RESULTS ON AN INTUITIONISTIC FUZZY TOPOLOGICAL SPACE

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ABSTRACT. In this paper, we introduce the concepts of r-gp-open map, weakly r-gp-open map, intuitionistic fuzzy r-compactness, nearly intuitionistic fuzzy r-compactness and almost intuitionistic fuzzy r-compactness defined by intuitionistic gradations of openness, and obtain some characterizations.

1. INTRODUCTION

In [8], Hazra, Samanta and Chattopadhyay introduced the concept of fuzzy topology redefined by a gradation of openness and investigated some fundamental properties, which is an extended concept of fuzzy topological spaces [2] in Chang’s sense. Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh’s sense [12]. Çoker [4] introduced Chang’s type intuitionistic fuzzy topological spaces, which it is an extended concept of fuzzy topological spaces redefined by a gradation of openness. In [10], Mondal and Samanta introduced and investigated the concept of intuitionistic gradation of openness which is a generalization of the concept of gradation of openness defined by Chattopadhyay et. al. In [9], we introduced the concepts of r-closure and r-interior defined by intuitionistic gradation of openness, which are the extended concepts of fuzzy closure and fuzzy interior of a fuzzy set [5, 6, 7].

In this paper, we introduce the concepts of r-gp-open map, weakly r-gp-open map, intuitionistic fuzzy r-compactness, nearly intuitionistic fuzzy r-compactness and almost intuitionistic fuzzy r-compactness defined by intuitionistic gradations.

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of openness, and obtain some characterizations in terms of r-closure and r-interior operators defined by intuitionistic gradation of openness

2. PRELIMINARIES

Let \( X \) be a set and \( I = [0, 1] \) be the unit interval of the real line. \( I^X \) will denote the set of all fuzzy sets of \( X \). \( 0_X \) and \( 1_X \) will denote the characteristic functions of \( \phi \) and \( X \), respectively.

**Definition 1** ([3,11]). Let \( X \) be a non-empty set and \( \tau : I^X \to I \) be a mapping satisfying the following conditions:

1. \( \tau(0_X) = \tau(1_X) = 1; \)
2. \( \forall A, B \in I^X, \tau(A \cap B) \geq \tau(A) \wedge \tau(B); \)
3. For every subfamily \( \{A_i : i \in J\} \subseteq I^X, \tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i). \)

Then the mapping \( \tau : I^X \to I \) is called a fuzzy topology (or gradation of openness [3]) on \( X \). We call the ordered pair \( (X, \tau) \) a fuzzy topological space. The value \( \tau(A) \) is called the degree of openness of \( A \).

**Definition 2** ([1]). An intuitionistic fuzzy set \( A \) in a set \( X \) is an object having the form \( A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \} \) where the functions \( \mu_A : X \to I \) and \( \gamma_A : X \to I \) denote the degree of membership and the degree of nonmembership of each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \).

**Definition 3** ([10]). An intuitionistic gradation of openness (briefly IGO) of fuzzy subsets of a set \( X \) is an ordered pair \( (\tau, \tau^*) \) of functions \( \tau, \tau^* : I^X \to I \) such that

- (IGO1) \( \tau(A) + \tau^*(A) \leq 1 \), for all \( A \in I^X; \)
- (IGO2) \( \tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0; \)
- (IGO3) \( \forall A, B \in I^X, \tau(A \cap B) \geq \tau(A) \wedge \tau(B) \) and \( \tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B); \)
- (IGO4) For every subfamily \( \{A_i : i \in J\} \subseteq I^X, \tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i) \) and \( \tau^*(\bigcup_{i \in J} A_i) \leq \bigvee_{i \in J} \tau^*(A_i). \)

Then the triplet \( (X, \tau, \tau^*) \) is called an intuitionistic fuzzy topological space (briefly IFTS) on \( X \). \( \tau \) and \( \tau^* \) may be interpreted as gradation of openness and gradation of nonopenness, respectively.

