

## ON SOMEWHAT PAIRWISE FUZZY CONTINUOUS FUNCTIONS

M. K. UMA, E. ROJA, AND G. BALASUBRAMANIAN

ABSTRACT. In this paper the concept of somewhat pairwise fuzzy continuous functions and somewhat pairwise fuzzy open functions are introduced. Some interesting properties of these functions are investigated besides giving some characterizations of these functions.

### 1. Introduction and preliminaries

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy sets by L. A. Zadeh [7]. Fuzzy sets have applications in many fields such as information [5] and control[6]. The theory of fuzzy topological spaces was introduced and developed by C. L. Chang [2] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of somewhat continuous functions was introduced by Karl R. Gentry and Hughes B. Hoyle III in [4]. In 1989 Kandil [3] introduced the concept of fuzzy bitopological spaces. In this paper we introduce the concepts of somewhat pairwise fuzzy continuous functions and somewhat pairwise fuzzy open functions and study their properties.

The product related spaces and the graph of a function were found in Azad [1].

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## 2. Main Results

### 2.1. Somewhat pairwise Fuzzy contiuous functions

DEFINITION 2.1.1. Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. A function  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is *pairwise\* fuzzy continuous* if for each  $S_1$ -fuzzy open set or  $S_2$ -fuzzy open set  $\lambda$  in  $(Y, S_1, S_2)$ , the inverse image  $f^{-1}(\lambda)$  is a  $T_1$ -fuzzy open set or  $T_2$ -fuzzy open set in  $(X, T_1, T_2)$ .

DEFINITION 2.1.2. Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. A function  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is called *somewhat pairwise fuzzy continuous* if  $\lambda \in S_1$  or  $\lambda \in S_2$  and  $f^{-1}(\lambda) \neq 0 \Rightarrow$  there exists  $\mu \in T_1$  or  $\mu \in T_2$  such that  $\mu \neq 0$  and  $\mu \leq f^{-1}(\lambda)$ .

It is clear from the definition that every pairwise\* fuzzy continuous function is somewhat pairwise fuzzy continuous, but however the converse is not true as the following example shows.

EXAMPLE 2.1.1. Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1\}$ ,  $T_2 = \{0, 1, \mu\}$ ,  $S_1 = \{0, 1, \mu, \lambda\}$ ,  $S_2 = \{0, 1, \lambda\}$ ,  $Q_1 = \{0, 1, \delta\}$  and  $Q_2 = \{0, 1, \delta, \mu\}$  where  $\mu: X \rightarrow [0, 1]$  is defined as  $\mu(a) = 0$ ,  $\mu(b) = 1$ ,  $\mu(c) = 0$ ,  $\lambda: X \rightarrow [0, 1]$  is defined as  $\lambda(a) = 1$ ,  $\lambda(b) = 0$ ,  $\lambda(c) = 1$  and  $\delta: X \rightarrow [0, 1]$  is defined as  $\delta(a) = 1$ ,  $\delta(b) = 1$ ,  $\delta(c) = 0$ . Clearly,  $T_1, T_2, S_1, S_2, Q_1$  and  $Q_2$  are fuzzy topologies on  $X$ . Define  $f: (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  as  $f(a) = a$ ,  $f(b) = c$ ,  $f(c) = c$  and let  $g: (X, S_1, S_2) \rightarrow (X, Q_1, Q_2)$  be the identity function. Then

$$g^{-1}(0)(x) = 0(g(x)) = 0(x),$$

$$g^{-1}(1)(x) = 1(g(x)) = 1(x),$$

$$g^{-1}(\delta)(x) = \delta(g(x)) = \delta(x),$$

but  $\delta \notin S_1$  and  $\delta \notin S_2$  and so  $g$  is not pairwise\* fuzzy continuous. However  $g$  is somewhat pairwise fuzzy continuous since  $\mu \leq g^{-1}(\delta)$  and  $\mu \leq g^{-1}(\mu)$ . It is easy to verify that  $f$  is pairwise\* fuzzy continuous. Now, consider  $g \circ f: (X, T_1, T_2) \rightarrow (X, Q_1, Q_2)$ . Then  $g^{-1}(0) = 0$ ,  $g^{-1}(1) = 1$  and  $(g \circ f)^{-1}(\delta) \neq 0$ , there exists no  $\mu \neq 0$ ,  $\mu \in T_2$  such that  $\mu \leq (g \circ f)^{-1}(\delta)$ . This shows that  $g \circ f$  is not somewhat pairwise fuzzy continuous. This example also shows that the composition of

pairwise\* fuzzy continuous and somewhat pairwise fuzzy continuous functions need not be somewhat pairwise fuzzy continuous.

**DEFINITION 2.1.3.** A fuzzy set  $\lambda$  in a fuzzy bitopological space  $(X, T_1, T_2)$  is called *pairwise dense fuzzy set* if there exists no  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set  $\mu$  in  $(X, T_1, T_2)$  such that  $\lambda < \mu < 1$ .

**EXAMPLE 2.1.2.** Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1, \delta\}$ ,  $T_2 = \{0, 1, \gamma\}$  where  $\delta : X \rightarrow [0, 1]$  is such that  $\delta(a) = 1$ ,  $\delta(b) = 1/4$ ,  $\delta(c) = 1/3$  and  $\gamma : X \rightarrow [0, 1]$  is such that  $\gamma(a) = 1$ ,  $\gamma(b) = 1/4$ ,  $\gamma(c) = 1/3$ . Define a fuzzy set  $\lambda : X \rightarrow [0, 1]$  is such that  $\lambda(a) = 0$ ,  $\lambda(b) = 3/4$ ,  $\lambda(c) = 3/4$ . Clearly  $\lambda$  is a pairwise dense fuzzy set.

**PROPOSITION 2.1.1.** Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. Let  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be any function. Then the following are equivalent.

- (a)  $f$  is somewhat pairwise fuzzy continuous.
- (b) If  $\lambda$  is  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed such that  $f^{-1}(\lambda) \neq 1$ , then there exists a proper  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set  $\mu$  such that  $\mu > f^{-1}(\lambda)$ .
- (c) If  $\lambda$  is a pairwise dense fuzzy set in  $(X, T_1, T_2)$ , then  $f(\lambda)$  is a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ .

*Proof.* (a)  $\Rightarrow$  (b). Suppose  $f$  is somewhat pairwise fuzzy continuous and  $\lambda$  is  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set in  $(Y, S_1, S_2)$  such that  $f^{-1}(\lambda) \neq 1$ . Therefore clearly,  $1 - \lambda \in S_1$  or  $1 - \lambda \in S_2$  and  $f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda) \neq 0$ . By (a), there exists a  $\eta \in T_1$  or  $\eta \in T_2$  such that  $\eta \neq 0$  and  $\eta < f^{-1}(1 - \lambda)$ . That is,  $1 - \eta$  is  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed and

$$1 - \eta > 1 - f^{-1}(1 - \lambda) = 1 - (1 - f^{-1}(\lambda)) = f^{-1}(\lambda).$$

Taking up  $\mu = 1 - \eta$ , (b) is proved.

(b)  $\Rightarrow$  (c). Let  $\lambda$  be any pairwise dense fuzzy set in  $(X, T_1, T_2)$  and suppose that  $f(\lambda)$  is not a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ . Then there exists a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set  $\eta$  such that  $f(\lambda) \leq \eta < 1$ . Since  $\eta < 1$ ,  $f^{-1}(\eta) \neq 1$  and so by (b), there exists a proper  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set  $\gamma$  such that  $\gamma > f^{-1}(\eta)$ . But  $f^{-1}(\eta) \geq f^{-1}(f(\lambda)) \geq \lambda$ . That is, there exists a proper  $T_1$ -fuzzy

closed or  $T_2$ -fuzzy closed set  $\gamma$  such that  $\gamma > \lambda$ , which is a contradiction since  $\lambda$  is pairwise dense fuzzy. Therefore (c) is proved.

