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ON MAXIMAL OPERATORS BELONGING TO THE MUCKENHOUPT'S CLASS A_1

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Abstract. We study a maximal operator defined on spaces of homogeneous type, and we prove that this operator is of weak type (1,1). As a consequence we show that the maximal operator belongs to the Muckenhoupt's class A_1.

1. Introduction

In this paper we first introduce a space of homogeneous type X, which is a more general setting than a Euclidean space $\mathbb{R}^n$, and we also consider the generalized upper half-space $X \times (0, \infty)$. Then we shall consider a maximal operator M_p defined on X as follows. For a measurable function f defined on $X \times (0, \infty)$ and $x \in X$, we define a maximal function of f, as

$$M_p(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)} \int_{T(B)} |f(y,t)|^p \frac{d\mu(y)dt}{t} \right)^{1/p}, \quad 1 \leq p < \infty,$$

where the supremum is taken over all balls B containing x. Here μ denotes surface area measure on X, and $T(B)$ denotes the tent over B.

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In this paper we study that a new kind of a maximal operator M_p gives rise to weights in A_1 [4]. To prove this we need to prove that this operator M_p is of weak type (p, p) for some $p, 1 \leq p < \infty$.

2. Some basic notations and preliminary materials

We begin by introducing the notion of a space of homogeneous type [1]: Let X be a topological space endowed with Borel measure μ. Assume that d is a pseudo-metric on X, that is, a nonnegative function on $X \times X$ satisfying

(i) $d(x, x) = 0; d(x, y) > 0$ if $x \neq y$,
(ii) $d(x, y) = d(y, x)$, and
(iii) $d(x, z) \leq K(d(x, y) + d(y, z))$, where K is some fixed constant.

Assume further that

(a) the balls $B(x, \rho) = \{y \in X : d(x, y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$,

and that μ satisfies the doubling property:

(b) $0 < \mu(B(x, 2\rho)) \leq A\mu(B(x, \rho)) < \infty$, where A is some fixed constant.

Then we call (X, d, μ) a *space of homogeneous type.*

Property (iii) will be referred to as the "triangle inequality." Note that property (b) implies that for every $C > 0$ there is a constant $A_C < \infty$ such that

$$\mu(B(x, C\rho)) \leq A_C \mu(B(x, \rho)) \tag{1}$$

for all $x \in X$ and $\rho > 0$.

Now consider the space $X \times (0, \infty)$, which is a kind of generalized upper half-space over X. We then define the analogue of nontangential or conoical regions as follows. For $x \in X$, set

$$\Gamma(x) = \{(y, t) \in X \times (0, \infty) : x \in B(y, t)\}.$$

For any set $\Omega \subset X$, the *tent* over Ω is the set

$$T(\Omega) = \{(x,t) \in X \times (0,\infty) : B(y,t) \subset \Omega\}.$$

It is then very easy to check that

$$T(\Omega) = (X \times (0,\infty)) \setminus \bigcup_{x \notin \Omega} \Gamma(x).$$

For a measurable function f defined on $X \times (0,\infty)$ and $x \in X$, we define a maximal function of f, as
(2)

$$M_p(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)} \int_{T(B)} |f(y,t)|^p \frac{d\mu(y)dt}{t} \right)^{1/p}, \quad 1 \leq p < \infty,$$

where the supremum is taken over all balls B containing x.

3. Main result

The following covering lemma can be found in [1, p.69].

LEMMA 1. *Let Ω be a bounded subset of X. Suppose that $\rho(x)$ is a positive number for each $x \in \Omega$. Then there is a (finite or infinite) sequence of disjoint balls $\{B(x_i, \rho(x_i)\}, x_i \in \Omega$, such that*

$$\Omega \subset \bigcup_i B(x_i, 4K\rho(x_i)),$$

where K is the constant in the triangle inequality. Furthermore, every $x \in \Omega$ is contained in some ball $B(x_i, 4K\rho(x_i))$ satisfying $\rho(x) \leq 2\rho(x_i)$.

LEMMA 2. *The maximal operator M_p, defined as in (2), is of weak type (p,p) for some $p, 1 \leq p < \infty$.*

Proof. Fix $\lambda > 0$, set

$$\Omega_\lambda = \{x \in X : M_p(f)(x) > \lambda\}.$$

For each $x \in \Omega_\lambda$, let

$$\rho(x) = \sup \left\{ \rho > 0 : \left(\frac{1}{\mu(B(x,\rho))} \int_{T(B(x,\rho))} |f(y,t)|^p \frac{d\mu(y)dt}{t} \right)^{1/p} > \lambda \right\}.$$

Thus for each $x \in \Omega_\lambda$, we have $\rho(x) > 0$ and

$$\frac{1}{\mu(B(x,\rho(x)))} \int_{T(B(x,\rho(x)))} |f(y,t)|^p \frac{d\mu(y)dt}{t} \geq \lambda^p. \tag{3}$$

