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STRONG AND WEAK DOMINATION IN FUZZY GRAPHS

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Abstract. In this paper, we introduce the concept of strong and weak domination in fuzzy graphs, and provide some examples to explain various notions introduced. Also some properties discussed.

1. Introduction

The basic idea of a fuzzy relation was defined by Zadeh [7]. Rosenfeld [5] considered fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Somasundaram et al [6] introduced the concept of domination in fuzzy graphs. In this view, we obtain the analog of strong and weak domination in fuzzy graphs.

2. Preliminaries

We recall some basic definitions, which can be found in [2,3,5,6,7]. A fuzzy relation on a finite and non-empty set V is a function $\rho : V \times V \to [0,1]$. A fuzzy graph $G = (V, \mu, \rho)$ is a non empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\rho : V \times V \to [0,1]$ such that $\rho(v,w) \leq \mu(v) \wedge \mu(w)$ for all $v, w \in V$, where $\mu(v) \wedge \mu(w)$ denotes the minimum of $\mu(v)$ and $\mu(w)$. A fuzzy graph $G = (V, \mu, \rho)$ is complete if $\rho(v,w) = \mu(v) \wedge \mu(w)$ for all $v, w \in V$. The order and

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size of a fuzzy graph $G = (V, \mu, \rho)$ are defined $O(G) = \sum \mu(v)$ and $S(G) = \sum \rho(v, w)$, respectively. The fuzzy cardinality of a fuzzy graph $G = (V, \mu, \rho)$ is defined to be $\sum \mu(v)$. An edge $e = (v, w)$ of a fuzzy graph $G = (V, \mu, \rho)$ is called an effective edge if $\rho(v, w) = \mu(v) \wedge \mu(w)$. $N(v) = \{w \in V : \rho(v, w) = \mu(v) \wedge \mu(w)\}$ is called the neighborhood of v, and $N[v] = N(v) \cup v$ is called the closed neighborhood of v.

The degree of a vertex can be generalized in different ways for a fuzzy graph. The effective degree of a vertex v is defined as the sum of the membership value of the effective edges incident with v, and is denoted by $d_E(v)$. That is, $d_E(v) = \sum \rho_E(v, w)$. The minimum and maximum effective degrees are defined by $\delta_E(G) = \wedge\{d_E(v) : v \in V\}$ and $\Delta_E(G) = \vee\{d_E(v) : v \in V\}$, respectively. The neighborhood degree of a vertex is defined as the sum of the membership value of the neighborhood vertices of v, and is denoted by $d_N(v)$. The minimum and maximum neighborhood degrees are defined by $\delta_N(G) = \wedge\{d_N(v) : v \in V\}$ and $\Delta_N(G) = \vee\{d_N(v) : v \in V\}$, respectively.

3. Domination in Fuzzy Graphs

The fundamental mathematical definition of domination (crisp) was given by Ore [4]. Somasundaram et al [6] introduced the concept of domination in fuzzy graphs.

DEFINITION 3.1 ([6]). Let $G = (V, \mu, \rho)$ be a fuzzy graph. Then we say that v *dominates* w in G if $\rho(v, w) = \mu(v) \wedge \mu(w)$ for all $v, w \in V$. A subset S of V is called a *dominating set* in G if for every $y \notin S$, there exists $x \in S$ such that x dominates y. The minimum fuzzy cardinality of a dominating set in G is called the *domination number* of G, and is denoted by$\gamma(G)$ or simply γ.

NOTE. (1) For any $v, w \in V$, if v dominates w then w dominates v and hence domination is a symmetric relation on V.

(2) For any $v \in V, N(v)$ is precisely the set of all vertices in V which are dominated by v.

(3) If $\rho(v, w) < \mu(v) \wedge \mu(w)$ for all $v, w \in V$, then obviously the only dominating set of G is V. Conversely, if V is the only dominating set G, then $\rho(v, w) < \mu(v) \wedge \mu(w), \forall v, w \in V$.