(c)  $\Rightarrow$  (a). Suppose  $\lambda \in S_1$  or  $\lambda \in S_2$ ,  $\lambda \neq 0$  and  $f^{-1}(\lambda) \neq 0$ . We want to show that  $\text{int}_{T_1} f^{-1}(\lambda) \neq 0$  or  $\text{int}_{T_2} f^{-1}(\lambda) \neq 0$ . Suppose that  $\text{int}_{T_1} f^{-1}(\lambda) = 0$  and  $\text{int}_{T_2} f^{-1}(\lambda) = 0$ . Then,  $\text{cl}_{T_1}(1 - f^{-1}(\lambda)) = 1$ . Similarly,  $\text{cl}_{T_2}(1 - f^{-1}(\lambda)) = 1$ . This shows that  $1 - f^{-1}(\lambda)$  is a pairwise dense fuzzy set in  $(X, T_1, T_2)$ . Therefore by (c),  $f(1 - f^{-1}(\lambda))$  is a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ . But  $f(1 - f^{-1}(\lambda)) = f(f^{-1}(1 - \lambda)) \leq 1 - \lambda < 1$ . Since  $1 - \lambda$  is  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed and  $f(1 - f^{-1}(\lambda)) \leq 1 - \lambda$ , we must have

$$\text{cl}_{S_1} f(1 - f^{-1}(\lambda)) \leq 1 - \lambda, \quad \text{or} \quad \text{cl}_{S_2} f(1 - f^{-1}(\lambda)) \leq 1 - \lambda.$$

That is,  $1 \leq 1 - \lambda$  implies  $\lambda = 0$ . Contradiction. Therefore,  $\text{int}_{T_1} f^{-1}(\lambda) \neq 0$  or  $\text{int}_{T_2} f^{-1}(\lambda) \neq 0$ . This proves that  $f$  is somewhat pairwise fuzzy continuous.  $\square$

**PROPOSITION 2.1.2.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces and  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be a somewhat pairwise fuzzy continuous function. Let  $A \subset X$  be such that  $\chi_A \wedge \mu \neq 0$  for all  $0 \neq \mu \in T_1 \cup T_2$ . Let  $T_1/A$  and  $T_2/A$  be the induced fuzzy topologies on  $A$ . Then the function  $f/A : (A, T_1/A, T_2/A) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous.*

*Proof.* Suppose  $\lambda \in S_1$  or  $\lambda \in S_2$  is such that  $f^{-1}(\lambda) \neq 0$ . Since  $f$  is somewhat pairwise fuzzy continuous, there exists  $\mu \in T_1$  or  $\mu \in T_2$  such that  $\mu \neq 0$  and  $\mu \leq f^{-1}(\lambda)$ . Now clearly,  $\mu/A \in T_1/A$  or  $\mu/A \in T_2/A$  and  $\mu/A \neq 0$  (since  $\chi_A \wedge \mu \neq 0$  for all  $\mu \in T_1 \cup T_2$ ). Also,

$$\begin{aligned} (f/A)^{-1}(\lambda)(x) &= \lambda(f/A)(x) \\ &= \lambda f(x) \quad (\text{for } x \in A) \\ &> \mu(x) \quad (\text{for } x \in A) \\ &= (\mu/A)(x). \end{aligned}$$

That is,  $\mu/A < (f/A)^{-1}(\lambda)$ . This shows  $f/A$  is somewhat pairwise fuzzy continuous.  $\square$

The following example shows that the condition on  $A$  in Proposition 2.1.2 cannot be omitted.

EXAMPLE 2.1.3. Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1, \lambda, \mu\}$ ,  $T_2 = \{0, 1, \lambda\}$ ,  $S_1 = \{0, 1, \delta\}$  and  $S_2 = \{0, 1, \delta, \mu\}$  where

$\lambda : X \rightarrow [0, 1]$  is such that  $\lambda(a) = 1, \lambda(b) = 1, \lambda(c) = 0$ ,

$\mu : X \rightarrow [0, 1]$  is such that  $\mu(a) = 0, \mu(b) = 0, \mu(c) = 1$ ,

and

$\delta : X \rightarrow [0, 1]$  is such that  $\delta(a) = 0, \delta(b) = 1, \delta(c) = 1$ .

Let  $A = \{a, b\}$ . Since  $\chi_A \wedge \mu = 0$ ,  $A$  does not satisfy the hypothesis of Proposition 2.1.2. Let  $f : (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  be the identity function. Then  $f$  is somewhat pairwise fuzzy continuous. For  $f^{-1}(1) = 1 \neq 0$ ,  $f^{-1}(\delta) = \delta \neq 0$ ,  $\mu \in T_1$ ,  $\mu \neq 0$  and  $\mu \leq f^{-1}(\delta)$ . That is,  $f$  is somewhat pairwise fuzzy continuous. Consider  $f/A : (A, T_1/A, T_2/A) \rightarrow (X, S_1, S_2)$  defined as  $(f/A)(a) = f(a) = a$ ,  $(f/A)(b) = f(b) = b$ ,  $T_1/A = \{0/A, 1/A, \lambda/A, \mu/A\}$  and  $T_2/A = \{0/A, 1/A, \lambda/A\}$ . Now,  $T_1/A = \{0_A, 1_A\}$  and  $T_2/A = \{0_A, 1_A\}$ . Clearly,  $(f/A)^{-1}(\delta) \neq 0$ . There exists no  $\mu \neq 0$  and  $\mu \in T_1/A$  or  $\mu \in T_2/A$  such that  $\mu \leq (f/A)^{-1}(\delta)$ . Therefore  $f/A$  is not somewhat pairwise fuzzy continuous.

The following example shows that an extension of a somewhat pairwise fuzzy continuous function need not be somewhat pairwise fuzzy continuous.

EXAMPLE 2.1.4. Let  $X = \{a, b\}$ . Define  $T_1 = \{0, 1, \mu\}$ ,  $T_2 = \{0, 1\}$ ,  $S_1 = \{0, 1, \lambda\}$  and  $S_2 = \{0, 1\}$  where  $\lambda : X \rightarrow [0, 1]$  is such that  $\lambda(a) = 0, \lambda(b) = 1$  and  $\mu : X \rightarrow [0, 1]$  is such that  $\mu(a) = 1, \mu(b) = 0$ . Let  $A = \{a\}$ . Then,  $T_1/A = \{0/A, 1/A\}$  and  $\chi_A = \mu$  is  $T_1$ -fuzzy open in  $(X, T_1, T_2)$  and  $\chi_A \wedge \mu = \mu \neq 0$ . Define  $f : (A, T_1/A, T_2/A) \rightarrow (X, S_1, S_2)$  by  $f(a) = a$ . Then  $f$  is pairwise fuzzy continuous. Define  $F : (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  as follows:  $F(a) = f(a)$ ,  $F(b) = b$ . Clearly,  $F$  is an extension of  $f$  onto  $(X, S_1, S_2)$ . We claim that  $F$  is not somewhat pairwise fuzzy continuous. For  $F^{-1}(0) = 0$ ,  $F^{-1}(1) = 1 \neq 0$ . Clearly,  $F^{-1}(\lambda) = \lambda$  and there is no non-zero  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set in  $(X, T_1, T_2)$  which is less than  $F^{-1}(\lambda) = \lambda$ . Therefore  $F$  is not somewhat pairwise fuzzy continuous.