Assume first that Ω_λ is bounded. Apply Lemma 1 to the balls $B(x, \rho(x))$ to obtain a sequence of disjoint balls $B(x_i, \rho(x_i))$, so that

$$\Omega_\lambda \subset \bigcup_i B(x_i, 4K\rho(x_i)).$$

Then

$$\begin{aligned}
\mu(\Omega_\lambda) &\leq \sum_i \mu(B(x_i, 4K\rho(x_i))) \\
&\leq A_{4K} \sum_i \mu(B(x_i, \rho(x_i))) \qquad \text{(by (1))} \\
&\leq \frac{A_{4K}}{\lambda^p} \sum_i \int_{T(B(x_i,\rho(x_i)))} |f(y,t)|^p \frac{d\mu(y)dt}{t} \qquad \text{(by (3))} \\
&\leq \frac{A_{4K}}{\lambda^p} \int_{X\times(0,\infty)} |f(y,t)|^p \frac{d\mu(y)dt}{t} \\
&= A_{4K}(||f||_p/\lambda)^p,
\end{aligned}$$

since the balls $B(x_i, \rho(x_i))$ are disjoint. Thus M_p is of weak type (p, p).

Assume second that Ω_λ is not bounded. Fix $a \in X$ and $R > 0$. Then $\Omega_\lambda \cap B(a, R)$ is a bounded set, and so, as in the above argument, we can apply Lemma 1 to the balls $\{B(x, \rho(x)) : x \in \Omega_\lambda \cap B(a, R)\}$ to obtain

$$\mu(\Omega_\lambda \cap B(a, R)) \le A_{4K}(||f||_p/\lambda)^p.$$

Letting $R \to \infty$ we obtain the same weak type estimate as before. Thus the proof is complete. □

We now state and prove the main result of this paper.

THEOREM 3. *Let M_p be defined as in* (2) *and* $1 \le p < \infty$. *Then M_p is in the class A_1* [4], *that is, there is a constant C such that*

$$\frac{1}{\mu(B)} \int_B M_p(f)(x) d\mu(x) \le C \inf_{x \in B} M_p(f)(x),$$

where the infimum is taken over all balls B containing x.

Proof. Let

$$M_1(u)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_{T(B)} u(y, t) \frac{d\mu(y)dt}{t},$$

where $u(y, t) = |f(y, t)|^p$. For any ball B in X, decompose

$$u(y, t) = u_1(y, t) + u_2(y, t),$$

where $u_1(y, t) = u(y, t)\chi_{T(3B)}(y, t)$. Since M_1 is of weak type $(1, 1)$ by Lemma 2, it follows from the Kolmogorov's inequality [6] that

$$\int_B M_1(u_1)(x)^{1/p} d\mu(x)$$
$$\le C\mu(B)^{1-1/p} \left(\int_{X \times (0,\infty)} u_1(y, t) \frac{d\mu(y)dt}{t} \right)^{1/p}$$

for some constant C, that is, for any $x \in B$

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_B M_1(u_1)(x)^{1/p} d\mu(x) \\
&\quad \le C \left(\frac{1}{\mu(B)} \int_{X \times (0,\infty)} u_1(y,t) \frac{d\mu(y)dt}{t} \right)^{1/p} \\
&\quad \le C \left(\frac{1}{\mu(3B)} \int_{T(3B)} u(y,t) \frac{d\mu(y)dt}{t} \right)^{1/p} \\
&\quad \le C M_1(u)(x)^{1/p}.
\end{aligned}$$

Thus

$$\frac{1}{\mu(B)} \int_B M_1(u_1)(x)^{1/p} d\mu(x) \le C \inf_{x \in B} M_1(u)(x)^{1/p}. \tag{4}$$

On the other hand, for any $x, z \in B$ we have

$$M_1(u_2)(x) \le C M_1(u_2)(z). \tag{5}$$

In fact, if $M_1(u_2)(x) \neq 0$, then clearly $z \in 3B$. Thus

$$\begin{aligned}
M_1(u_2)(x) &\le \sup_{x \in B} \frac{1}{\mu(B)} \int_{T(3B)} |u_2(y,t)| \frac{d\mu(y)dt}{t} \\
&\le C M_1(u_2)(z),
\end{aligned}$$

as desired. Thus it follows from (5) that

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B M_1(u_2)(x)^{1/p} d\mu(x) &\le C \inf_{z \in B} M_1(u_2)(z)^{1/p} \\
&\le C \inf_{z \in B} M_1(u)(z)^{1/p}.
\end{aligned} \tag{6}$$

Since

$$M_1(u)(x)^{1/p} \le C \left(M_1(u_1)(x)^{1/p} + M_1(u_2)(x)^{1/p} \right),$$

it follows from (4) and (6) that

$$\begin{aligned} &\frac{1}{\mu(B)}\int_B M_1(u)(x)^{1/p}d\mu(x) \\ &\quad \le \frac{C}{\mu(B)}\left(\int_B M_1(u_1)(x)^{1/p}d\mu(x) + \int_B M_1(u_2)(x)^{1/p}d\mu(x)\right) \\ &\quad \le C \inf_{x\in B} M_1(u)(x)^{1/p}, \end{aligned}$$

that is, $M_1(u)^{1/p} = M_p(f) \in A_1$. The proof is therefore complete.□

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