Journal of Korean
Data & Information Science Society
2007, Vol. 18, No. 1, pp. 211 ~ 218

On Strict Stationarity of Nonlinear Time Series Models without Irreducibility or Continuity Condition[1)]

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Abstract

Nonlinear ARMA model $X_n = h(X_{n-1},\cdots,X_{n-p},e_{n-1},\cdots,e_{n-q}) + e_n$ is considered and easy-to-check sufficient condition for strict stationarity of $\{X_n\}$ without some irreducibility or continuity assumption is given. Threshold ARMA(p, q) and momentum threshold ARMA(p, q) models are examined as special cases.

Keywords : Nonlinear ARMA(p,q) Model, Stationarity, Uniform, Countable Additivity

1. Introduction

Let $\{Y_n : n = 0,1,2,\cdots\}$ be a discrete time Markov chain defined on $R^k (k \geq 1)$ with homogeneous n-step transition probabilities $P^n(y,A) = \mathrm{Prob}(Y_n \in A | Y_0 = y)$, $y \in R^k$, $A \in B(R^k)$ where $B(R^k)$ is the Borel σ-field of R^k. Let $P^1(y,A)$ be simply written as $P(y,A)$.

It is known that the following Foster-Lyapounov drift condition can be used to prove stationarity, ergodicity and geometric ergodicity of general nonlinear ARMA(p, q) models under some irreducibility or continuity condition (see, e.g., Meyn and Tweedie(1993)).

Drift condition: There exists an extended real valued nonnegative measurable function g with $g(y) < \infty$ for at least one y, such that for some constants $b < \infty$, $0 < \lambda < 1$ and a compact set K in $B(R^k)$,

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$$\int g(z)P(y,dz) \leq \lambda g(y) + bI_K(y), \tag{1.1}$$

where I_K denotes the indicator function of K.

To prove the irreducibility of autoregressive processes with additive errors is relatively simple in many cases. But the irreducibility of ARMA(p, q)-type models with $q \geq 1$ is far from easy to check. When the generating function of a process is continuous, the process has weak Feller property. However, in general, evaluating continuity condition is also an awkward task or requires rather strong assumptions. There are many papers on these subjects in the literature. For example, Bhattacharya and Lee(1995) consider the process $X_n = h(X_{n-1}, \cdots, X_{n-p}) + e_n$ and give sufficient condition for φ-irreducibility and geometric ergodicity of the process. Nonlinear ARMA process given by $X_n = h(X_{n-1}, \cdots, X_{n-p}, e_n, e_{n-1}, \cdots, e_{n-p})$ is examined by Cline and Pu(1998) and a sufficient condition for irreducibility of the auxiliary Markov chain is given, in which bijective continuity with differentiable inverse on h is required.

Liu and Susko(1992) derives a necessary and sufficient condition for strict stationarity of a general Markov chain without irreducibility and continuity assumption. Following so called uniform countable additivity condition is adopted in place of irreducibility or continuity condition.

Uniform countable additivity: If A_n is any sequence in $B(R^k)$ with $A_n \downarrow \varnothing$, then

$$\lim_{n \to \infty} sup_{y \in K} P(y, A_n) = 0 \tag{1.2}$$

for every compact set K in $B(R^k)$.

Verifying the uniform countable additivity condition appear to be simpler in many cases than verifying some irreducibility or continuity condition.

Liu and Susko(1992) (see, also Fonseca and Tweedie(2002)) show that the existence of a Lyapounov function g satisfying the inequality in (1.1) together with uniform countable additivity assumption ensures the existence of a stationary solution.

Theorem 1. (Theorem 1.1, Liu and Susko(1992)) Suppose that the transition probability function of Y_n holds uniform countable additivity condition (1.2) and that there is a function g which is nonnegative and measurable with $g(y) < \infty$ for at least one y satisfies (1.1) for some $\lambda < 1$, $b < \infty$ and a compact set K in R^k, and that for some sequence of compact sets $K_n \uparrow R^k$, $\lim_{y \notin K_n} g(y) \to \infty$. Then Y_n

has a stationary probability measure π and $\int g(y)\pi(dy) < \infty$.

In this paper, we consider a nonlinear ARMA(p, q) $(p \geq 1, q \geq 0)$ model given by

$$X_n = h(X_{n-1}, \cdots, X_{n-p}, e_{n-1}, \cdots, e_{n-q}) + e_n \tag{1.3}$$

where $\{e_n\}$ is a sequence of independent and identically distributed random variables and $h(\cdot)$ is some measurable function on R^{p+q}.

The goal of this paper is to give a sufficient condition for the existence of a stationary solution of the process given by (1.3). Because of difficulty of proving irreducibility, Feller continuity or T-chain property for the model without continuity assumption on h, we drive the desired result via uniform countable additivity condition. Threshold ARMA model and momentum threshold ARMA model are considered as special cases.