PROPOSITION 2.1.3. Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces.  $X = A \cup B$ , where  $A$  and  $B$  are such that  $\chi_A, \chi_B \in T_1 \cap T_2$ . Let  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be such that

$f/A$  and  $f/B$  are somewhat pairwise fuzzy continuous. Then  $f$  is somewhat pairwise fuzzy continuous.

*Proof.* Let  $\lambda \in S_1$  or  $\lambda \in S_2$  be such that  $f^{-1}(\lambda) \neq 0$ . Consider  $(f/A)^{-1}(\lambda)$  and  $(f/B)^{-1}(\lambda)$ . Since  $f^{-1}(\lambda) \neq 0$ , we must have at least  $(f/A)^{-1}(\lambda) \neq 0$  or  $(f/B)^{-1}(\lambda) \neq 0$ . To be specific, let us suppose that  $(f/A)^{-1}(\lambda) \neq 0$ . Therefore by assumption, there exists  $\mu/A \in T_1/A \cup T_2/A$  such that  $\mu/A \neq 0$  and  $\mu/A < (f/A)^{-1}(\lambda)$ . Then  $\mu \in T_1$  or  $\mu \in T_2$ ,  $\mu \neq 0$  and  $\mu < f^{-1}(\lambda)$ . This proves that  $f$  is somewhat pairwise fuzzy continuous.  $\square$

**DEFINITION 2.1.4.** Let  $X$  be any set and let  $(T_1, T_2)$  and  $(T_1^*, T_2^*)$  be any two fuzzy bitopologies for  $X$ . We say that  $(T_1, T_2)$  is *weakly pairwise equivalent* to  $(T_1^*, T_2^*)$  if  $\lambda \in T_1$  or  $\lambda \in T_2$  and  $\lambda \neq 0$ , then there is a  $\mu \in T_1^*$  or  $\mu \in T_2^*$  such that  $\mu \neq 0$  and  $\mu < \lambda$  and if  $\lambda \in T_1^*$  or  $\lambda \in T_2^*$  and  $\lambda \neq 0$ , then there is a  $\mu \in T_1$  or  $\mu \in T_2$  such that  $\mu \neq 0$  and  $\mu < \lambda$ .

It is easy to observe that two fuzzy bitopologies  $(T_1, T_2)$  and  $(T_1^*, T_2^*)$  are weakly pairwise equivalent iff the identity function from  $(X, T_1, T_2)$  onto  $(X, T_1^*, T_2^*)$  is somewhat pairwise fuzzy continuous in both directions.

**PROPOSITION 2.1.4.** Let  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be a somewhat pairwise fuzzy continuous function from a fuzzy bitopological space  $(X, T_1, T_2)$  into another fuzzy bitopological space  $(Y, S_1, S_2)$ . If  $(T_1, T_2)$  is weakly pairwise equivalent to  $(T_1^*, T_2^*)$ , then  $f : (X, T_1^*, T_2^*) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous.

*Proof.* Let  $\lambda \in S_1$  or  $\lambda \in S_2$  and  $f^{-1}(\lambda) \neq 0$ . Since  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous, there is a  $\mu \in T_1$ , or  $\mu \in T_2$ ,  $\mu \neq 0$  such that  $\mu < f^{-1}(\lambda)$ . But by hypothesis,  $(T_1^*, T_2^*)$  is weakly pairwise equivalent to  $(T_1, T_2)$ . There exists  $\eta \in T_1^*$  or  $\eta \in T_2^*$  such that  $\eta \neq 0$  and  $\eta < \mu$ . But  $\mu < f^{-1}(\lambda) \Rightarrow \eta < f^{-1}(\lambda)$ . Therefore,  $f : (X, T_1^*, T_2^*) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous.  $\square$

**PROPOSITION 2.1.5.** Let  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be a somewhat pairwise fuzzy continuous function from a fuzzy bitopological space  $(X, T_1, T_2)$  onto a fuzzy bitopological space  $(Y, S_1, S_2)$ . If  $(T_1^*, T_2^*)$

and  $(S_1^*, S_2^*)$  are fuzzy bitopologies for  $X$  and  $Y$  respectively such that  $(T_1^*, T_2^*)$  is weakly pairwise equivalent to  $(T_1, T_2)$  and  $(S_1^*, S_2^*)$  is weakly pairwise equivalent to  $(S_1, S_2)$ , then  $f : (X, T_1^*, T_2^*) \rightarrow (Y, S_1^*, S_2^*)$  is somewhat pairwise fuzzy continuous.

*Proof.* Let  $\lambda \in S_1^*$  or  $\lambda \in S_2^*$ , and  $f^{-1}(\lambda) \neq 0$ . Since  $(S_1^*, S_2^*)$  is weakly pairwise equivalent to  $(S_1, S_2)$ , there exists  $\eta \in S_1$  or  $\eta \in S_2$  and  $\eta \neq 0$  such that  $\eta < \lambda$ . Since  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous, there is a  $\mu \in T_1$  or  $\mu \in T_2$ ,  $\mu \neq 0$  such that  $\mu < f^{-1}(\eta)$ . Since  $(T_1^*, T_2^*)$  is weakly pairwise equivalent to  $(T_1, T_2)$ , there exists  $\gamma \in T_1^*$  or  $\gamma \in T_2^*$  such that  $\gamma \neq 0$  and  $\gamma < \mu < f^{-1}(\eta) < f^{-1}(\lambda)$ . Hence, the theorem.  $\square$

**DEFINITION 2.1.5.** A fuzzy bitopological space  $(X, T_1, T_2)$  is said to be *pairwise fuzzy separable* if there exists a pairwise dense fuzzy set  $\lambda$  in  $(X, T_1, T_2)$  such that  $\lambda \neq 0$  for atmost countably many points of  $(X, T_1, T_2)$ .

**EXAMPLE 2.1.5.** Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1, \delta\}$ ,  $T_2 = \{0, 1, \gamma\}$  where

$$\begin{aligned} \delta : X &\rightarrow [0, 1] \text{ is such that } \delta(a) = 1/2, \delta(b) = 1/4, \delta(c) = \\ &1/3 \text{ and} \\ \gamma : X &\rightarrow [0, 1] \text{ is such that } \gamma(a) = 1/3, \gamma(b) = 1/4, \gamma(c) = \\ &1/3. \end{aligned}$$

Define a fuzzy set  $\lambda : X \rightarrow [0, 1]$  is such that  $\lambda(a) = 3/4$ ,  $\lambda(b) = 3/4$ ,  $\lambda(c) = 3/4$ . Clearly the fuzzy bitopological space  $(X, T_1, T_2)$  is pairwise fuzzy separable.

**PROPOSITION 2.1.6.** *If  $f$  is somewhat pairwise fuzzy continuous from  $(X, T_1, T_2)$  onto  $(Y, S_1, S_2)$  and  $(X, T_1, T_2)$  is pairwise fuzzy separable, then  $(Y, S_1, S_2)$  is pairwise fuzzy separable.*

*Proof.* Since  $(X, T_1, T_2)$  is pairwise fuzzy separable, there exists a pairwise dense fuzzy set  $\lambda$  such that  $\lambda \neq 0$  for atmost countably many points of  $(X, T_1, T_2)$ . Since  $f$  is somewhat pairwise fuzzy continuous, Proposition 2.1.1 implies  $f(\lambda)$  is pairwise dense fuzzy in  $(Y, S_1, S_2)$ . Since  $\lambda \neq 0$  for atmost countably many points,  $f(\lambda) \neq 0$  for at most countably many points. Thus,  $(Y, S_1, S_2)$  is pairwise fuzzy separable.  $\square$

## 2.2. Somewhat pairwise fuzzy open functions

DEFINITION 2.2.1. Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. A mapping  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is called *pairwise fuzzy open* if  $\lambda \in T_1$  or  $\lambda \in T_2$  implies  $f(\lambda) \in S_1$  or  $f(\lambda) \in S_2$ .