EXAMPLE 3.1. Consider a fuzzy graph $G = (V, \mu, \rho)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $\mu(v_1) = 0.3$, $\mu(v_2) = 0.6$, $\mu(v_3) = 0.8$, $\mu(v_4) = 0.9$, $\mu(v_5) = 0.8$, and $\rho(v_1, v_2) = 0.3$, $\rho(v_2, v_3) = 0.5$, $\rho(v_2, v_4) = 0.4$, $\rho(v_3, v_4) = 0.8$, $\rho(v_3, v_5) = 0.8$, $\rho(v_4, v_5) = 0.6$, $\rho(v_5, v_1) = 0.2$. Here v_3 dominates v_4 and v_5, and v_2 dominates v_1. Clearly, $S = \{v_1, v_3\} \subset V$, is the minimum dominating set of G, and therefore $\gamma(G) = 1.1$.

EXAMPLE 3.2. In a complete fuzzy graph G for all $v \in V, \{v\}$ is a dominating set, we have $\gamma(G) = \wedge\mu(v)$.

DEFINITION 3.2 ([6]). A dominating set S of a fuzzy graph is said to be a *minimal set* if no proper subset of S is a dominating set of G.

THEOREM 3.1 ([6]). *A dominating set D of G is a minimal dominating set if and only if for each $d \in D$ one of the following two conditions holds:*

(a) $N(d) \cap D = \phi$;
(b) *There exists a vertex $c \in V \backslash D$ such that* $N(c) \cap D = \{d\}$.

DEFINITION 3.3 ([6]). A vertex v of a fuzzy graph is said to be an *isolated vertex* if $\rho(v, w) < \mu(v) \wedge \mu(w)$ for all $w \in V \backslash v$, that is, $N(v) = \phi$.

Thus an isolated vertex does not dominate any other vertex in G.

THEOREM 3.2 ([6]). *Let G be a fuzzy graph without isolated vertices. Let D be a minimal dominating set of G. Then $V \backslash D$ is a dominating set of G.*

COROLLARY 3.3 ([6]). *For any fuzzy graph G without isolated vertices, $\gamma(G) \leq O(G)/2$.*

THEOREM 3.4 ([6]). *For any fuzzy graph $G, \gamma(G) \leq O(G) - \Delta_N(G)$.*

DEFINITION 3.4 ([6]). Let $G = (\mu, \rho)$ be a fuzzy graph on V. A subset S of V is called a vertex cover of G if for every effective edge $e = (v, w)$, at least one of v, w is in S. The minimum fuzzy cardinality of a vertex cover is called the covering number of G, and is denoted by $\alpha_0(G)$ or simply α_0.

DEFINITION 3.5 ([6]). Let $G = (\mu, \rho)$ be a fuzzy graph on V. A subset S of V is said to be an *independent set* if $\rho(x, y) < \mu(x) \wedge \mu(y) \forall x, yS$. The set S is said to *maximal independent set* if $S \cup \{y\}$ is not an independent set for any $y \in V \backslash S$. The minimum fuzzy cardinality of an independent set in G is called the *independence number* of G, and is denoted by $\beta_0(G)$ or simply β_0.

THEOREM 3.5 ([6]). *If D is an independent dominating set of a fuzzy graph G then D is both a minimal dominating set and a maximal independent set. Conversely any maximal independent set D in G is an independent dominating set of G.*

EXAMPLE 3.3. Consider a fuzzy graph $G = (V, \nu, \rho)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ with $\mu(v_1) = 0.4$, $\mu(v_2) = 0.6$, $\mu(v_3) = 0.8$, $\mu(v_4) = 0.1$, $\mu(v_5) = 0.2$, $\mu(v_6) = 0.3$ and $\rho(v_1, v_2) = 0.4$, $\rho(v_2, v_3) = 0.6$, $\rho(v_2, v_6) = 0.3$, $\rho(v_3, v_4) = 0.1$, $\rho(v_3, v_5) = 0.2$, $\rho(v_4, v_5) = 0.1$, $\rho(v_5, v_6) = 0.2$, $\rho(v_3, v_6) = 0.3$, $\rho(v_6, v_1) = 0.3$. Here $\{v_1, v_3\}$ is an independent set of maximum fuzzy cardinality and therefore $\beta_0(G) = 1.2.\{v_2, v_4, v_5, v_6\}$ is a vertex cover of minimum fuzzy cardinality and therefore $\alpha_0(G) = 1.2$.

