Honam Mathematical J. **29** (2007), No. 4, pp. 511–521

A UNITARY LINEAR SYSTEM ON THE BIDISK

MEEHYEA YANG† AND BUM IL HONG*

ABSTRACT. Let $S(z_1, z_2)$ be a power series with operator coefficients such that multiplication by $S(z_1, z_2)$ is a contractive transformation in the Hilbert space $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. In this paper we show that there exists a Hilbert space $\mathcal{D}(\mathbb{D}, \tilde{S})$ which is the state space of extended canonical linear system with a transfer fucntion $\tilde{S}(z)$.

1. Introduction

Let $\mathcal{C}$ be a Hilbert space and let $\mathbb{D}$ be the open unit disk in the complex plain $\mathbb{C}$. Let $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$ be a Hilbert space defined by

$$\begin{aligned}&\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})\\ =&\{F(z_1, z_2) = \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j \ : \ z_i, z_j \in \mathbb{D}, \quad a_{ij} \in \mathcal{C}, \sum_{i,j=0}^{\infty} \|a_{ij}\|_{\mathcal{C}}^2 < \infty\}\end{aligned}$$

with the scalar product $\|F\|^2_{\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})} = \sum_{i,j=0}^{\infty} \|a_{ij}\|^2_{\mathcal{C}}$. It is a reproducing kernel Hilbert space with reproducing kernel function

$$K(w_1, w_2; z_1, z_2) = \frac{I_{\mathcal{C}}}{(1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)}.$$

It is well known that the reproducing kernel function is uniquely determined for a Hilbert space if it exists [3].

Complementation theory can be used to construct a reproducing kernel Hilbert space which is the state space of a canonical linear system with given transfer function.

Received June 18, 2007. Revised October 11, 2007.
2000 Mathematics Subject Classification : 47B32, 15A48.
Key words and phrases : Reproducing kernel function, Unitary linear system.
* Corresponding Author.
† This research was supported by the University of Incheon Research Grant in 2006.

THEOREM 1.1. *If P is contractive self-adjoint transformation in a Hilbert space $\mathcal{H}$, then unique Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ exist, which are contained continuously and contractively in $\mathcal{H}$ such that the inequality*

$$\|c\|^2_{\mathcal{H}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds.

Proof. See [6]. □

Assume that $S(z_1, z_2)$ is a power series with operator coefficients such that multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. Let the space $\mathbf{H}_2(\mathbb{D}^2, S)$ be the set of elements in $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$ which satisfies

$$m(F) = sup_{G\in \mathbf{H}_2(\mathbb{D}^2,\mathcal{C})}\{\|F + SG\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})} - \|G\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})}\}$$

is finite. Then the space $\mathbf{H}_2(\mathbb{D}^2, S)$ becomes a Hilbert space with the scalar product

$$\|F\|^2_{\mathbf{H}_2(\mathbb{D}^2,S)} = m(F).$$

D. Alpay [1] has shown that the space $\mathbf{H}_2(\mathbb{D}^2, S)$ has the following properties: $\frac{F(z_1,z_2)-F(0,z_2)}{z_1}$ and $\frac{[S(z_1,z_2)-S(0,z_2)]c}{z_1}$ are in the space $\mathbf{H}_2(\mathbb{D}^2, S)$ for every $c \in \mathcal{C}$ and the inequalities

$$\|\frac{F(z_1, z_2) - F(0, z_2)}{z_1}\|_{\mathbf{H}_2(\mathbb{D}^2,S)} \leq \|F(z_1, z_2)\|_{\mathbf{H}_2(\mathbb{D}^2,S)} - \|F(0, z_2)\|_{\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})}$$

and

$$\|\frac{[S(z_1, z_2) - S(0, z_2)]c}{z_1}\|_{\mathbf{H}_2(\mathbb{D}^2,S)} \leq \|c\|_{\mathcal{C}} - \|S(0, z_2)c\|_{\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})}$$

hold. Also $\frac{F(z_1,z_2)-F(z_1,0)}{z_2}$ and $\frac{[S(z_1,z_2)-S(z_1,0)]c}{z_2}$ are in the space $\mathbf{H}_2(\mathbb{D}^2, S)$ and similar inequalities are satisfied.

Complementation theory can be applied to prove the following theorem [1, 5, 6].

THEOREM 1.2. *The Hilbert space* $\mathbf{H}_2(\mathbb{D}^2, S)$ *is contained contractively in the space* $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$ *and has the reproducing kernel function*

$$K(w_1, w_2; z_1, z_2) = \frac{I_{\mathcal{C}} - S(z_1, z_2)S^*(w_1, w_2)}{(1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)}.$$

Every element $H(z_1, z_2)$ *in* $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$ *has a unique minimal decomposition,*

$$H(z_1, z_2) = F(z_1, z_2) + S(z_1, z_2)G(z_1, z_2)$$

with $F(z_1, z_2)$ *in* $\mathbf{H}_2(\mathbb{D}^2, S)$, $G(z_1, z_2)$ *in* $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$ *and*

$$\|H\|^2_{\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})} = \|F\|^2_{\mathbf{H}_2(\mathbb{D}^2, S)} + \|G\|^2_{\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})}.$$

