Commun. Korean Math. Soc. **22** (2007), No. 2, pp. 315–321

ON THE EXPONENTIAL INEQUALITY FOR NEGATIVE DEPENDENT SEQUENCE

Tae-Sung Kim and Hyun-Chull Kim

Abstract. We show an exponential inequality for negatively associated and strictly stationary random variables replacing an uniform boundedness assumption by the existence of Laplace transforms. To obtain this result we use a truncation technique together with a block decomposition of the sums. We also identify a convergence rate for the strong law of large number.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. We start with definition. A finite family $\{X_1, \ldots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on R^A, g on R^B,

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0.$$

Infinite family of random variables is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [4].

As pointed out and proved by Joag-Dev and Proschan [4], a number of well-known multivariate distributions possess the NA property, such as multinomial distribution, multivariate hypergeometric distribution, negatively correlated normal distribution, permutation distribution, and joint distribution of ranks. Because of their wide applications in multivariate statistical analysis and reliability theory, the concept of negatively associated random variables has received extensive attention recently. We refer to Joag-Dev and Proschan [4] for fundamental properties, Newman [8] for the central limit theorem, Matula [7] for the three series theorem, Roussas [9] for the Hoeffding inequality, Shao [10] for Rosenthal type inequality and the Kolmogorov exponential inequality, Shao and Su [11] for the law of the iterated logarithm, Liang [6] for complete

Received January 23, 2007.

2000 *Mathematics Subject Classification.* 60F15.

Key words and phrases. exponential inequality, negatively associated, strong law of large numbers, convergence rate, decomposition.

This work was partially supported by the Korea Research Foundation Grant funded by the Korean Government(KRF-2006-251-C00026).

convergence of weighted sum and Shao [10] for maximal inequality and weak convergence.

One of the main tools used for characterizing convergence rates in nonparametric estimation has been convenient versions of Bernstein type exponential inequalities. There exist several versions available in the literature for independent sequences of variables with assumptions of uniform boundedness or some, quite relaxed, control on their moments. If the independent case is classical in the literature, the treatment of dependent variables is more recent. The extension to dependent variables was first studied considering m-dependence or different mixing conditions. An inequality for strong mixing variables eventually was proved in Carbon [2] using the same type, as for the treatment of the independent case, of assumptions on the variables, besides the strong mixing: uniformly bounded or some control on the moments. Azuma [1] proved a version of exponential inequalities is also available for martingale differences supposing the variables to be uniformly bounded and, more recently, Lesigne and Volný [5] obtained an extension assuming only the existence of Laplace transforms.

In this article we derive a rate of almost sure convergence for NA random variables without the boundedness assumption, which is replaced by the existence of Laplace transforms(see Theorem in section 2).

2. Notation and main result

Next we introduce the notation that will be used throughout the paper. Let $\{c_n, n \geq 1\}$ be a sequence of nonnegative real numbers such that $c_n \to +\infty$ and define, for each $i, n \geq 1$,

$$\begin{aligned} X_{1,i,n} &= -c_n 1_{(-\infty,-c_n)}(X_i) + X_i 1_{[-c_n,c_n]}(X_i) + c_n 1_{(c_n,+\infty)}(X_i), \\ X_{2,i,n} &= (X_i - c_n) 1_{(c_n,+\infty)}(X_i), \ X_{3,i,n} = (X_i + c_n) 1_{(-\infty,-c_n)}(X_i), \end{aligned} \tag{1}$$

where 1_A represents the characteristic function of the set A. For each $n \geq 1$ fixed, the variables $X_{1,1,n}, \ldots, X_{1,n,n}$ are uniformly bounded. Note that, for each $n \geq 1$ fixed, all these variables are monotone transformations of the initial variable X_n. This implies that a negative association assumption is preserved by this construction. The derivation of a convergence rate will use, besides the truncation introduced before, a convenient decomposition of the sums into blocks. This block decomposition is the mean to an approximation to independence technique on the truncated variables. The tails will be treated directly using Laplace transforms.