2. Stationarity for Nonlinear ARMA Models

Consider a nonlinear ARMA(p, q) $(p \geq 1, q \geq 0)$ model given by

$$X_n = h(X_{n-1}, \cdots, X_{n-p}, e_{n-1}, \cdots, e_{n-q}) + e_n, \tag{2.1}$$

where $\{e_n\}$ is a sequence of independent and identically distributed random variables and $h(\cdot)$ is some measurable function on R^{p+q}.

Define

$$Y_n = (X_n, e_n, X_{n-1}, \cdots, X_{n-p+1}, e_{n-1}, \cdots, e_{n-q+1}). \tag{2.2}$$

Then

$$Y_{n+1} = H(Y_n) + C \cdot e_{n+1}, \tag{2.3}$$

where

$$H(Y_n) = (h^*(Y_n), 0, X_n, \cdots, X_{n-p+2}, e_n, \cdots, e_{n-q+2}),,$$
$$C = (1, 1, 0, 0, \cdots, 0),$$
$$h^*(Y_n) = h(T(Y_n)),$$
$$T(Y_n) = (X_n, X_{n-1}, \cdots, X_{n-p+1}, e_n, e_{n-1}, \cdots, e_{n-q+1}).$$

Then Y_n is a Markov chain with R^{p+q} as its state space and $P(y, dz)$ its one-step transition probability function.

Lemma 1. Consider the process X_n given in (2.1). Assume that (1) h is locally bounded (2) e_n has a probability density function $f(\cdot)$ with respect to the Lebesgue measure. Then the uniform countable additivity condition (1.2) holds for Y_n in (2.1)-(2.3).

Proof. Recall That $B(R^{p+q}) = \sigma(\{A_1 \times \cdots \times A_{p+q} | A_i \in B(R^1),\ i = 1, \cdots, p+q\})$. To prove the uniform countable additivity for Y_n, it suffices to show the validity of (1.2) for the sets of the form $A_n = A_1^{(n)} \times A_2^{(n)} \times R^{p+q-2}$ decreasing to the empty set where $A_1^{(n)}, A_2^{(n)} \in B(R^1)$. Note that $A_n \downarrow \varnothing$ implies that $A_1^{(n)} \downarrow \varnothing$ or $A_2^{(n)} \downarrow \varnothing$.

For $y = (x_1, u_1, x_2, \cdots, x_p, u_2, \cdots, u_q)$,

$$
\begin{aligned}
P(y, A_n) &= \mathrm{Prob}((h^*(y) + e_1,\ e_1,\ x_1,\ \cdots,\ x_{p-1}, u_1, \cdots, u_{q-1}) \in A_n) \\
&\le \mathrm{Prob}((h^{*(y)} + e_1,\ e_1) \in A_1^{(n)} \times A_2^{(n)}) \\
&= \int_{(h^{*(y)} + z,\, z) \in A_1^{(n)} \times A_2^{(n)}} f(z) dz \\
&= \int_{h^*(y) + z \in A_1^{(n)}} I_{A_2^{(n)}}(z) f(z) dz. \qquad (2.4)
\end{aligned}
$$

By local boundedness of h, there is a compact set $L \subset R$ such that $\{h^*(y) | y \in K\} \subset L$. Therefore,

$$
\begin{aligned}
\sup_{y \in K} P(y, A_n) &= \sup_{y \in K} \int I_{A_1^{(n)} - h^*(y)}(z) I_{A_2^{(n)}}(z) f(z) dz \\
&\le \sup_{v \in L} \int I_{A_1^{(n)} - v}(z) I_{A_2^{(n)}}(z) f(z) dz. \qquad (2.5)
\end{aligned}
$$

If $A_2^{(n)} \downarrow \varnothing$, from integrability of f,

$$
\sup_{y \in K} P(y, A_n) \le \int I_{A_2^{(n)}}(z) f(z) dz \to 0 \quad \text{as } n \to \infty.
$$

Suppose $A_1^{(n)} \downarrow \varnothing$. Then we have from (2.5) that

$$
\sup_{y \in K} P(y, A_n) \le \sup_{v \in L} \int I_{A_1^{(n)} - v}(z) f(z) dz. \qquad (2.6)
$$

The proof that the righthand side of (2.6) converges to 0 is similar to that of Theorem 3 in Fonseca and Tweedie(2002). For ease of reading, sketch of the proof is given as follows: Let $g_n(v) = \int I_{A_1^{(n)} - v}(z) f(z) dz$. Then $g_n(v)$ is uniformly bounded in n and v and continuous and hence $\{g_n\}$ is equicontinuous. For any $\epsilon > 0$, by compactness of L and equicontinuity of g_n, we can choose finite points

$v_1, v_2, \cdots, v_m$ such that

$$\sup_{v \in L} g_n(v) \le \max_{1 \le i \le m} g_n(v_i) + \epsilon.$$

But for fixed $v_i (1 \le i \le m)$, $g_n(v_i) \downarrow 0$ as $n \to \infty$ and hence the proof is completed.