**Definition 4** ([10]). Let \( X \) be a nonempty set and \( \mathcal{F}, \mathcal{F}^* : I^X \to I \) be two functions satisfying
(IGC1) $\mathcal{F}(A) + \mathcal{F}^*(A) \leq 1$, for all $A \in I^X$;
(IGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1, \mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = 0$;
(IGC3) $\forall A, B \in I^X, \mathcal{F}(A \cup B) \geq \mathcal{F}(A) \land \mathcal{F}(B)$ and $\mathcal{F}^*(A \cup B) \leq \mathcal{F}^*(A) \lor \mathcal{F}^*(B)$;
(IGC4) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\mathcal{F}(\bigwedge_{i \in J} A_i) \geq \bigwedge_{i \in J} \mathcal{F}(A_i)$ and $\mathcal{F}^*(\bigvee_{i \in J} A_i) \leq \bigvee_{i \in J} \mathcal{F}^*(A_i)$.

Then the ordered pair $(\mathcal{F}, \mathcal{F}^*)$ is called an intuitionistic gradation of closedness [10] (briefly IGC) on $X$. $\mathcal{F}$ and $\mathcal{F}^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

**Theorem 5 ([10]).** Let $X$ be a nonempty set. If $(\tau, \tau^*)$ is an IGO on $X$, then the pair $(\mathcal{F}, \mathcal{F}^*)$, defined by $\mathcal{F}_\tau(A) = \tau(A^c)$, $\mathcal{F}^*_{\tau^*}(A) = \tau^*(A^c)$ where $A^c$ denotes the complement of $A$, is an IGC on $X$. And if $(\mathcal{F}, \mathcal{F}^*)$ is an IGC on $X$, then the pair $(\tau_{\mathcal{F}}, \tau^*_{\mathcal{F}^*})$, defined by $\tau_{\mathcal{F}}(A) = \mathcal{F}(A^c)$, $\tau^*_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c)$ is an IGO on $X$.

**Definition 6 ([10]).** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs. A mapping $f : X \rightarrow Y$ is called a gp-map if $\tau(f^{-1}(A)) \geq \sigma(A)$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$ for every $A \in I^Y$.

**Definition 7 ([9]).** Let $(X, \tau, \tau^*)$ be an IFTS, $A \in I^X$ and $r \in [0, 1)$. Then the r-closure (resp., r-interior) of $A$, denoted by $\text{cl}_r A$ (resp., $i_r A$), is defined by $\text{cl}_r A = \cap\{K \in I^X : \mathcal{F}_\tau(K) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(K) \leq r, A \subseteq K\}$ (resp., $i_r A = \cup\{K \in I^X : \tau(K) > 0 \text{ and } \tau^*(K) \leq r, K \subseteq A\}$).

**Theorem 8 ([9]).** Let $(X, \tau, \tau^*)$ be an IFTS and $A, B \in I^X$, $r \in [0, 1)$. Then

1. $\text{cl}_r(0_X) = 0_X$,
2. $A \subseteq \text{cl}_r A$,
3. $\text{cl}_r A = \text{cl}_r(\text{cl}_r A)$,
4. $\text{cl}_r A \cup \text{cl}_r B \subseteq \text{cl}_r(A \cup B)$.

**Definition 9 ([9]).** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs, and $r \in [0, 1)$. A mapping $f : X \rightarrow Y$ is a r-gp-map iff $\sigma(A) \leq \tau(f^{-1}(A))$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$, for each a fuzzy set $A$ in $Y$ such that $\sigma(A) > 0$ and $\sigma^*(A) \leq r$. A mapping $f : X \rightarrow Y$ is a weakly r-gp-map iff $\tau(f^{-1}(A)) > 0$ and $\tau^*(f^{-1}(A)) \leq r$, for each fuzzy set $A \in I^Y$ such that $\sigma(A) > 0$ and $\sigma^*(A) \leq r$.

**Theorem 10 ([9]).** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs, $r \in [0, 1)$. If a mapping $f : X \rightarrow Y$ is a weakly r-gp-map, then we have

1. $f(\text{cl}_r A) \subseteq \text{cl}_r f(A)$ for every $A \in I^X$,,
(2) \( cl_r(f^{-1}(A)) \subseteq f^{-1}(cl_r A) \) for every \( A \in I^Y \),
(3) \( f^{-1}(i_r A) \subseteq i_r(f^{-1}(A)) \) for every \( A \in I^Y \).