Consider a sequence of natural numbers p_n such that, for each $n \geq 1, p_n < n/2$ and define r_n as the greatest integer less or equal to $n/2p_n$. Define then, for $q = 1, 2, 3$, and $j = 1, \ldots, 2r_n$

$$Y_{q,j,n} = \sum_{l=(j-1)p_n+1}^{jp_n} (X_{q,l,n} - E(X_{q,l,n})). \tag{2}$$

Finally, for each $q = 1, 2, 3$, and $n \geq 1$, define

$$Z_{q,n,od} = \sum_{j=1}^{r_n} Y_{q,2j-1,n},\ Z_{q,n,ev} = \sum_{j=1}^{r_n} Y_{q,2j,n},$$

$$R_{q,n} = \sum_{l=2r_np_n+1}^{n} (X_{q,l,n} - E(X_{q,l,n})). \tag{3}$$

Note that $Y_{q,j,n}$, $Z_{q,n,od}$, $Z_{q,n,ev}$ and $R_{q,n}$ are divided into the bounded terms, corresponding to the index $q = 1$, and the unbounded terms that correspond to the indices $q = 2$ and 3.

The following theorem is the main result which provides strong convergence rate:

Theorem. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables. Suppose that, for some $\alpha > 0$*

$$\epsilon = 18(\frac{\alpha p_n \log n}{n})^{1/2} c_n \tag{4}$$

and that there exists $\delta > \alpha$ satisfying

$$\sup_{|t|\leq\delta} E(e^{tX_1}) \leq M_\delta < \infty. \tag{5}$$

Then, for n large enough,

$$P(|\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))| > \epsilon) \leq (4 + \frac{M_\delta n^2}{9\alpha^3 p_n (\log n)^3})n^{-\alpha}. \tag{6}$$

3. Proof of Theorem

We start with a general lemma used to control some of the terms appearing in the course of proof.

Lemma 3.1 ([3]). *Let W be a centered random variable. If there exist $a, b \in \mathbb{R}$ such that $P(a \leq W \leq b) = 1$, then, for every $\lambda > 0$,*

$$E(e^{\lambda W}) \leq \exp(\frac{\lambda^2(b-a)^2}{8}). \tag{7}$$

Lemma 3.2 ([8]). *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables. Then*

$$E(\exp\sum_{j=1}^{n} X_j) \leq \prod_{j=1}^{n}[E(e^{X_j})]. \tag{8}$$

From definitions (1) *and* (2) *it is obvious that $|Y_{1,j,n}| \leq 2p_nc_n$, $j = 1, \ldots, r_n$. This enables us to use Lemma 3.1 to control the Laplace transform of these variables. A straightforward application of Lemmas 3.1 and 3.2 produces the following upper bounds.*

Lemma 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables. If $Y_{1,j,n}, j = 1, \dots, 2r_n$ are defined by (2) then, for every $\lambda > 0$*

$$E(e^{\lambda Z_{1,n,od}}) \leq \exp(\lambda^2 n p_n c_n^2) \tag{9}$$

and analogously for the term corresponding to $Z_{1,n,ev}$.

Proof. According to (3) and the fact that the variables defined in (1) are NA we have, from a direct application of Lemmas 3.1 and 3.2

$$\begin{aligned} E(e^{\lambda Z_{1,n,od}}) &= E(\exp \lambda(\sum_{j=1}^{r_n} Y_{1,2j-1,n})) \\ \leq \prod_{j=1}^{r_n} E(e^{\lambda Y_{1,2j-1,n}}) &\leq \exp(\frac{\lambda^2 (4c_n p_n)^2}{8} \cdot r_n) \leq \exp(\lambda^2 n p_n c_n^2) \end{aligned}$$

since $2r_n p_n \leq n$. Similarly, the result for the term corresponding to $Z_{1,n,ev}$ is derived. □

We may now prove an exponential inequality for the sum of odd indexed or even indexed terms.

Lemma 3.4. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables. Then, for every $\epsilon \in (0, 1)$,*

$$P(\frac{1}{n}|Z_{1,n,od}| > \frac{\epsilon}{9}) \leq 2\exp(-\frac{n\epsilon^2}{324 p_n c_n^2}) \tag{10}$$

and analogously for the term corresponding to $Z_{1,n,ev}$.

Proof. Applying Markov's inequality and using the previous lemma we find that, for every $\lambda > 0$,

$$\begin{aligned} P(\frac{1}{n}|Z_{1,n,od}| > \frac{\epsilon}{9}) &= P(\frac{1}{n}Z_{1,n,od} > \frac{\epsilon}{9}) + P(\frac{1}{n}(-Z_{1,n,od}) > \frac{\epsilon}{9}) \\ &= P(e^{\lambda Z_{1,n,od}} > e^{\frac{\lambda n \epsilon}{9}}) + P(e^{-\lambda Z_{1,n,od}} > e^{\frac{\lambda n \epsilon}{9}}) \\ &\leq 2\exp(\lambda^2 n p_n c_n^2 - \lambda n \frac{\epsilon}{9}). \end{aligned} \tag{11}$$

Optimizing the exponent in the term of this upper bound we find $\lambda = \epsilon/18 p_n c_n^2$, so that this exponent becomes equal to $-n\epsilon^2/324 p_n c_n^2$. The proof is complete. □

To complete the treatment of the bounded terms it remains to consider the sum corresponding to the indices following $2r_n p_n$, that is, $R_{1,n}$.

