FUZZY SET CONNECTED FUNCTIONS

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Abstract. The purpose of this paper is to introduce the concept of fuzzy set connected functions and investigate their properties.

1. Introduction

Throughout this paper, $I$ will denote the closed unit interval $[0, 1]$. $X$ and $Y$ will be non-empty sets. Zadeh [15] generalized characteristic functions into fuzzy sets and Chang [5] introduced the topological structure in a class of fuzzy sets in a given set. For $X$, $I^X$ denotes the collection of all functions from $X$ into $I$. A member $\lambda$ of $I^X$ is called a fuzzy set of $X$. We will denote fuzzy sets in $X$ by $\lambda$, $\mu$, $\delta$, $\xi$ and etc. The fuzzy null set 0 and the fuzzy whole set 1 denote constant functions taking 0 and 1 for each $x \in X$, respectively. If there are confusions in using 1, we will use the whole set $X$ instead of 1. A fuzzy set $\lambda$ is said to be contained in a fuzzy set $\mu$ (denoted by $\lambda \leq \mu$) iff $\lambda(x) \leq \mu(x)$ for each $x \in X$. The complement $1 - \lambda$ of a fuzzy set $\lambda$ of $X$ is defined by $(1 - \lambda)(x)$ for each $x \in X$.

If $\lambda$ is a fuzzy set of $X$ and $\delta$ is a fuzzy set of $Y$, then $\lambda \times \delta$ is a fuzzy set of $X \times Y$, defined by $(\lambda \times \delta)(x, y) = \min \{\lambda(x), \delta(y)\}$ for each $(x, y) \in X \times Y$.

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A **fuzzy point** $x_\beta$ of $X$ is a fuzzy set of $X$ such that it takes the value 0 for all $y \in X$ except $x$, that is, it is defined as

$$x_\beta(y) = \begin{cases} \beta, & \text{if } y = x \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in X$. A fuzzy point $x_\beta$ is said to be contained in a fuzzy set $\lambda$ (denoted by $x_\beta \in \lambda$) if $\beta \leq \lambda(x)$. Let $f : X \to Y$ be a function and $\lambda$ be a fuzzy set of $X$. Then a fuzzy set $f(\lambda)$ of $Y$ is defined as

$$f(\lambda)(y) = \begin{cases} \lor_{x \in f^{-1}(y)} \lambda(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$, and if $\mu$ is a fuzzy set of $Y$, then a fuzzy set of $X$ $f^{-1}(\mu)$ is defined as

$$f^{-1}(\mu)(x) = (\mu \circ f)(x) = \mu(f(x))$$

for each $x \in X$.

### 2. Definition and Terminology

From now on, $f : (X, \tau) \to (Y, \tau^*)$ denotes a function from a fuzzy topological space $(X, \tau)$ into a fuzzy topological space $(Y, \tau^*)$, and in this section we describe definitions and theorems without their proofs, which we need in the last section. For concepts which is not defined here, we refer to [4], [5], [8], [10], [12] and [13].

Let $(X, \tau)$ be a fuzzy topological space (written as $fts$). For a fuzzy set $\lambda$ of $X$, its closure $\text{Cl}(\lambda)$ and its interior $\text{Int}(\lambda)$ are defined by $\text{Cl}(\lambda) = \lor \{\nu : \lambda \leq \nu, 1-\nu \in \tau(X)\}$ and $\text{Int}(\lambda) = \lor \{\nu : \nu \leq \lambda, \nu \in \tau(X)\}$, respectively. A fuzzy set which is both fuzzy open and fuzzy closed is called a fuzzy clopen set, and a class of all fuzzy clopen set of a fts $X$ will be denoted by $\text{FCO}(X)$.

**Definition 2.1.** A fts $X$ is said to be **fuzzy connected** if 1 and 0 are the only fuzzy subsets of $X$ which are both fuzzy open and fuzzy closed.

**Theorem 2.2.** A fts $X$ is not fuzzy connected if there are fuzzy clopen sets $\lambda(\neq 1)$ and $\delta(\neq 1)$ in $X$ such that $\lambda \lor \delta = 1$ and $\lambda \land \delta = 0$. 
In the above theorem, the condition of $\lambda(\neq 1)$ and $\delta(\neq 1)$ can not be dropped, as shown by the following example.

**Example 2.3.** For a fts $(X, \tau)$ where $\tau = \{0, 1\}$. Taking $\lambda = 1$ and $\delta = 0$, $\lambda, \delta \in \text{FCO}(X)$, $\lambda \lor \delta = 1$ and $\lambda \land \delta = 0$. But $(X, \tau)$ is fuzzy connected.