3. Main Results

We introduce the concepts of \( r \)-gp-open map, weakly \( r \)-gp-open map and several types compactness in intuitionistic topological spaces and investigate some properties of them.

Definition 11. Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTs, \( r \in [0,1)\). A mapping \( f : X \to Y \) is called

(1) a \( r \)-gp-open map if \( \tau(A) \leq \sigma(f(A)) \) and \( \sigma^*(f(A)) \leq \tau^*(A) \), for every \( A \in I^X \) such that \( \tau(A) > 0 \) and \( \tau^*(A) \leq r \);
(2) a \( r \)-gp-closed map if \( F_r(A) \leq F_{\sigma}(f(A)) \) and \( F_{\sigma^*}(f(A)) \leq F_{\tau^*}(A) \), for every \( A \in I^X \) such that \( F_r(A) > 0 \) and \( F_{\tau^*}(A) \leq r \).

Definition 12. Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTs, \( r \in [0,1)\). A mapping \( f : X \to Y \) is called

(1) a weakly \( r \)-gp-open map if \( \sigma(f(A)) > 0 \) and \( \sigma^*(f(A)) \leq r \), for every \( A \in I^X \) such that \( \tau(A) > 0 \) and \( \tau^*(A) \leq r \);
(2) a weakly \( r \)-gp-closed map if \( F_{\sigma}(f(A)) > 0 \) and \( F_{\sigma^*}(f(A)) \leq r \), for every \( A \in I^X \) such that \( F_r(A) > 0 \) and \( F_{\tau^*}(A) \leq r \).

Every \( r \)-gp-open (resp., \( r \)-gp-closed) maps are weakly \( r \)-gp-open (resp., \( r \)-gp-closed) maps but the converse may not be true.

Example 13. Let \( X = I \) and let \( N \) denote the set of all natural numbers. For each \( n \in N \), we consider \( \mu_n \in I^X \) such that \( \mu_n(x) = \frac{1}{n} x \) for \( x \in X \).

Define \( \tau, \tau^* : I^X \to I \) by
\[
\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0; \\
\tau(\mu_n) = \frac{n}{n+2}, \tau^*(\mu_n) = \frac{2}{n+2} \quad \text{for each} \quad n \in N; \\
\tau(\mu) = 0, \tau^*(\mu) = 1 \quad \text{for all other fuzzy set} \quad \mu \in I^X.
\]
And define \( \sigma, \sigma^* : I^X \to I \) by
\[
\sigma(0_X) = \sigma(1_X) = 1, \sigma^*(0_X) = \sigma^*(1_X) = 0; \\
\sigma(\mu_n) = \frac{1}{n+1}, \sigma^*(\mu_n) = \frac{1}{n+1} \quad \text{for each} \quad n \in N;
\]
\[ \sigma(\mu) = 0, \sigma^*(\mu_r) = 1 \] for all other fuzzy set \( \mu \in I^X \).

Then the pairs \((\tau, \tau^*)\) and \((\sigma, \sigma^*)\) are two intuitionistic gradations of openness on \(X\).

Let \( r = \frac{1}{2} \) and \( f : (X, \tau, \tau^*) \to (X, \sigma, \sigma^*) \) be the identity mapping. Then \( f \) is a weakly r-gp-open map but not a r-gp-open map. In the same way, we can show that a weakly r-gp-closed map may not be a r-gp-closed map.

**Theorem 14.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, and \( r \in [0, 1) \). If \( f : X \to Y \) is a weakly r-gp-open map, then \( f(i_r(A)) \subseteq i_r(f(A)) \) for every \( A \in I^X \).