DEFINITION 2.2.2. Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. A mapping  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is called *somewhat pairwise fuzzy open* if  $\lambda \in T_1$  or  $\lambda \in T_2$ ,  $\lambda \neq 0$  implies there exists a  $\mu \in S_1$  or  $\mu \in S_2$  such that  $\mu \neq 0$  and  $\mu \leq f(\lambda)$ . That is,  $\text{int}_{S_1} f(\lambda) \neq 0$  or  $\text{int}_{S_2} f(\lambda) \neq 0$ .

Clearly, every pairwise fuzzy open mapping is somewhat pairwise fuzzy open, but the converse is not true as the following example shows.

EXAMPLE 2.2.1. Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1\}$ ,  $T_2 = \{0, 1, \lambda\}$ ,  $S_1 = \{0, 1, \mu, \delta\}$ ,  $S_2 = \{0, 1, \delta\}$  where

$\lambda: X \rightarrow [0, 1]$  is such that  $\lambda(a) = 1$ ,  $\lambda(b) = 1$ ,  $\lambda(c) = 0$ .

$\mu: X \rightarrow [0, 1]$  is such that  $\mu(a) = 1$ ,  $\mu(b) = 0$ ,  $\mu(c) = 0$ ,

and

$\delta: X \rightarrow [0, 1]$  is such that  $\delta(a) = 0$ ,  $\delta(b) = 1$ ,  $\delta(c) = 0$ .

Define  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be the identity function. Since  $f(\lambda) = \lambda$  and  $\lambda \in T_2$  whereas  $\lambda \notin S_1$  and  $\lambda \notin S_2$ ,  $f$  is not pairwise fuzzy open. We now claim that  $f$  is somewhat pairwise fuzzy open.  $1$  and  $\lambda$  are non-zero  $T_1$ -fuzzy open sets or  $T_2$ -fuzzy open sets and  $f(1) = 1$ ,  $f(\lambda) = \lambda$ . Also,  $\delta \in S_1$ ,  $\delta \in S_2$  and  $\delta \neq 0$  such that  $\delta < 1$  and  $\delta < \lambda = f(\lambda)$ . Therefore  $f$  is somewhat pairwise fuzzy open.

PROPOSITION 2.2.1. Let  $(X, T_1, T_2)$ ,  $(Y, S_1, S_2)$  and  $(Z, R_1, R_2)$  be any three fuzzy bitopological spaces. If  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  and  $g: (Y, S_1, S_2) \rightarrow (Z, R_1, R_2)$  are somewhat pairwise fuzzy open functions, then  $g \circ f: (X, T_1, T_2) \rightarrow (Z, R_1, R_2)$  is somewhat pairwise fuzzy open function.

*Proof.* Let  $\lambda \in T_1$  or  $\lambda \in T_2$ . Since  $f$  is somewhat pairwise fuzzy open, there exists  $\mu \in S_1$  or  $\mu \in S_2$  such that  $\mu \leq f(\lambda)$ . Now,  $\text{int}_{S_1} f(\lambda) \in S_1$  and  $\text{int}_{S_2} f(\lambda) \in S_2$ . Also, since  $g$  is somewhat pairwise fuzzy open, there exists  $\gamma \in R_1$  or  $\gamma \in R_2$  such that



$\gamma < g(\text{int}_{S_1} f(\lambda))$  and  $\gamma < g(\text{int}_{S_2} f(\lambda))$ . But  $g(\text{int}_{S_1} f(\lambda)) < g(f(\lambda))$  and  $g(\text{int}_{S_2} f(\lambda)) < g(f(\lambda))$ . Thus, there exists  $\gamma \in R_1$  or  $\gamma \in R_2$  such that  $\gamma < (g \circ f)(\lambda)$ . That is,  $g \circ f$  is somewhat pairwise fuzzy open.  $\square$

**PROPOSITION 2.2.2.** *Suppose  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. Let  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be a one-to-one and onto function. Then the following conditions are equivalent.*

- (a)  $f$  is somewhat pairwise fuzzy open.
- (b) If  $\lambda$  is a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ , then  $f^{-1}(\lambda)$  is a pairwise dense fuzzy set in  $(X, T_1, T_2)$ .

*Proof.* (a)  $\Rightarrow$  (b). Assume (a). Suppose  $\lambda$  is a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ . We want to show that  $f^{-1}(\lambda)$  is a pairwise dense fuzzy set in  $(X, T_1, T_2)$ . Suppose not. Then there exists a  $T_1$ -fuzzy closed set or  $T_2$ -fuzzy closed set  $\mu$  such that  $f^{-1}(\lambda) < \mu < 1$ . That is,  $\lambda = f(f^{-1}(\lambda)) < f(\mu) < f(1)$  (since  $f$  is onto). Since  $f$  is somewhat pairwise fuzzy open,  $f(\mu) < \text{cl}_{S_1} f(\mu) < 1$  or  $f(\mu) < \text{cl}_{S_2} f(\mu) < 1$ . That is,  $\lambda < \text{cl}_{S_1} f(\mu) < 1$  or  $\lambda < \text{cl}_{S_2} f(\mu) < 1$ . This is a contradiction to our assumption that  $\lambda$  is a pairwise dense fuzzy set in  $(Y, S_1, S_2)$ . Hence,  $f^{-1}(\lambda)$  must be a pairwise dense fuzzy set.

(b)  $\Rightarrow$  (a). Assume (b). Suppose  $\lambda \in T_1$  or  $\lambda \in T_2$ ,  $\lambda \neq 0$ . We want to show that  $\text{int}_{S_1} f(\lambda) \neq 0$  or  $\text{int}_{S_2} f(\lambda) \neq 0$ . Suppose  $\text{int}_{S_1} f(\lambda) = 0$  and  $\text{int}_{S_2} f(\lambda) = 0$ . Then  $\text{cl}_{S_1}(1 - f(\lambda)) = 1 - \text{int}_{S_1} f(\lambda) = 1$  and  $\text{cl}_{S_2}(1 - f(\lambda)) = 1 - \text{int}_{S_2} f(\lambda) = 1$ . Therefore by assumption (b),  $f^{-1}(1 - f(\lambda))$  is a pairwise dense fuzzy set in  $(X, T_1, T_2)$ . But  $f^{-1}(1 - f(\lambda)) \leq 1 - \lambda$ . Since  $1 - \lambda$  is  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed and  $f^{-1}(1 - f(\lambda)) \leq 1 - \lambda$ ,

$$\text{cl}_{T_1}(f^{-1}(1 - f(\lambda))) \leq 1 - \lambda \quad \text{or} \quad \text{cl}_{T_2}(f^{-1}(1 - f(\lambda))) \leq 1 - \lambda.$$

That is,  $1 \leq 1 - \lambda$  implies  $\lambda = 0$ . Contradiction. Therefore,  $\text{int}_{S_1} f(\lambda) \neq 0$  or  $\text{int}_{S_2} f(\lambda) \neq 0$ . This proves that  $f$  is somewhat pairwise fuzzy open.  $\square$

**EXAMPLE 2.2.2.** Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1, \lambda\}$ ,  $T_2 = \{0, 1, \mu\}$ ,  $S_1 = \{0, 1, \mu\}$  and  $S_2 = \{0, 1\}$  where  $\lambda: X \rightarrow [0, 1]$  is such that  $\lambda(a) = 0$ ,  $\lambda(b) = 0$ ,  $\lambda(c) = 1$  and  $\mu: X \rightarrow [0, 1]$  is such that

$\mu(a) = 1, \mu(b) = 0, \mu(c) = 0$ . Define  $f : (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  as the identity function. Clearly,  $\mu$  is a pairwise dense fuzzy set in  $(X, S_1, S_2)$ . Further,  $f^{-1}(\mu) = \mu$  and  $\mu$  is not a pairwise dense fuzzy set in  $(X, T_1, T_2)$ . For  $\mu < 1 - \lambda < 1$  and  $1 - \lambda$  is  $T_1$ -fuzzy closed. Also  $f$  is not pairwise fuzzy open and  $f$  is not somewhat pairwise fuzzy open.