4. Strong and Weak Domination

Firstly, we recall the idea of strong and weak domination in graph (crisp) theory.

Let $G = (V, E)$ be an undirected connected loop-free graph. Given two adjacent vertices u and v. We say that u strongly dominates v if $d(u) \geq d(v)$. Similarly, we say that v weakly dominates u if $d(u) \geq d(v)$. A set $D \subseteq V(G)$ is a strong-dominating set if every vertex in $V \backslash D$ is strongly dominated by at least one vertex in D. Similarly,D is a weak-dominating set if every vertex in $V \backslash D$ is weakly dominated by at least one vertex in D. The strong domination number $\gamma_S(G)$ is the minimum cardinality of a strong dominating set of G. The weak domination number$\gamma_W(G)$ is the minimum cardinality of a weak dominating set of G.

We now introduce the concept of strong and weak domination in fuzzy graphs.

DEFINITION 4.1. Let $G = (V, \mu, \rho)$ be a fuzzy graph. For any $u, v \in V, u$ *strongly dominates* v if $\rho(u, v) = \mu(u) \wedge \mu(v)$ and $d(u) \geq d(v)$. Similarly, u *weakly dominates* v if $\rho(u, v) = \mu(u) \wedge \mu(v)$ and $d(v) \geq d(u)$.

DEFINITION 4.2. A set D V is a *strong-dominating set* of G if every vertex in $V \backslash D$ is strongly dominated by at least one vertex in D. Similarly, a set $D \subseteq V$ is a *weak-dominating set* of G if every vertex in $V \backslash D$ is weakly dominated by at least one vertex in D.

DEFINITION 4.3. The minimum fuzzy cardinality of a strong dominating set is called the *strong domination number*, and is denoted by$\gamma_S(G)$. Similarly, The minimum fuzzy cardinality of a weak dominating set is called the *weak domination number*, and is denoted by $\gamma_W(G)$.

EXAMPLE 4.1. Consider a fuzzy graph $G = (V, \mu, \rho)$, where $V = \{v_1, v_2, v_3\}$ with $\mu(v_1) = 1.0$, $\mu(v_2) = 0.7$, $\mu(v_3) = 0.6$, and $\rho(v_1, v_2) = 0.7$, $\rho(v_2, v_3) = 0.5$, $\rho(v_3, v_1) = 0.6$. Suppose $D = v_1$, we have $V \backslash D = \{v2, v3\}$. Here v_1 dominates v_2 and v_3, also $d(v_1) > d(v_2)$ and $d(v_1) > d(v_3)$. Therefore, v_1 strongly dominates both v_2 and v_3. There is no other strong-dominating set. Thus $D = v_1$ is the strong dominating set. Therefore, we have $\gamma_S(G) = \mu(v_1) = 1.0$. Suppose, we have $D = \{v_2, v_3\}, V \backslash D = \{v_1\}$. D is weak dominating set and $\gamma_W(G) = 1.3$.

EXAMPLE 4.2. Consider a complete fuzzy graph $G = (V, \mu, \rho)$, where $V = \{v_1, v_2, v_3, v_4\}$ with $\mu(v_1) = 0.9$, $\mu(v_2) = 0.7$, $\mu(v_3) = 0.5$, $\mu(v_4) = 1.0$ and $\rho(v_1, v_2) = 0.7$, $\rho(v_2, v_3) = 0.5$, $\rho(v_3, v_1) = 0.5$, $\rho(v_1, v_4) = 0.9$, $\rho(v_2, v_4) = 0.7$, $\rho(v_3, v_4) = 0.5$. Let $D = \{v_1\}$, then $V \backslash D = \{v_2, v_3, v_4\}$. We have every vertex of $V \backslash D$ is strongly dominated by the set D. Therefore, D is a strong dominating set. Suppose we have $D = \{v_4\}, V \backslash D = \{v_1, v_2, v_3\}$, which is also a strongly dominating set. Thus $\gamma_S(G) = \mu(v_1) = 0.9$. Similarly, $\gamma_W(G) = \mu(v_3) = 0.5$.