**Proof.** For \( A \in I^X \), we have that

\[
\begin{align*}
f(i_r(A)) &= f(\{U \in I^X : \tau(U) > 0 \text{ and } \tau^*(U) \leq r, U \subseteq A\}) \\
&\subseteq \{U \in I^Y : \tau(U) > 0 \text{ and } \tau^*(U) \leq r, f(U) \subseteq f(A)\} \\
&\subseteq \{K \in I^Y : \sigma^*(K) > 0 \text{ and } \sigma^*(K) \leq r, f(U) \subseteq f(A)\} \\
&= i_r(f(A)).
\end{align*}
\]

Thus the proof is obtained. \( \square \)

**Corollary 1.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, and \( r \in [0, 1) \). If \( f : X \to Y \) is a r-gp-open map then \( f(i_r(A)) \subseteq i_r(f(A)) \) for every \( A \in I^X \).

**Theorem 15.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, and \( r \in [0, 1) \). If \( f : X \to Y \) is an injective weakly r-gp-closed map, then \( cl_r(f(A)) \subseteq f(cl_r(A)) \) for every \( A \in I^X \).

**Proof.** Let \( A \in I^X \); then since \( f \) is an injective weakly r-gp-closed map, we have

\[
\begin{align*}
f(cl_r(A)) &= f(\{U \in I^X : F_r(U) > 0 \text{ and } F^*_{\tau}(U) \leq r, A \subseteq U\} \\
&= \{f(U) \in I^Y : F_{\tau}(U) > 0 \text{ and } F^*_{\tau}(U) \leq r, f(A) \subseteq f(U)\} \\
&\supseteq \{f(U) \in I^X : F_{\tau}(f(U)) > 0 \text{ and } F^*_{\tau}(f(U)) \leq r, f(A) \subseteq f(U)\} \\
&\supseteq cl_r(f(A)).
\end{align*}
\]

Thus it follows \( cl_r(f(A)) \subseteq f(cl_r(A)) \) for every \( A \in I^X \). \( \square \)

Since every r-gp-close map is a weakly r-gp-closed map, we get the following theorem.

**Theorem 16.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, and and \( r \in [0, 1) \). If \( f : X \to Y \) is an injective r-gp-closed map, then \( cl_r(f(A)) \subseteq f(cl_r(A)) \) for every \( A \in I^X \).
Definition 17. Let \((X, \tau, \tau^*)\) be an IFTS, and \(r \in [0, 1)\). A family \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) is called an intuitionistic fuzzy \(r\)-cover if \(\bigcup_{i \in J} A_i = 1_X\).

Definition 18. For \(r \in [0, 1)\), an IFTS \((X, \tau, \tau^*)\) is said to be intuitionistic fuzzy \(r\)-compact if for every intuitionistic fuzzy \(r\)-cover \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) of \(X\), there exists a finite subset \(J_o\) of \(J\) such that \(\bigcup_{i \in J_o} A_i = 1_X\).

Theorem 19. Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, and \(r \in [0, 1)\) and let \(f : X \to Y\) be a surjective weakly \(r\)-gp-map. If \((X, \tau, \tau^*)\) is intuitionistic fuzzy \(r\)-compact, then so is \((Y, \sigma, \sigma^*)\).

Proof. Obvious. \(\square\)

Definition 20. For \(r \in [0, 1)\), an IFTS \((X, \tau, \tau^*)\) is called nearly intuitionistic fuzzy \(r\)-compact if for every intuitionistic fuzzy \(r\)-cover \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) of \(X\), there exists a finite subset \(J_o\) of \(J\) such that \(\bigcup_{i \in J_o} i_r(\text{cl}_r(A_i)) = 1_X\).

Theorem 21. For \(r \in [0, 1)\), an intuitionistic fuzzy \(r\)-compact IFTS \((X, \tau, \tau^*)\) is nearly intuitionistic fuzzy \(r\)-compact.

Proof. Let \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) be an intuitionistic fuzzy \(r\)-cover of \(X\); then there exists a finite subset \(J_o\) of \(J\) such that \(\bigcup_{i \in J_o} A_i = 1_X\). Since \(\tau(A_i) > 0\) for all \(i \in J\), by Theorem 2.8 we have \(A_i = i_r(\text{cl}_r(A_i))\). Thus \(1_X = \bigcup_{i \in J_o} A_i \subseteq \bigcup_{i \in J_o} i_r(\text{cl}_r(A_i))\). Hence \((X, \tau, \tau^*)\) is nearly intuitionistic fuzzy \(r\)-compact. \(\square\)

Remark 22. In Theorem 3.12, the converse of implication may not be true. For if \((X, \tau, \tau^*)\) is an IFTS and \(\tau^*(\mu) = 0\) for all \(\mu \in I^X\), then the \((X, \tau, \tau^*)\) is a fuzzy topological space in Sostak’s sense. Since a nearly fuzzy compact space is not fuzzy compact, so we can say a nearly intuitionistic fuzzy \(r\)-compact IFTS is not always intuitionistic fuzzy \(r\)-compact.