**Definition 2.4.** A fts $X$ is said to be *fuzzy connected between fuzzy sets* $\lambda$ and $\delta$ [9] if there is no fuzzy clopen set $\eta$ such that $\lambda \leq \eta$ and $\eta \land \delta = 0$.

Let a fts $Y$ be a subspace of a fts $X$. It is shown [9] that if $Y$ is fuzzy connected between fuzzy sets $\lambda$ and $\delta$, then $X$ is fuzzy connected between fuzzy sets $\lambda$ and $\delta$.

**Theorem 2.5.** If a fts $X$ is not fuzzy connected between fuzzy sets $\lambda$ and $\delta$, then there is a $\xi \in \text{FCO}(X)$ such that $\lambda \leq \xi \leq (1 - \delta)$.

The converse of the above theorem may not be true, as shown by the next example.

**Example 2.6.** Let $I = [0, 1]$ and $\tau = \{0, \xi, 1\}$ where $\xi(x) = \frac{1}{2}$ for all $x \in I$. Then $\xi \in \text{FCO}(I)$. Let $\lambda(x) = \frac{1}{3}$ and $\delta(x) = \frac{1}{2}$ for all $x \in I$. Then $\lambda \leq \xi \leq 1 - \delta$, but $(I, \tau)$ is fuzzy connected between $\lambda$ and $\delta$ since there is no $\eta \in \text{FCO}(I)$ such that $\lambda \leq \eta$ and $\eta \land \delta = 0$.

**Definition 2.7.** A fts $(X, \tau)$ is said to be *fuzzy extremally disconnected* [3] if $\lambda \in \tau$ implies $\text{Cl}(\lambda) \in \tau$.

**Definition 2.8.** $f : (X, \tau) \to (Y, \tau^*)$ is said to be *fuzzy slightly continuous* if for each $\mu \in \text{FCO}(Y)$, there exists a $\lambda \in \tau$ such that $f(\lambda) \leq \mu$.

**Theorem 2.9.** $f : (X, \tau) \to (Y, \tau^*)$ is fuzzy slightly continuous if for each $\mu \in \text{FCO}(Y)$, $f^{-1}(\mu) \in \tau$.

The converse of Theorem 2.9 may not be true, as shown by the following.

**Example 2.10.** Let $I = [0, 1]$ and $\tau = \{0, \lambda, 1\}$ and $\tau^* = \{0, \mu, 1\}$ where $\lambda(x) = \frac{1}{3}$ and $\mu(x) = \frac{1}{2}$ for all $x \in I$. And let $f : (I, \tau) \to (I, \tau^*)$ be the identity. Then $f$ is fuzzy slightly continuous and $\mu \in \text{FCO}(I)$, but $f^{-1} \notin \tau$ because $f^{-1}(\mu)(x) = \mu(f(x)) = \mu(x)$ for all $x \in X$. 

Definition 2.11. $f : (X, \tau) \to (Y, \tau^*)$ is said to be fuzzy continuous [5] if for each $\mu \in \tau^*$, $f^{-1}(\mu) \in \tau$.

Definition 2.12. $f : (X, \tau) \to (Y, \tau^*)$ is said to be fuzzy weakly continuous [1] if for each $\mu \in \tau^*$, $f^{-1}(\mu) \leq \text{Int}(f^{-1}(\text{Cl}(\mu)))$.

Let $\lambda$ be a fuzzy set of an fts $X$. Then $\lambda$ is said to be fuzzy regular open [4] if $\text{Int}(\text{Cl}(\lambda)) = \lambda$.

Definition 2.13. $f : (X, \tau) \to (Y, \tau^*)$ is said to be fuzzy almost continuous [1] if for each fuzzy regular open set $\mu$ of $Y$, $f^{-1}(\mu) \in \tau$.

3. Fuzzy set connected functions

In this section $X$, $Y$ and $Z$ will denote fts’ without specification.

Definition 3.1. A function $f : X \to Y$ is said to be fuzzy set connected provided that $f(X)$ is fuzzy connected between fuzzy sets $f(\lambda)$ and $f(\delta)$ with respect to relative fuzzy topology if $X$ is fuzzy connected between fuzzy sets $\lambda$ and $\delta$.