THEOREM 4.1. *In any complete fuzzy graph $G = (V, \mu, \rho)$, the following inequality holds*

$$\gamma_W(G) \leq \gamma_S(G).$$

Proof. Let $G = (V, \mu, \rho)$ be a complete fuzzy graph.
Case(i): Suppose for all $v_i \in V, \mu(v_i)$ are equal. Since G is complete fuzzy graph, $\rho(v_i, v_j) = \mu(v_i) \wedge \mu(v_j), \forall v_i, v_j \in V$.

We have $\rho(v_i, v_i) = \mu(v_i), \forall v_i \in V$. Thus

$$\gamma_W(G) = \gamma_S(G) = \wedge \mu(v_i) = \mu(v_i). \tag{1}$$

Case(ii): Suppose for all $v_i \in V, \mu(v_i)$ are not equal. In a complete fuzzy graph, any one of the vertices dominates all other vertices; if it is least among them then the dominating set with that vertex is called weak dominating set. Thus the fuzzy cardinality of the set is the weak domination number. That is, $\gamma_W(G) = \wedge \mu(v_i)$. Obviously, the strong dominating set has a vertex other than the least value of the vertex set. Therefore, the strong domination number is strictly greater than weak domination number.

$$\gamma_W(G) < \gamma_S(G). \tag{2}$$

From the equations (1) and (2), we get $\gamma_W(G) \leq \gamma_S(G)$. □

EXAMPLE 4.3. In example 4.2, $\gamma_W(G) = 0.5; \gamma_S(G) = 0.9$. Therefore, $\gamma_W(G) < \gamma_S(G)$.

COROLLARY 4.2. *For any fuzzy graph* $G, \gamma_W(G) \leq \gamma_S(G)$ *(or)* $\gamma_W(G) \geq \gamma_S(G)$.

THEOREM 4.3. *In any fuzzy graph* $G = (V, \mu, \rho)$, *the following inequalities hold*

(a) $\gamma(G) \leq \gamma_S(G) \leq O(G) - \Delta(G)$;
(b) $\gamma(G) \leq \gamma_W(G) \leq O(G) - \delta(G)$.

Proof. (a) By definitions 3.1 & 4.3, we have

$$\gamma(G) \leq \gamma_S(G). \tag{3}$$

Also, by definitions [3], $O(G) = \sum \mu(v)$ and $S(G) = \sum \rho(v, w)$. Therefore,

$$O(G) - S(G) = \text{The sum of the degrees of } G \text{ excluding the maximum degree of a vertex.} \tag{4}$$

From (3) and (4), we get, $\gamma(G) \leq \gamma_S(G) \leq O(G) - \Delta(G)$.

(b) By definitions 3.1 & 4.3, the weakly domination number of G has greater or equal weight of a domination number of G, because the vertices of weakly dominating set D, it weakly dominates any one of the vertices of $V \backslash D$. But, the dominating set has no limitations, it should be minimum fuzzy cardinality of a dominating set. Therefore, it should be lesser or equal to the weakly domination number. That is,

$$\gamma(G) \leq \gamma_W(G). \tag{5}$$

Also, we have

$$O(G) - \delta(G) = \sum \mu(v) - \wedge\{d(v_i)\} = \text{ The sum of the degress of } G \text{ exclusing the minimum degree of a vertex.} \tag{6}$$

From the equations (5) and (6), we have $\gamma(G) \leq \gamma_W(G) \leq O(G) - \delta(G)$. □

EXAMPLE 4.4. In example 4.3, $\gamma(G) = 0.5$; $\gamma_W(G) = 0.5$; $\gamma_S(G) = 0.9$; $O(G) = 3.1$; $\delta(G) = 1.5$; $\Delta(G) = 2.1$.

Hence the above theorem is verified.

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