Theorem 2. For the process X_n given in (2.1), assume that e_n has a probability density function with respect to the Lebesgue measure. Suppose that for $x = (x_1, \cdots, x_p, u_1, \cdots, u_q)$,

$$|h(x)| \le \sum_{j=1}^{l} (a_0^{(j)} + \sum_{i=1}^{p} a_i^{(j)} |x_i| + \sum_{i=1}^{q} b_i^{(j)} |u_i|) I_{\omega(x) \in R_j},$$

for some $a_i^{(j)} \ge 0 (i = 0, 1, \cdots, p)$, $b_k^{(j)} \ge 0$ $(k = 1, \cdots, q)$, $j = 1, \cdots, l$, where $\omega(x)$ is a function on R^{p+q} to R and $R_j, j = 1, \cdots, l$ is a partition of R. If $\max_j \sum_{i=1}^{p} a_i^{(j)} < 1$, then a strictly stationary solution of (2.1) exists and $E_\pi |X_n| < \infty$ with π as its stationary distribution.

Proof of Theorem 2 : We first show the existence of a Lyapounov function which satisfies the inequality given in (1.1). Choose $\rho > 0$ and $\delta > 0$ so that

$$\max_{1 \le j \le l} \sum_{i=1}^{p} a_i^{(j)} < \rho < \rho^{\frac{1}{p}} < \delta < 1.$$

Choose $\gamma_1 > 0$ arbitrarily and fix. Then define

$$\gamma_i = \rho^{\frac{1}{p}} \gamma_{i-1}, \quad i = 2, \cdots, p$$
$$b_i = \max_j b_i^{(j)}, \ 1 \le i \le q$$

and

$$\gamma_{p+i} = \gamma_1 \left(\frac{b_i}{\delta} + \frac{b_{i+1}}{\delta^2} + \cdots + \frac{b_q}{\delta^{q-i+1}} \right), \ i = 1, \cdots, q.$$

Now define a Lyapounov function $g^*(\cdot)$ by

$$g^*(y) = g(T(y)), \tag{2.7}$$

where the function g: $R^{p+q} \to R$ is given by

$$g(x_1, \cdots, x_{p+q}) = 1 + \max_{1 \le i \le p}\{\gamma_i|x_i|\} + \sum_{i=p+1}^{p+q} \gamma_i|x_i|.$$

Let $y = (x_1, u_1, x_2, \cdots, x_p, u_2, \cdots, u_q)$ and $x = (x_1, x_2, \cdots, x_p, u_1, u_2, \cdots, u_q)$. Then $T(y) = x$, and we have that

$$\begin{aligned}
&E[g^*(Y_n)|Y_{n-1} = y] \\
&= E[g^*(h^*(y) + e_n, e_n, x_1, \cdots, x_{p-1}, u_1, \cdots, u_{q-1})] \\
&= E[\max \gamma_1|h(x) + e_n|, \gamma_2|x_1|, \cdots, \gamma_p|x_{p-1}|] \\
&\quad + \gamma_{p+1}E|e_n| + \gamma_{p+2}|u_1| + \cdots + \gamma_{p+q}|u_{q-1}| \\
&\le \max\{\gamma_1 \sum_{j=1}^{l}(\sum_{i=1}^{p} a_i^{(j)}|x_i|) I_{\omega(x)\in R_j}, \gamma_2|x_1|, \cdots, \gamma_p|x_{p-1}|\} \\
&\quad + \gamma_1 \sum_{j=1}^{l}(\sum_{i=1}^{q} b_i^{(j)}|u_i|) I_{\omega(x)\in R_j} + \gamma_{p+2}|u_1| + \cdots + \gamma_{p+q}|u_{q-1}| + c, \qquad (2.8)
\end{aligned}$$

where $c = (\gamma_1 + \gamma_{p+1})E|e_n| + \gamma_1(\max_j a_0^{(j)})$.