**PROPOSITION 2.2.3.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. If  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is a one-to-one and onto function, then the following conditions are equivalent.*

- (a)  $f$  is somewhat pairwise fuzzy open.
- (b) If  $\lambda$  is a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set in  $(X, T_1, T_2)$  such that  $f(\lambda) \neq 1$ , then there exists a  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set  $\mu$  in  $(Y, S_1, S_2)$  such that  $\mu \neq 1$  and  $\mu > f(\lambda)$ .

*Proof.* (a)  $\Rightarrow$  (b). Let  $\lambda$  be any  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set in  $(X, T_1, T_2)$  such that  $f(\lambda) \neq 1$ . Then  $1 - \lambda$  is a  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set such that  $f(1 - \lambda) = 1 - f(\lambda) \neq 0$ . As  $f$  is somewhat pairwise fuzzy open there exists a  $\theta \in S_1$  or  $\theta \in S_2$  such that  $\theta \neq 0$  and  $\theta < f(1 - \lambda)$ . Since  $\theta \in S_1$  or  $\theta \in S_2$  such that  $\theta \neq 0$ ,  $1 - \theta$  is a  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set in  $(Y, S_1, S_2)$  such that  $1 - \theta \neq 1$  and  $1 - \theta > f(\lambda)$ .

(b)  $\Rightarrow$  (a). Let  $\lambda$  be any  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set such that  $\lambda \neq 0$ . Then  $1 - \lambda$  is  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed and  $1 - \lambda \neq 1$ . And  $f(1 - \lambda) = 1 - f(\lambda) \neq 1$  (for, if  $1 - f(\lambda) = 1$ , then  $f(\lambda) = 0 \Rightarrow \lambda = 0$ ). Hence by (b), there exists a  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set  $\mu$  in  $(Y, S_1, S_2)$  such that  $\mu \neq 1$  and  $\mu > f(1 - \lambda) = 1 - f(\lambda)$ , that is,  $f(\lambda) > 1 - \mu$ . Clearly,  $1 - \mu$  is  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set in  $(Y, S_1, S_2)$  such that  $1 - \mu < f(\lambda)$  and  $1 - \mu \neq 0$ . This proves (a).  $\square$

**EXAMPLE 2.2.3.** Let  $X = \{a\}$  and  $Y = \{a, b\}$ . Define  $T_1 = \{0, 1\}$ ,  $T_2 = \{0, 1, \lambda_1\}$ ,  $S_1 = \{0, 1, \lambda_2\}$ ,  $S_2 = \{0, 1\}$ ,  $Q_1 = \{0, 1, \lambda_3\}$ ,  $Q_2 = \{0, 1\}$  where

- $\lambda_1 : X \rightarrow [0, 1]$  is such that  $\lambda_1(a) = 1/4$ ,
- $\lambda_2 : Y \rightarrow [0, 1]$  is such that  $\lambda_2(a) = 1/2, \lambda_2(b) = 0$  and
- $\lambda_3 : Y \rightarrow [0, 1]$  is such that  $\lambda_3(a) = 0, \lambda_3(b) = 1$ .

Clearly,  $S_1, S_2, Q_1, Q_2, T_1$  and  $T_2$  are fuzzy topologies. Define  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  as  $f(a) = a$ . Then

- (1)  $f$  is one-to-one but not onto.
- (2)  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy open. And  $f$  does not satisfy (b) of above Proposition 2.2.3. For  $1 - \lambda_1$  is a  $T_2$ -fuzzy closed set and  $f(1 - \lambda) \neq 1$ . There is no  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set  $\mu \neq 1$  such that  $\mu > f(1 - \lambda_1)$ . Therefore, condition (b) of the Proposition 2.2.3 is violated.
- (3) Now define  $g : (X, T_1, T_2) \rightarrow (Y, Q_1, Q_2)$  as  $g(a) = a$ .  $g$  is one-to-one but not onto.  $g$  satisfies (b) of Proposition 2.2.3, for  $g(1 - \lambda_1) \neq 1$  and  $1 - \lambda_3$  is  $Q_1$ -fuzzy closed set such that  $1 - \lambda_3 > g(1 - \lambda_1)$ .
- (4)  $g$  is not somewhat pairwise fuzzy open. For, the non-zero  $T_2$ -fuzzy open set  $\lambda_1$ , there is no  $Q_1$ -fuzzy open or  $Q_2$ -fuzzy open set  $\mu$  such that  $\mu < g(\lambda_1)$ . That is,  $g$  is not somewhat pairwise fuzzy open.

**PROPOSITION 2.2.4.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces and  $X = A \cup B$  where  $A$  and  $B$  are subsets of  $X$  and  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is a function such that  $f/A$  and  $f/B$  are somewhat pairwise fuzzy open, then  $f$  is somewhat pairwise fuzzy open.*

*Proof.* Let  $\lambda \in T_1$  or  $\lambda \in T_2$  be such that  $f(\lambda) \neq 0$ . Consider  $(f/A)(\lambda)$  and  $(f/B)(\lambda)$ . Since  $f(\lambda) \neq 0$ , we must have atleast  $(f/A)(\lambda) \neq 0$  or  $(f/B)(\lambda) \neq 0$ . To be specific, let us suppose that  $(f/A)(\lambda) \neq 0$ . Therefore by assumption, there exists  $\mu \in S_1$  or  $\mu \in S_2$  such that  $\mu \neq 0$  and  $\mu \leq (f/A)(\lambda)$ . That is,  $\mu \leq f(\lambda)$ .  $\square$

**PROPOSITION 2.2.5.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. If  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy open and  $(T_1^*, T_2^*)$  and  $(S_1^*, S_2^*)$  are fuzzy bitopologies for any two sets  $X$  and  $Y$  respectively such that  $(T_1^*, T_2^*)$  is weakly pairwise equivalent to  $(T_1, T_2)$  and  $(S_1^*, S_2^*)$  is weakly pairwise equivalent to  $(S_1, S_2)$ , then  $f : (X, T_1^*, T_2^*) \rightarrow (Y, S_1^*, S_2^*)$  is somewhat pairwise fuzzy open.*

*Proof.* Let  $\lambda \in T_1^*$  or  $\lambda \in T_2^*$ , and  $f^{-1}(\lambda) \neq 0$ . Since  $(T_1^*, T_2^*)$  is weakly pairwise equivalent to  $(T_1, T_2)$ , there exists  $\eta \in T_1$  or  $\eta \in T_2$

and  $\eta \neq 0$  such that  $\eta < \lambda$ . Since  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is somewhat pairwise fuzzy continuous, there is a  $\mu \in S_1$  or  $\mu \in S_2$ ,  $\mu \neq 0$  such that  $\mu < f(\eta)$ . By hypothesis,  $(S_1^*, S_2^*)$  is weakly pairwise equivalent to  $(S_1, S_2)$ , there exists  $\gamma \in S_1^*$  or  $\gamma \in S_2^*$  such that  $\gamma \neq 0$  and  $\gamma < \mu < f(\eta) < f(\lambda)$ . Hence, the result.  $\square$

**DEFINITION 2.2.3.** A function  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  is said to be *somewhat pairwise fuzzy homeomorphism* if  $f$  is one-to-one, onto, somewhat pairwise fuzzy continuous and somewhat pairwise fuzzy open.