Lemma 3.5. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables and $R_{1,n}$ be defined as in (3). Suppose that*

$$\frac{n}{c_n} \to +\infty. \tag{12}$$

Then, for n large enough and every $\epsilon > 0$, we have

$$P(|R_{1,n}| > n\epsilon) = 0. \tag{13}$$

Proof. As $R_{1,n} = \sum_{l=2r_n p_n+1}^{n}(X_{1,l,n} - E(X_{1,l,n}))$ it follows that

$$|R_{1,n}| \leq 2(n - 2r_n p_n)c_n \leq 2c_n,$$

according to the construction of the sequences r_n and p_n. Now

$$P(|R_{1,n}| > n\epsilon) \leq P(2 > n\epsilon/c_n)$$

and, using (12), this is zero for n large enough. □

In order to prove the almost sure convergence of $\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n} - E(X_{1,i,n}))$ and identify a convergence rate we will allow ϵ in the previous lemmas to depend on n in such a way as to define a convergent series in the upper bound.

Taking ϵ as in (4) and tracking back the proof Lemma 3.4, the choice of ϵ means that the optimizing value of λ would now be $\lambda = \frac{1}{c_n}(\alpha \log n/np_n)^{1/2}$.

Inserting these expressions in (11) and repeating the arguments would lead to the following result.

Lemma 3.6. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables. Then, for ϵ as in (4), we have*

$$P(\frac{1}{n}|Z_{1,n,od}| > \frac{\epsilon}{9}) \leq 2n^{-\alpha}, \tag{14}$$

and analogously for $Z_{1,n,ev}$.

As for the term $R_{1,n}$, it is treated exactly as in Lemma 3.5.

Repeating the arguments used in that lemma we would be left with the term $P(2 > n\epsilon/c_n)$. But $n\epsilon/c_n \sim (np_n \log n)^{1/2} \to +\infty$, so the argument of Lemma 3.5 still applies.

We may now state a theorem summarizing the partial results described in the lemmas of this section.

Theorem. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables. Then, for ϵ as in (4) and n large enough,*

$$P(\frac{1}{n}|\sum_{i=1}^{n}(X_{1,i,n} - E(X_{1,i,n}))| > \frac{\epsilon}{3}) \leq 2n^{-\alpha}. \tag{15}$$

Proof. It suffices to write

$$\begin{aligned} & P(\frac{1}{n}|\sum_{i=1}^{n}(X_{1,i,n} - E(X_{1,i,n}))| > \epsilon) \\ \leq\ & P(\frac{1}{n}|Z_{1,n,od}| > \frac{\epsilon}{3}) + P(\frac{1}{n}|Z_{1,n,ev}| > \frac{\epsilon}{3}) + P(|R_{1,n}| > \frac{n\epsilon}{3}) \end{aligned}$$

and apply the previous lemmas. □

Note that the result just proved implies the convergence to zero of the upper bound in (15) but implies the almost sure convergence to zero of

$$\frac{1}{n}|\sum_{i=1}^{n}(X_{1,i,n}-E(X_{1,i,n}))|$$

only if we may choose $\alpha > 1$.

The variables $X_{2,i,n}$ and $X_{3,i,n}$ are NA but not bounded, even for fixed n. This means that Lemma 3.1 may not be applied to the sums of such terms. But we may note that these variables depend only on the tails of the distribution of the original variables. So, by controlling the decrease rate of these tails we may prove an exponential inequality for sums of $X_{2,i,n}$ or $X_{3,i,n}$. For this control we will not use of the block decomposition of the sums $\sum_{i=1}^{n}(X_{q,i,n}-E(X_{q,i,n}))$ as the condition derived would be exactly the same as the one obtained with a direct treatment.