From (2.8) and following relations

$$\frac{\gamma_{i+1}}{\gamma_i} < \delta, \qquad \max_j \sum_{i=1}^{p} a_i^{(j)} \frac{\gamma_1}{\gamma_i} < \delta < 1, \ \ 1 \le i \le p-1$$

and

$$\delta\gamma_{p+i} = \gamma_1 b_i + \gamma_{p+i+1}, \ \ i = 1, 2, \cdots, q-1, \qquad \delta\gamma_{p+q} = \gamma_1 b_q,$$

we obtain that

$$E[g^*(Y_n)|Y_{n-1} = y] \le \delta g(x) + c = \delta g^*(y) + c.$$

Then we can show that for any given $\epsilon > 0$, there exist nonnegative constants $\delta < \delta' < 1$ and $M < \infty$ such that

$$E[g^*(Y_n)|Y_{n-1} = y] \le \delta' g^*(y) - \epsilon, \ \ \text{if} \quad y \in B_M^c$$

and

$$\sup_{y\in B_M} E[g^*(Y_n)|Y_{n-1} = y] < \infty$$

where $B_M = \{y \in R^{p+q} \mid \| y \| \le M\}$. Since local boundedness of h is clear, applying Theorem 1 and Lemma 1 yields the desired results.

For next two corollaries, we assume that e_n has a probability density function with respect to the Lebesgue measure.

As a special case of processes given in (2.1), we consider the following well known threshold ARMA(p, q) model:

$$X_n = \sum_{j=1}^{l} (\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_{1i}^{(j)} X_{n-i} + \sum_{i=1}^{q} \alpha_{2i}^{(j)} e_{n-i}) I_{(X_{n-d} \in R_j)} + e_n, \tag{2.9}$$

where $d \in \{1,\cdots,p\}$, $-\infty = r_0 < r_1 < \cdots < r_l = \infty$, $R_j = (r_{j-1}, r_j]$, $j = 1,2,\cdots,l$.

Corollary 1. For the process $\{X_n\}$ given in (2.9), a strictly stationary solution exists if $\max_{1 \le j \le l} \sum_{i=1}^{p} |\alpha_{1i}^{(j)}| < 1$ and in this case $E_\pi |X_n| < \infty$.

Proof. Adopting the same test function as in (2.7), we can show that the drift condition in (1.1) holds.

Remark An auxiliary Markov chain of the process in (2.9) is weak Feller if

$$\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_{1i}^{(j)} X_{n-i} + \sum_{i=1}^{q} \alpha_{2i}^{(j)} e_{n-i} = \alpha_0^{(j+1)} + \sum_{i=1}^{p} \alpha_{1i}^{(j+1)} X_{n-i} + \sum_{i=1}^{q} \alpha_{2i}^{(j+1)} e_{n-i}$$

holds whenever $X_{n-d} = r_j$, $j = 1,\cdots,l-1$.

Now consider the momentum threshold ARMA (MTARMA(p, q)) model given by:

$$X_n = \sum_{j=1}^{2} (\sum_{i=1}^{p} \alpha_{1i}^{(j)} X_{n-i} + \sum_{i=1}^{q} \alpha_{2i}^{(j)} e_{n-i}) I_{nj} + e_n, \tag{2.10}$$

where $I_{n1} = I_{(X_{n-1} \ge X_{n-2})}$, $I_{n2} = 1 - I_{n1}$.

Corollary 2. Consider a momentum threshold ARMA(p, q) process $\{X_n\}$ given by (2.10). If $\max_{1 \le j \le 2} \sum_{i=1}^{p} |\alpha_{1i}^{(j)}| < 1$, the conclusion of Theorem 2 holds.

Proof. The proof is essentially the same as those of Theorem 2 and we omit it.

Remark It is difficult to find a sufficient condition for Feller continuity for the associated Markov chain of (2.10). Geometric ergodicity of MTAR(1) model is studied by Lee and Shin(2000).

References

1. Bhattacharya, R.N. and Lee, C. (1995) On geometric ergodicity of nonlinear autoregressive models. *Statistics and Probability Letters* 22, 311-315.
2. Cline, D.B.H. and Pu, H.H. (1998) Verifying irreducibility and continuity of a nonlinear time series. *Statistics and Probability Letters* 40, 139-148.
3. Fonseca, G. and Tweedie, R.L.(2002) Stationary measures for non-irreducible non-continuous Markov chains with time series applications. *Statistica Sinica* 12, 651-660.
4. Lee, O. and Lee, M. (2001) On stationarity of TARMA(p, q) process. *Journal of Korean Statistical Society* 30, 115-125.
5. Lee, O. and Shin, D.W. (2000) On geometric ergodicity of the MTAR process. *Statistics and Probability Letters* 48, 229-237.
6. Liu, J. and Susko, Ed.(1992) On strict stationarity and ergodicity of a nonlinear ARMA model. *Journal of Applied Probability* 21, 270-286.
7. Meyn, S. P. and Tweedie, R. L. (1993) *Markov Chains and Stochastic Stability*. Springer, London.

[received date : Nov. 2006, accepted date : Dec. 2006]