Definition 23. For \(r \in [0, 1)\), an IFTS \((X, \tau, \tau^*)\) is said to be almost intuitionistic fuzzy \(r\)-compact if for every intuitionistic fuzzy \(r\)-cover \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) of \(X\), there exists a finite subset \(J_o\) of \(J\) such that \(\bigcup_{i \in J_o} \text{cl}_r(A_i) = 1_X\).

Theorem 24. For \(r \in [0, 1)\), a nearly intuitionistic fuzzy \(r\)-compact IFTS \((X, \tau, \tau^*)\) is almost intuitionistic fuzzy \(r\)-compact.

Proof. Let \(\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq r, i \in J\}\) be an intuitionistic fuzzy
r-cover $X$; then there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i\in J_0} i_r(\text{cl}_r(A_i)) = 1_X$.

Since $i_r(\text{cl}_r(A_i)) \subseteq \text{cl}_r(A_i)$ for each $i \in J$, $1_X = \bigcup_{i\in J_0} i_r(\text{cl}_r(A_i)) \subseteq \bigcup_{i\in J_0} \text{cl}_r(A_i)$. So $\bigcup_{i\in J_0} \text{cl}_r(A_i) = 1_X$. Hence $(X, \tau, \tau^*)$ is almost intuitionistic fuzzy r-compact. □

As Remark 3.13, we can say that an almost intuitionistic fuzzy r-compact IFTS is not always a nearly intuitionistic fuzzy r-compact IFTS.

**Theorem 25.** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs, $r \in [0,1)$ and $f : X \to Y$ a surjective, weakly r-gp-map. If $X$ is almost intuitionistic fuzzy r-compact, then so is $Y$.

**Proof.** It is obvious. □

**Corollary 2.** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs, and $r \in [0,1)$ and let $f : X \to Y$ be a surjective, weakly r-gp-map. If $X$ is nearly intuitionistic fuzzy r-compact, then $Y$ is nearly intuitionistic fuzzy r-compact.

**Theorem 26.** Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two IFTSs, $r \in [0,1)$ and $f : X \to Y$ a surjective, weakly r-gp-map and r-gp-open map. If $X$ is nearly intuitionistic fuzzy r-compact, then so is $Y$.

**Proof.** Let $\{A_i \in I^Y : \sigma(A_i) > 0 \text{ and } \sigma^*(A_i) \leq r, i \in J\}$ be an intuitionistic fuzzy r-cover of $Y$. Then $1_Y = f^{-1}(1_Y) = \bigcup_{i \in J} f^{-1}(A_i)$. Since $f$ is a weakly r-gp-map, we have an intuitionistic fuzzy r-cover $\{f^{-1}(A_i) \in I^X : \tau(f^{-1}(A_i)) > 0 \text{ and } \tau^*(f^{-1}(A_i)) \leq r, i \in J\}$ of $X$. And since $X$ is nearly intuitionistic fuzzy r-compact, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0} i_r(\text{cl}_r(f^{-1}(A_i))) = 1_X$. Thus by hypothesis, we have

$$1_Y = \bigcup_{i \in J_0} i_r(\text{cl}_r(f^{-1}(A_i))))$$

$$\subseteq \bigcup_{i \in J_0} i_r(f \text{cl}_r(f^{-1}(A_i)))$$

$$\subseteq \bigcup_{i \in J_0} i_r(f(f^{-1}(\text{cl}_r(A_i))))$$

$$= \bigcup_{i \in J_0} i_r(\text{cl}_r(A_i)).$$

Thus $(Y, \sigma, \sigma^*)$ is nearly intuitionistic fuzzy r-compact. □

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