We have, for $q = 2, 3$, recalling that the variables are identically distributed,

$$\begin{aligned} & P(|\sum_{i=1}^{n}(X_{q,i,n}-E(X_{q,i,n})) > n\epsilon) \\ \leq \; & nP(X_{q,1,n}-E(X_{q,1,n}))| > \epsilon) \\ \leq \; & \frac{n}{\epsilon^2}Var(X_{q,1,n}) \leq \frac{n}{\epsilon^2}E(X_{q,1,n}^2). \end{aligned}$$

Lemma 3.7. *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of NA random variables such that there exists $\delta > 0$ satisfying*

$$\sup_{|t|\leq\delta} E[e^{tX_1}] \leq M_\delta < +\infty.$$

Then, for $t \in (0, \delta)$,

$$P(|\sum_{i=1}^{n}(X_{q,i,n}-E(X_{q,i,n}))| > n\epsilon) \leq \frac{2M_\delta n e^{-tc_n}}{t^2\epsilon^2}, \quad q = 2, 3. \tag{16}$$

Proof. According to the inequality stated before this lemma it remains to control $E(X_{q,1,n}^2)$ for $q = 2, 3$. Let us fix $q = 2$, the other possible choice being treated analogously. We will set $\bar{F}(x) = P(X_1 > x)$. Now, using Markov's inequality it follows that, for $t \in (0, \delta)$, $\bar{F}(x) \leq e^{-tx}E(e^{tX_1}) \leq M_\delta e^{-tx}$. Writing the mathematical expectation as a Stieljes integral and integrating by parts we find

$$E(X_{2,1,n}^2) = -\int_{(c_n,\infty)}(x-c_n)^2\bar{F}(dx) = \int_{c_n}^{+\infty}2(x-c_n)\bar{F}(x)dx \leq 2M_\delta\frac{e^{-tc_n}}{t^2}$$

from which the result (16) follows. □

Note that for this step the negative association of variables is irrelevant. Finally, we show the main theorem by using the above results.

Proof of Theorem. Separate the sum in left of (6) into three terms as in section 2 and apply (15) and (16) with $\epsilon/3$ in place of ϵ for the latter. Then choose $t = \alpha$ and $c_n = \log n$ in (16), so that the exponents are equal, and recalculate ϵ for this choice of c_n. Then the result (6) follows. □

Remark. Notice that this result requires some extra assumptions on the choice of α in order to derive the almost sure convergence with rate

$$\epsilon^{-1} \sim n^{1/2}/p_n^{1/2}(\log n)^{3/2}.$$

References

[1] K. Azuma, *Weighted sums of certain dependent random variables*, Tohoku Math. J. **19** (1967), 357–367.
[2] M. Carbon, *Une nouvelle inégalité de grandes deviations Applications,* Publ. IRMA Lille **32** XI, 1993.
[3] L. Devroye, *Exponential inequalities in nonparametric estimation,* In : Roussas, G. (Ed.) Nonparametric Functional Estimation and Related Topics. Kluwer Academic Publishers, Dordrecht, pp. 31–44, (1991).
[4] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications,* Ann. Statist. **11** (1983), 286–295.
[5] E. Lesigne and D. Volný, *Large deviations for martingales,* Stochastic. Process. Appl. **96** (2001), 143–159.
[6] H. Y. Liang, *Complete convergence for weighted sums of negatively associated ranldom variables,* Statist. Prob. Lett. **48** (2000), 317–325.
[7] P. Matula, *A note on the almost sure convergence of sums of negatively dependent random variables,* Statist. Prob. Lett. **15** (1992), 209–213.
[8] C. M. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables,* In: Tong, Y. L.(Ed.), Inequalities in Statistics and Probability. IMS Lectures Notes-Monograph Series **5** (1984), 127–140, Hayward CA.
[9] G. G. Roussas, *Exponential probability inequalities with some applications,* In:Ferguson, T. S., Shapely, L. S., MacQueen, J. B.(Ed.),Statistics, Probability and Game Theory. IMS Lecture Notes-Monograph Series, **30** (1996), 303–309, Hayward, CA.
[10] Q. M. Shao, *A comparison theorem on maximum inequalities between negatively associated and independent random variables,* J. Theor. Probab. **13** (2000), 343–356.
[11] Q. M. Shao and C. Su, *The law of the iterated logarithm for negatively associated random variables,* Stoch. Process. **83** (1999), 139–148.

Tae-Sung Kim
Department of Mathematics
WonKwang University
Jeonbuk 570-749, Korea
E-mail address: starkim@wonkwang.ac.kr

Hyun-Chull Kim
Department of Mathematics Education
Daebul University
Jeonnam 526-720, Korea
E-mail address: kimhc@mail.daebul.ac.kr