Clearly, if  $f$  is somewhat pairwise fuzzy homeomorphism, then  $f^{-1}$  is also somewhat pairwise fuzzy homeomorphism.

**EXAMPLE 2.2.4.** Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1\}$ ,  $T_2 = \{0, 1, \lambda\}$ ,  $S_1 = \{0, 1, \mu, \delta\}$ ,  $S_2 = \{0, 1, \delta\}$ , where

$$\lambda : X \rightarrow [0, 1] \text{ is defined as } \lambda(a) = 0, \lambda(b) = 1, \lambda(c) = 0,$$

$$\mu : X \rightarrow [0, 1] \text{ is defined as } \mu(a) = 1, \mu(b) = 0, \mu(c) = 0$$

and

$$\delta : X \rightarrow [0, 1] \text{ is defined as } \delta(a) = 1, \delta(b) = 1, \delta(c) = 0.$$

Clearly,  $T_1, T_2, S_1, S_2$  are fuzzy topologies on  $X$ . Define  $f : (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  as  $f(a) = b, f(b) = a, f(c) = c$ . Clearly  $f$  is one-one and onto. Now

$$f^{-1}(\mu)(a) = 0, f^{-1}(\mu)(b) = 1, f^{-1}(\mu)(c) = 0,$$

$$f^{-1}(\delta)(a) = 1, f^{-1}(\delta)(b) = 1, \text{ and } f^{-1}(\delta)(c) = 0.$$

$\lambda$  is a non-zero  $T_2$ -fuzzy set such that  $\lambda \leq f^{-1}(\mu)$ . Therefore  $f$  is somewhat pairwise fuzzy continuous. Now,  $f(\lambda)(a) = 1, f(\lambda)(b) = 0$  and  $f(\lambda)(c) = 0$ . There exists a non-zero  $S_1$ -fuzzy set  $\mu$  such that  $\mu \leq f(\lambda)$ . Therefore  $f$  is somewhat pairwise fuzzy open. Hence  $f$  is somewhat pairwise fuzzy homeomorphism.

**DEFINITION 2.2.4.** Let  $(X, T_1, T_2)$  be a fuzzy bitopological space. A fuzzy set  $\lambda$  in  $(X, T_1, T_2)$  is called *pairwise nowhere dense fuzzy set* if there exists no non-zero fuzzy set  $\mu \in T_1$  or  $\mu \in T_2$  such that  $\mu < \text{cl}_{T_1}(\lambda)$  or  $\mu < \text{cl}_{T_2}(\lambda)$ .

**EXAMPLE 2.2.5.** Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, 1, \delta\}$ ,  $T_2 = \{0, 1, \mu\}$  where

$\delta: X \rightarrow [0, 1]$  is such that  $\delta(a) = 0$ ,  $\delta(b) = 1$ ,  $\delta(c) = 0$  and  
 $\mu: X \rightarrow [0, 1]$  is such that  $\mu(a) = 1/4$ ,  $\mu(b) = 1$ ,  $\mu(c) = 1/4$ .

Define a fuzzy set  $\lambda: X \rightarrow [0, 1]$  is such that  $\lambda(a) = 1/2$ ,  $\lambda(b) = 0$ ,  $\lambda(c) = 1/2$ . Clearly the fuzzy set  $\lambda$  is nowhere dense in  $(X, T_1, T_2)$ .

**Notation:** (i)  $\text{cl}_{T_1/T_2}(\lambda)$  denotes the  $T_1$ -fuzzy closure or  $T_2$ -fuzzy closure of a fuzzy set  $\lambda$  in a fuzzy bitopological space  $(X, T_1, T_2)$ .

(ii)  $\text{int}_{T_1/T_2}(\lambda)$  denotes the  $T_1$ -fuzzy interior or  $T_2$ -fuzzy interior of a fuzzy set  $\lambda$  in a fuzzy bitopological space  $(X, T_1, T_2)$ .

**PROPOSITION 2.2.6.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. Let  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be a somewhat pairwise fuzzy homeomorphism. If  $\lambda$  is a pairwise nowhere dense fuzzy set in  $(X, T_1, T_2)$ , then  $f(\lambda)$  is a pairwise nowhere dense fuzzy set in  $(Y, S_1, S_2)$ .*

*Proof.* Let  $\lambda$  be a pairwise nowhere dense fuzzy set in  $(X, T_1, T_2)$ . We want to show that  $f(\lambda)$  is pairwise nowhere dense fuzzy set in  $(Y, S_1, S_2)$ . Suppose not. Then there exists a non-zero  $S_1$ -fuzzy open set or  $S_2$ -fuzzy open set  $\theta$  such that  $0 \neq \theta < \text{cl}_{S_1} f(\lambda)$  or  $0 \neq \theta < \text{cl}_{S_2} f(\lambda)$  and  $0 = f^{-1}(0) \neq f^{-1}(\theta) < f^{-1}(\text{cl}_{S_1} f(\lambda))$  or  $0 = f^{-1}(0) \neq f^{-1}(\theta) < f^{-1}(\text{cl}_{S_2} f(\lambda))$ . Since  $f$  is somewhat pairwise fuzzy homeomorphism,  $f^{-1}(\text{cl}_{S_1} f(\lambda)) = \text{cl}_{T_1/T_2}(f^{-1}(f(\lambda))) = \text{cl}_{T_1/T_2}(\lambda)$  or  $f^{-1}(\text{cl}_{S_2} f(\lambda)) = \text{cl}_{T_1/T_2}(f^{-1}(f(\lambda))) = \text{cl}_{T_1/T_2}(\lambda)$  and  $f^{-1}(\theta)$  is a non-zero  $T_1$ -fuzzy open set or  $T_2$ -fuzzy open set such that  $0 \neq f^{-1}(\theta) < \text{cl}_{T_1/T_2}(\lambda)$ . This shows that  $\lambda$  is not a pairwise nowhere dense fuzzy set and this contradiction proves the proposition.  $\square$

**PROPOSITION 2.2.7.** *Let  $f: (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be any function from a fuzzy bitopological space  $(X, T_1, T_2)$  to another fuzzy bitopological space  $(Y, S_1, S_2)$ . If the graph  $g: X \rightarrow X \times Y$  of  $f$  is somewhat pairwise fuzzy continuous, then  $f$  is also somewhat pairwise fuzzy continuous.*

*Proof.* Let  $R_1$  and  $R_2$  be fuzzy topologies on  $X \times Y$ . Let  $\lambda$  be any non-zero  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set in  $(Y, S_1, S_2)$ . By Lemma 2.4 of Azad [1], we have  $f^{-1}(\lambda) = 1 \wedge f^{-1}(\lambda) = g^{-1}(1 \times \lambda)$ . Since  $g$  is somewhat pairwise fuzzy continuous and  $(1 \times \lambda) \neq 0$  is a  $R_1$ -fuzzy

open or  $R_2$ -fuzzy open set in  $X \times Y$ , there exists a  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set  $\mu \neq 0$  of  $X$  such that  $\mu \leq g^{-1}(1 \times \lambda) = f^{-1}(\lambda)$ . This proves that  $f$  is a somewhat pairwise fuzzy continuous function.  $\square$

**PROPOSITION 2.2.8.** *Let  $(X, T_1, T_2)$ ,  $(X_1, S_1, S_2)$  and  $(X_2, R_1, R_2)$  be any three fuzzy bitopological spaces and  $p_i : X_1 \times X_2 \rightarrow X_i$  ( $i = 1, 2$ ) be the projection mappings. If  $f : X \rightarrow X_1 \times X_2$  is a somewhat pairwise fuzzy continuous function, then  $p_i \circ f$  is also a somewhat pairwise fuzzy continuous function for  $i = 1, 2$ .*

