} \leq d < 2^{\mu_0}$. Let $\tilde{A}_j(x) = A_j(x) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(D^\gamma A_j)_{B_{\mu_0}} x^\gamma$, then $R_m(A_j;x,y) = R_m(\tilde{A}_j;x,y)$ and $D^\gamma \tilde{A}_j = D^\gamma A_j - (D^\gamma A_j)_{B_{\mu_0}}$ for $|\gamma| = m_j$. We choose x_0 such that $2d < |x_0| < 3d$. It is only to prove that

$$2^{\mu_0(\alpha+n/q_2)} \left(\frac{1}{2^{\mu_0 n}} \int_{|x|\leq 2^{\mu_0}} |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

We write, for $f_1 = f\chi_{4B_{\mu_0}}$ and $f_2 = f\chi_{R^n\setminus 4B_{\mu_0}}$,

$$\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| = \left| ||F_t^A(f)(x)|| - ||F_t^{\tilde{A}}(f_2)(x_0)|| \right|$$
$$\leq \ ||F_t^A(f)(x) - F_t^{\tilde{A}}(f_2)(x_0)|| \leq \left|T^{\tilde{A}}(f_1)(x)\right| + \left|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)\right|,$$

then

$$\begin{aligned}
& 2^{\mu_0(\alpha+n/q_2)}\left(\frac{1}{2^{\mu_0 n}}\int_{|x|\le 2^{\mu_0}}\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right|^{q_2}dx\right)^{1/q_2}\\
\le\ & 2^{\mu_0(\alpha+n/q_2)}\left(\frac{1}{2^{\mu_0 n}}\int_{|x|\le 2^{\mu_0}}\left|T^{\tilde{A}}(f_1)(x)\right|^{q_2}dx\right)^{1/q_2}\\
& +2^{\mu_0(\alpha+n/q_2)}\left(\frac{1}{2^{\mu_0 n}}\int_{|x|\le 2^{\mu_0}}\left|T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)\right|^{q_2}dx\right)^{1/q_2}\\
:=\ & J_1+J_2.
\end{aligned}$$

For J_1, by the (L^{q_1}, L^{q_2})-boundedness of T^A and Lemma 3, we get

$$\begin{aligned}
J_1 &\le C2^{\mu_0(\alpha+n/q_2)}2^{-\mu_0 n/q_2}\left(\int_{R^n}|f_1(x)|^{q_1}dx\right)^{1/q_1}\\
&\le C2^{\mu_0\alpha}||f\chi_{B_{\mu_0}}||_{L^{q_1}}\\
&\le C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}$$

For J_2, similar to the estimates of Theorem 1, we obtain, by Hölder's inequality and recall that $\max(-n/q_2-1, -n/q_2-\varepsilon)<\alpha$, $1/q_2=1/q_1-(2\beta+\delta)/n$,

$$\begin{aligned}
& |T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)|\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)\\
&\quad\times\sum_{k=1}^{\infty}\left(\frac{2^{\mu_0}}{2^{(\mu_0+k)(n+1-\delta-2\beta)}}+\frac{2^{\varepsilon\mu_0}}{2^{(\mu_0+k)(n+\varepsilon-\delta-2\beta)}}\right)\int_{C_{\mu_0+k}}|f(y)|dy\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)\\
&\quad\times\sum_{k=1}^{\infty}\left(\frac{2^{\mu_0}}{2^{(\mu_0+k)(n+1-\delta-2\beta)}}+\frac{2^{\varepsilon\mu_0}}{2^{(\mu_0+k)(n+\varepsilon-\delta-2\beta)}}\right)||f\chi_{\mu_0+k}||_{L^{q_1}}2^{(\mu_0+k)n(1-\frac{1}{q_1})}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)\\
&\quad\times\sum_{k=1}^{\infty}\left(\frac{2^{\mu_0}}{2^{(\mu_0+k)(1-\delta-2\beta)}}+\frac{2^{\varepsilon\mu_0}}{2^{(\mu_0+k)(\varepsilon-\delta-2\beta)}}\right)\frac{2^{(\mu_0+k)(-n/q_1)}}{2^{\alpha(\mu_0+k)}}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}
\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)$$

$$\times\sum_{k=1}^{\infty}\left(2^{k(\delta+2\beta-1-\alpha-n/q_1)}+2^{k(\delta+2\beta-\varepsilon-\alpha-n/q_1)}\right)2^{\mu_0(\delta+2\beta-\alpha-n/q_1)}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)2^{\mu_0(\delta+2\beta-\alpha-n/q_1)}||f||_{\dot{K}^{\alpha,\infty}_{q_1}},$$

thus

$$J_2 \leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)2^{\mu_0(\alpha+n/q_2)}2^{\mu_0(\delta+2\beta-\alpha-n/q_1)}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{\dot{\Lambda}_\beta}\right)||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

This completes the proof of the theorem. □

4. Applications

Now we give some applications of results in this paper.

Application 1. Littlewood-Paley operators.

Fixed $0 \leq \delta < n-1$, $\varepsilon > 0$ and $\mu > 1$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n}\psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y)-\psi(x)| \leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n}\frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [20]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$g_\mu^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right|\right|.$$

It is easily to see that g_ψ^A, S_ψ^A and g_μ^A satisfy the conditions of Theorem 1 and 2(see [8-12]), thus Theorem 1 and 2 hold for g_ψ^A, S_ψ^A and g_μ^A.

Application 2. Marcinkiewicz operators.

Fixed $0 \le \delta < n-1$, $\lambda > 1$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators(see [21]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right\|, \quad \mu_S(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|$$

and

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|,$$

$$\mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

It is easily to see that μ_Ω^A, μ_S^A and μ_λ^A satisfy the conditions of Theorem 1 and 2 (see [8-10], [13]), thus Theorem 1 and 2 hold for μ_Ω^A, μ_S^A and μ_λ^A.

Application 3. Bochner-Riesz operators.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t>0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator (see [14]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorem 1 and 2, thus Theorem 1 and 2 hold for $B_{\delta,*}^A$.

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