Theorem 3.2. A function $f : X \to Y$ is fuzzy set connected if and only if for each $\xi \in \text{FCO}(f(X))$, $f^{-1}(\xi) \in \text{FCO}(f(X))$.

Proof. Let $X$ be fuzzy connected between fuzzy sets $\lambda$ and $\delta$. Suppose $f(X)$ is not fuzzy connected between fuzzy sets $f(\lambda)$ and $f(\delta)$. Then by Theorem 2.5 there exists a $\xi \in \text{FCO}(f(X))$ such that $f(\lambda) \leq \xi \leq (1 - f(\delta))$. By hypothesis, $f^{-1}(\xi) \in \text{FCO}(X)$ and $\lambda \leq f^{-1}(\xi) \leq (1 - \delta)$. Therefore, $X$ is not fuzzy connected between $\lambda$ and $\delta$. It is a contradiction. Hence $f$ is fuzzy set connected.

To show the converse, let $f : X \to Y$ be fuzzy set connected and $\xi \in \text{FCO}(f(X))$. Suppose $f^{-1}(\xi)$ is not a fuzzy clopen set of $X$. Then $X$ is fuzzy connected between fuzzy sets $f^{-1}(\xi)$ and $1 - f^{-1}(\xi)$. Therefore, $f(X)$ is fuzzy connected between fuzzy sets $f(f^{-1}(\xi))$ and $f(1 - f^{-1}(\xi))$. But $f(f^{-1}(\xi)) = \xi \wedge f(X) = \xi$ and $f(1 - f^{-1}(\xi)) = f(X) \wedge (1 - \xi) = 1 - \xi$ implies that $\xi$ is not a fuzzy clopen set of $f(X)$. It is a contradiction.

Theorem 3.3. IF $f : (X, \tau) \to (Y, \tau^*)$ is fuzzy set connected, then $f^{-1}(\mu) \in \text{FCO}(X)$ for each $\mu \in \text{FCO}(Y)$. 
Notice that $f^{-1}(\xi) \in \text{FCO}(X)$ for any $\xi \in \text{FCO}(Y)$, if $f : X \to Y$ is fuzzy set connected.

**Theorem 3.4.** Every fuzzy continuous function is fuzzy set connected.

The converse of Theorem 3.4 may not be true, as shown by the following example.

**Example 3.5.** Let $X = \{x, y\}$, $Y = \{a, b\}$ and $\lambda \leq X$, $\mu \leq Y$ defined as follows:

- $\lambda(x) = 0.3$, $\lambda(y) = 0.4$
- $\mu(a) = 0.6$, $\mu(b) = 0.5$

Let $\tau = \{0, \lambda, X\}$ and $\tau^* = \{0, \mu, Y\}$. Then the function $f : (X, \tau) \to (Y, \tau^*)$ defined by $f(x) = a$ and $f(y) = b$ is fuzzy set connected, but not fuzzy continuous.

**Theorem 3.6.** Every function $f : X \to Y$ such that $f(X)$ is fuzzy connected is a fuzzy set connected function.

*Proof.* Let $f(X)$ be fuzzy connected. Then there is no non-empty proper fuzzy set of $f(X)$ which is clopen. Hence vacuously $f$ is fuzzy set connected. \hfill \Box

**Theorem 3.7.** Let $f : X \to Y$ be a fuzzy set connected function. If $X$ is fuzzy connected, then $f(X)$ is fuzzy connected.

*Proof.* Suppose $f(X)$ is fuzzy disconnected. Then there is a non-empty proper closed open fuzzy set $\xi$ of $f(X)$. Since $f$ is fuzzy set connected, by Theorem 3.2 $f^{-1}(\xi)$ is a non-empty proper closed open fuzzy set of $X$. Consequently $X$ is not fuzzy connected. It contradicts. \hfill \Box

**Theorem 3.8.** Let $f : X \to Y$ be a fuzzy set connected surjection and $g : Y \to Z$ be a fuzzy set connected function. Then $g \circ f : X \to Z$ is fuzzy set connected.

*Proof.* Let $\xi \in \text{FCO}(g(Y))$. Then $g^{-1}(\xi) \in \text{FCO}(Y) = \text{FCO}(f(X))$. Thus $f^{-1}(g^{-1}(\xi)) \in \text{FCO}(X)$. Now $(g \circ f)(X) = g(Y)$, and $(g \circ f)^{-1}(\xi) = f^{-1}(g^{-1}(\xi))$. So $g \circ f$ is fuzzy set connected by Theorem 3.2. \hfill \Box
Theorem 3.9. Let \( f : X \to Y \) be a function and \( g : X \to X \times Y \) be the graph function of \( f \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X \). If \( g \) is fuzzy set connected, then \( f \) is fuzzy set connected.