*Proof.* Let  $Q_1$  and  $Q_2$  be fuzzy topologies on  $X_1 \times X_2$ . Let us prove the theorem for the case  $p_1 \circ f$ . Let  $\lambda$  be a  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set in  $(X_1, S_1, S_2)$ . Now  $(p_1 \circ f)^{-1}\lambda = f^{-1}(p_1^{-1}(\lambda))$ . Since  $p_1$  is fuzzy continuous,  $p_1^{-1}(\lambda)$  is  $Q_1$ -fuzzy open or  $Q_2$ -fuzzy open in  $X_1 \times X_2$ . Since  $f$  is somewhat pairwise fuzzy continuous, there exists a  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set  $\mu$  such that  $\mu \leq f^{-1}(p_1^{-1}(\lambda)) = (p_1 \circ f)^{-1}(\lambda)$ . Therefore  $p_1 \circ f$  is a somewhat pairwise fuzzy continuous function.  $\square$

### 2.3. Pairwise fuzzy almost continuous functions

**DEFINITION 2.3.1.** A function  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  from a fuzzy bitopological space  $(X, T_1, T_2)$  to another fuzzy bitopological space  $(Y, S_1, S_2)$  is called *pairwise fuzzy almost continuous* if the inverse image of a  $S_1$ -fuzzy regular open or  $S_2$ -fuzzy regular open set in  $(Y, S_1, S_2)$  is a  $T_1$ -fuzzy open or  $T_2$  fuzzy open in  $(X, T_1, T_2)$ . That is,  $f^{-1}(\lambda)$  is  $T_1$ -fuzzy open or  $T_2$ -fuzzy open in  $(X, T_1, T_2)$  for each  $S_1$ -fuzzy regular open or  $S_2$ -fuzzy regular open set  $\lambda$  in  $(Y, S_1, S_2)$ .

**PROPOSITION 2.3.1.** *Let  $(X, T_1, T_2)$  and  $(Y, S_1, S_2)$  be any two fuzzy bitopological spaces. Let  $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$  be any function. Then the following are equivalent.*

- (a)  $f$  is pairwise fuzzy almost continuous.
- (b) For each  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set  $\lambda$  in  $(Y, S_1, S_2)$ ,

$$f^{-1}(\lambda) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1}(\lambda))$$

and

$$f^{-1}(\lambda) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_2} \text{cl}_{S_2}(\lambda)).$$

(c) For each  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set  $\mu$  in  $(Y, S_1, S_2)$ ,

$$\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_1} \text{int}_{S_1}(\mu)) \leq f^{-1}(\mu)$$

and

$$\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_2} \text{int}_{S_2}(\mu)) \leq f^{-1}(\mu).$$

(d)  $f^{-1}(\lambda)$  is a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set for each  $S_1$ -fuzzy regular closed or  $S_2$ -fuzzy regular closed set  $\lambda$  in  $(Y, S_1, S_2)$ .

*Proof.* (a)  $\Rightarrow$  (b). Let  $\lambda$  be any  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set in  $(Y, S_1, S_2)$ . Then,  $\lambda \leq \text{int}_{S_1} \text{cl}_{S_1} \lambda$  or  $\lambda \leq \text{int}_{S_2} \text{cl}_{S_2} \lambda$ . Therefore,  $f^{-1}(\lambda) \leq f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda)$  or  $f^{-1}(\lambda) \leq f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda)$ . Since the interior of a fuzzy closed set is a fuzzy regular open set (by Theorem 5.6 of [1]),  $\text{int}_{S_1} \text{cl}_{S_1} \lambda$  and  $\text{int}_{S_2} \text{cl}_{S_2} \lambda$  are  $S_1$ -fuzzy regular open set and  $S_2$ -fuzzy regular open set respectively. Since  $f$  is pairwise fuzzy almost continuous,  $f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda)$  and  $f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda)$  are  $T_1$ -fuzzy open or  $T_2$ -fuzzy open sets. Therefore,

$$f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda) = \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda)$$

and

$$f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda) = \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda).$$

Also,

$$f^{-1}(\lambda) \leq f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda) = \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \lambda)$$

and

$$f^{-1}(\lambda) \leq f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda) = \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_2} \text{cl}_{S_2} \lambda).$$

This proves (b).

(b)  $\Rightarrow$  (c). Let  $\mu$  be any  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed set in  $(Y, S_1, S_2)$ . Then  $1 - \mu$  is a  $S_1$ -fuzzy open or  $S_2$ -fuzzy open set in  $(Y, S_1, S_2)$ . Now by (b),  $f^{-1}(1 - \mu) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1}(1 - \mu))$  and  $f^{-1}(1 - \mu) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_2} \text{cl}_{S_2}(1 - \mu))$ . This implies that

$$1 - f^{-1}(\mu) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1}(1 - \mu))$$

and

$$1 - f^{-1}(\mu) \leq \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_2} \text{cl}_{S_2}(1 - \mu)).$$

Then,  $f^{-1}(\mu) \geq 1 - \text{int}_{T_1/T_2} f^{-1}(\text{int}_{S_1} \text{cl}_{S_1}(1 - \mu))$  implies  $f^{-1}(\mu) \geq \text{cl}_{T_1/T_2}(1 - f^{-1}(\text{int}_{S_1} \text{cl}_{S_1}(1 - \mu)))$ . That is,  $f^{-1}(\mu) \geq \text{cl}_{T_1/T_2}(f^{-1}(1 -$

$\text{int}_{S_1} \text{cl}_{S_1}(1 - \mu))$  and  $f^{-1}(\mu) \geq \text{cl}_{T_1/T_2}(f^{-1}(1 - \text{int}_{S_2} \text{cl}_{S_2}(1 - \mu)))$ .  
Therefore,

$$f^{-1}(\mu) \geq \text{cl}_{T_1/T_2}(f^{-1}(\text{cl}_{S_1} \text{int}_{S_1} \mu))$$

and

$$f^{-1}(\mu) \geq \text{cl}_{T_1/T_2}(f^{-1}(\text{cl}_{S_2} \text{int}_{S_2} \mu)).$$

This proves (c).

(c)  $\Rightarrow$  (d). Let  $\lambda$  be any  $S_1$ -fuzzy regular closed or  $S_2$ -fuzzy regular closed set in  $(Y, S_1, S_2)$ . Then,  $\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_1} \text{int}_{S_1} \lambda) = \text{cl}_{T_1/T_2}(f^{-1}(\lambda))$  and  $\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_2} \text{int}_{S_2} \lambda) = \text{cl}_{T_1/T_2}(f^{-1}(\lambda))$ . Since  $\lambda$  is  $S_1$ -fuzzy regular closed or  $S_2$ -fuzzy regular closed,  $\lambda$  is  $S_1$ -fuzzy closed or  $S_2$ -fuzzy closed. Therefore by (c),  $\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_1} \text{int}_{S_1} \lambda) \leq f^{-1}(\lambda)$  and  $\text{cl}_{T_1/T_2} f^{-1}(\text{cl}_{S_2} \text{int}_{S_2} \lambda) \leq f^{-1}(\lambda)$ . Therefore  $\text{cl}_{T_1/T_2} f^{-1}(\lambda) \leq f^{-1}(\lambda)$  (Since  $\lambda$  is  $S_1$ -fuzzy regular closed or  $S_2$ -fuzzy regular closed). Therefore,  $f^{-1}(\lambda) = \text{cl}_{T_1/T_2} f^{-1}(\lambda)$  which implies that  $f^{-1}(\lambda)$  is a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set. This shows (d).