Proof. Let \( \xi \) any fuzzy clopen set of the subspace \( f(X) \) of \( Y \). Then \( X \times \xi \) is a fuzzy clopen set of the subspace \( X \times f(X) \) of the fuzzy product space \( X \times Y \). Since \( g(X) \) is a subset of \( X \times f(X) \), \( (X \times \xi) \land g(X) \) is a fuzzy clopen set of the subspace \( g(X) \) of \( X \times Y \). By Theorem 3.2, \( g^{-1}((X \times \xi) \land g(X)) \subseteq \mathrm{FCO}(X) \). It follows from \( g^{-1}((X \times \xi) \land g(X)) = g^{-1}(X \times \xi) = f^{-1}(\xi) \) that \( f^{-1}(\xi) \subseteq \mathrm{FCO}(X) \). Hence \( f \) is fuzzy set connected.

Theorem 3.10. If a surjection \( f : (X, \tau) \to (Y, \tau^*) \) is fuzzy weakly continuous, then \( f \) is fuzzy set connected.

Proof. Let \( \mu \in \mathrm{FCO}(Y) \). Since \( f \) is fuzzy weakly continuous, \( f^{-1}(\mu) \subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(\mu))) = \mathrm{Int}(f^{-1}(\mu)) \). Hence \( f^{-1}(\mu) \in \tau \). Moreover, we obtain \( \mathrm{Cl}(f^{-1}(\mu)) \subseteq f^{-1}(\mathrm{Cl}(\mu)) = f^{-1}(\mu) \) from [14, Theorem 4.6]. This shows that \( f^{-1}(\mu) \) is a closed fuzzy set of \( X \). Since \( f \) is surjective, by Theorem 3.2 we obtain that \( f \) is a fuzzy set connected function.

The converse of Theorem 3.10 may be false, as shown by the following example.

Example 3.11. Let \( X = \{x, y\} \), \( Y = \{a, b\} \) and \( \lambda \leq X \), \( \mu \leq Y \) defined as follows:
\[
\begin{align*}
\lambda(x) &= 0.3, \quad \lambda(y) = 0.4 \\
\mu(a) &= 0.4, \quad \mu(b) = 0.5
\end{align*}
\]
Let \( \tau = \{0, \lambda, X\} \) and \( \tau^* = \{0, \mu, Y\} \). Then the function \( f : (X, \tau) \to (Y, \tau^*) \) defined by \( f(x) = f(y) = a \) is fuzzy set connected, but not fuzzy weakly continuous.

Theorem 3.12. Let \( Y \) be fuzzy extremally disconnected. If \( f : X \to Y \) is fuzzy set connected and surjective, then \( f \) is fuzzy almost continuous.

Proof. Let \( x_p \) be a fuzzy point of \( X \) and \( \mu \) be a fuzzy open set of \( Y \) containing \( f(x_p) \). Since \( Y \) is fuzzy extremally disconnected, \( \mathrm{Cl}(\mu) \in \mathrm{FCO}(Y) \). Thus \( \mathrm{Cl}(\mu) \leq f(X) \) and \( \mathrm{Cl}(\mu) \in \mathrm{FCO}(f(X)) \). Putting \( f^{-1}[\mathrm{Cl}(\mu) \land f(X)] = \lambda \), it follows from Theorem 3.2 that \( \lambda \in \mathrm{FCO}(X) \).
because $f$ is fuzzy set connected. Clearly $x_p \in \lambda$ and $f(\lambda) \leq \text{Cl}(\mu) \leq \text{Int}(\text{Cl}(\mu))$. Hence by [14, Theorem 4.12], $f$ is fuzzy almost continuous.

**Corollary 3.13.** Let $Y$ be fuzzy extremally disconnected. If a function $f : X \to Y$ is fuzzy set connected and is surjective, then $f$ is fuzzy weakly continuous.

**Theorem 3.14.** Let $Y$ be fuzzy extremally disconnected and $f : X \to Y$ be surjective. Then the follows are equivalent:

1. $f$ is fuzzy set connected.
2. $f$ is fuzzy almost continuous.
3. $f$ is fuzzy weakly continuous.

**Proof.** This is an immediate consequence of Theorem 3.10, Theorem 3.12 and Corollary 3.13.

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