(d)  $\Rightarrow$  (a). Let  $\lambda$  be any  $S_1$ -fuzzy regular open or  $S_2$ -fuzzy regular open set in  $(Y, S_1, S_2)$ . Then  $1 - \lambda$  is  $S_1$ -fuzzy regular closed or  $S_2$ -fuzzy regular closed in  $(Y, S_1, S_2)$ . Then by (d),  $f^{-1}(1 - \lambda)$  is a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set in  $(X, T_1, T_2)$ . That is,  $1 - f^{-1}(\lambda)$  is a  $T_1$ -fuzzy closed or  $T_2$ -fuzzy closed set in  $(X, T_1, T_2)$ . Therefore  $f^{-1}(\lambda)$  is a  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set in  $(X, T_1, T_2)$ . Hence,  $f$  is pairwise fuzzy almost continuous. This proves (a).  $\square$

## 2.4. Interrelations

In this section, interrelations between somewhat pairwise fuzzy continuous functions and pairwise fuzzy almost continuous functions are discussed with necessary counter examples.

It is clear that a function which is pairwise fuzzy continuous is obviously a pairwise fuzzy almost continuous, though the converse is false as shown by the following example.



EXAMPLE 2.4.1. Let  $\lambda$ ,  $\mu$  and  $\delta$  be fuzzy sets of  $I = [0, 1]$  defined as follows:

$$\begin{aligned}\lambda(x) &= x, & x \in I, \\ \mu(x) &= 1 - x, & x \in I \\ \delta(x) &= \begin{cases} x, & 0 \leq x \leq 1/2, \\ 0, & 1/2 \leq x \leq 1. \end{cases}\end{aligned}$$

Let  $T_1 = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ ,  $T_2 = \{0, \lambda, 1\}$ ,  $S_1 = \{0, \lambda, \mu, \delta, \lambda \vee \mu, \lambda \wedge \mu, 1\}$  and  $S_2 = \{0, \delta, 1\}$ . Clearly,  $T_1, T_2, S_1, S_2$  are fuzzy topologies on  $I$ . Consider,  $f: (I, T_1, T_2) \rightarrow (I, S_1, S_2)$  defined by  $f(x) = x$  for each  $x \in I$ . It is clear that  $\lambda, \mu, \lambda \vee \mu$  and  $\lambda \wedge \mu$  being both  $S_1$ -fuzzy open and  $S_1$ -fuzzy closed are  $S_1$ -fuzzy regular open sets while  $\delta$  is not  $S_1$ -fuzzy regular open. But  $\delta \notin T_1$  and  $\delta \notin T_2$ . Therefore  $f$  is pairwise fuzzy almost continuous mapping. Since  $f^{-1}(\delta) = \delta \notin T_1$  and  $f^{-1}(\delta) = \delta \notin T_2$ ,  $f$  is not pairwise fuzzy continuous.

EXAMPLE 2.4.2. Let  $X = \{a, b, c\}$ . Define  $T_1 = \{0, \lambda, 1\}$ ,  $T_2 = \{0, 1\}$ ,  $S_1 = \{0, \mu, 1\}$  and  $S_2 = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$  where  $\lambda: X \rightarrow [0, 1]$  is such that  $\lambda(a) = 0$ ,  $\lambda(b) = 2/3$ ,  $\lambda(c) = 1/2$  and  $\mu: X \rightarrow [0, 1]$  is such that  $\mu(a) = 1$ ,  $\mu(b) = 0$ ,  $\mu(c) = 0$ . Clearly  $T_1, T_2, S_1, S_2$  are fuzzy topologies on  $X$ . Consider the function  $f: (X, T_1, T_2) \rightarrow (X, S_1, S_2)$  defined by  $f(x) = x$  for each  $x \in X$ . 0 and 1 are the  $S_1$ -fuzzy regular open and  $S_2$ -fuzzy regular open sets. But  $f^{-1}(1) = 1$  and  $f^{-1}(0) = 0$ . Therefore  $f$  is pairwise fuzzy almost continuous and  $f^{-1}(1) = 1 \neq 0$ ,  $f^{-1}(\mu) \neq 0$  but there exists no  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set  $\delta$  such that  $\delta \leq f^{-1}(\mu)$ . Therefore  $f$  is not somewhat pairwise fuzzy continuous. Since  $f^{-1}(\mu) = \mu \notin T_1 \cap T_2$ ,  $f$  is not pairwise fuzzy continuous.

EXAMPLE 2.4.3. Let  $\mu_1, \mu_2$  and  $\mu_3$  be fuzzy sets on  $I = [0, 1]$  defined as follows:

$$\mu_1(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 \leq x \leq 1, \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leq x \leq 1/4, \\ -4x + 2, & 1/4 \leq x \leq 1/2, \\ 0, & 1/2 \leq x \leq 1, \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 1, & 0 \leq x \leq 1/4, \\ 2(2 - 2x)/3, & 1/4 \leq x \leq 1. \end{cases}$$

Let  $S_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ ,  $S_2 = \{0, 1\}$ ,  $T_1 = \{0, \mu_3, 1\}$  and  $T_2 = \{0, \mu_1, 1\}$ . Then, clearly  $T_1, T_2, S_1$  and  $S_2$  are fuzzy topologies on  $I$ . Let  $f: (I, T_1, T_2) \rightarrow (I, S_1, S_2)$  be defined by  $f(x) = x/2$  for each  $x \in I$ . Then,  $f^{-1}(0) = 0$ ,  $f^{-1}(1) = 1$ ,  $f^{-1}(\mu_1) = 0$ ,  $f^{-1}(\mu_2) = 1 - \mu_1$  and  $f^{-1}(\mu_1 \vee \mu_2) = 1 - \mu_1$ . Also,  $\mu_3 \leq 1 - \mu_1$  and  $\mu_3$  is  $T_1$ -fuzzy open. For each  $S_1$ -fuzzy open and  $S_2$ -fuzzy open set  $\lambda$ , there exists a non-zero  $T_1$ -fuzzy open or  $T_2$ -fuzzy open set  $\mu$  such that  $\mu \leq f^{-1}(\lambda)$ . Therefore  $f$  is somewhat pairwise fuzzy continuous. Now, consider  $S_1$ -fuzzy open set  $\mu_2$ . We claim that  $f^{-1}(\mu_2) \leq \text{int}_{T_1/T_2}(\text{int}_{S_1} \text{cl}_{S_1} \mu_2)$ . For,  $\text{cl}_{S_1} \mu_2 = 1 - \mu_1$  implies

$$\text{int}_{S_1} \text{cl}_{S_1} \mu_2 = \text{int}_{S_1} (1 - \mu_1) = \mu_2 \Rightarrow$$

$$f^{-1}(\text{int}_{S_1} \text{cl}_{S_1} \mu_2) = f^{-1}(\mu_2) \Rightarrow \text{int}_{T_1/T_2} f^{-1}(\mu_2) \leq f^{-1}(\mu_2).$$

In other words  $\text{int}_{T_1/T_2} f^{-1}(\mu_2) \geq f^{-1}(\mu_2)$ . Therefore  $f$  is not pairwise fuzzy almost continuous.

REMARK 2.4.1. From the above examples we find that the two concepts namely somewhat pairwise fuzzy continuity and pairwise fuzzy almost continuity are independent notions.

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M. K. Uma and E. Roja  
Department of Mathematics  
Sri Sarada College for Women  
Salem 636 016  
Tamil Nadu, India

G. Balasubramanian  
Ramanujan Institute for Advanced Study in Mathematics  
University of Madras  
Chennai, Tamil Nadu, India