2. The state space of a unitaty linear system $\mathbf{H}_2(\mathbb{D}, \tilde{S})$

Let $\mathcal{H}$ and $\mathcal{C}$ be Hilbert spaces. A continuous linear transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{C} \longrightarrow \mathcal{H} \oplus \mathcal{C}$$

is called a linear system. The underlying Hilbert space $\mathcal{H}$ is called the state space and the auxiliary Hilbert space $\mathcal{C}$ is called the coefficient space or the external space. A linear system is said to be contractive if the matrix is contractive and is said to be unitary if the matrix is unitary. The transfer function $S(z)$ of the linear system is defined by

$$S(z) = D + zC(I - zA)^{-1}B.$$

Let $F(z_1, z_2) = \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j \in \mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. The functions

$$f(z_1) = \begin{pmatrix} f_0(z_1) \\ f_1(z_1) \\ \vdots \end{pmatrix} \quad \text{and} \quad g(z_2) = \begin{pmatrix} g_0(z_2) \\ g_1(z_2) \\ \vdots \end{pmatrix}$$

with $f_j(z_1) = \sum_{i=0}^{\infty} a_{ij} z_1^i$ and $g_i(z_2) = \sum_{j=0}^{\infty} a_{ij} z_2^j$ belong to $\mathbf{H}(\mathbb{D}, l_2(\mathcal{C}))$. Then $F(z_1, z_2)$ can be written by

$$F(z_1, z_2) = E(z_2)f(z_1) = E(z_1)g(z_2)$$

where $E(z) = (I_{\mathcal{C}}, zI_{\mathcal{C}}, z^2 I_{\mathcal{C}}, \cdots)$ and $\|F\|_{\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})} = \|f\|_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))} = \|g\|_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))}$.

Let $S(z_1, z_2) = \sum_{i,j=0}^{\infty} s_{ij} z_1^i z_2^j$ be a power series with operator coefficients such that multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. $S(z_1, z_2) = E(z_2)s(z_1)$ where

$$s(z) = \begin{pmatrix} s_0(z) \\ s_1(z) \\ \vdots \end{pmatrix}$$

with $s_j(z) = \sum_{i=0}^{\infty} s_{ij} z^i$. Then $S(z_1, z_2)F(z_1, z_2) = E(z_2)\tilde{S}(z_1)f(z_1)$ where

$$\tilde{S}(z) = \begin{pmatrix} s_0(z) & 0 & 0 & 0 & \cdots \\ s_1(z) & s_0(z) & 0 & 0 & \cdots \\ s_2(z) & s_1(z) & s_0(z) & 0 & \cdots \\ \vdots & \vdots & & & \end{pmatrix}$$

and multiplication by $\tilde{S}(z)$ is contactive transformation in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$.

THEOREM 2.1. *Let $S(z_1, z_2)$ be a power series with operator coefficients such that multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. There exists a Hilbert space $\mathbf{H}_2(\mathbb{D}, \tilde{S})$ which is the state space of a unitary linear system whose transfer function is $\tilde{S}(z)$. The reproducing kernel fuction of the space $\mathbf{H}_2(\mathbb{D}, \tilde{S})$ is*

$$K(w, z) = \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(w)}{(1 - \bar{w}z)}.$$

Proof. Let $\mathbf{H}_2(\mathbb{D}, \tilde{S})$ be the subset of elements in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$ which satisfies

$$m(f) = sup_{g \in \mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))}\{\|f + \tilde{S}g\|^2_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))} - \|g\|^2_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))}\} < \infty.$$

Define the saclar product by $\|f\|^2_{\mathbf{H}_2(\mathbb{D},\tilde{S})} = m(f)$. The space $\mathbf{H}_2(\mathbb{D}, \tilde{S})$ becomes a Hilbert space which is contained contractively in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$. $F(z_1, z_2) = E(z_2)f(z_1) \in \mathbf{H}_2(\mathbb{D}^2, S)$ if and only if $f(z) \in \mathbf{H}_2(\mathbb{D}, \tilde{S})$ and

$$\|F\|_{\mathbf{H}_2(\mathbb{D}^2,S)} = \|f\|_{\mathbf{H}_2(\mathbb{D},\tilde{S})}.$$

Assume that $h(z)$ is in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$. Let $g(z)$ be in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$ which satisfies

$$\langle h(z), \tilde{S}(z)u(z)\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} = \langle g(z), u(z)\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))}$$

for every $u(z)$ in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$. Let $f(z) = h(z) - \tilde{S}(z)g(z)$. Then we show that $f(z)$ is in $\mathbf{H}_2(\mathbb{D}, \tilde{S})$. Since the inequality

$$\begin{aligned}
&\|f + \tilde{S}u\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} \\
\leq &\|h\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} + \|g - u\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \langle h(z), \tilde{S}(z)(g(z) - u(z))\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} \\
&\quad - \langle \tilde{S}(z)(g(z) - u(z)), h(z)\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} \\
\leq &\|h\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|g\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} \\
< &\infty
\end{aligned}$$

holds for every $u(z)$ in $\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$, we have $f \in \mathbf{H}_2(\mathbb{D}, \tilde{S})$ and

$$\|f\|^2_{\mathbf{H}_2(\mathbb{D}, \tilde{S})} \leq \|h\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|g\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))}.$$

Hence we have $\|f\|^2_{\mathbf{H}_2(\mathbb{D}, \tilde{S})} = \|h\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|g\|^2_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))}$.

From the identity

$$\langle \tilde{S}(z)u(z), \frac{I_{l_2(\mathcal{C})}}{1 - \bar{w}z}c\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))} = \langle u(z), \frac{\tilde{S}^*(w)c}{1 - \bar{w}z}\rangle_{\mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))},$$

we have that

$$K(w, z) = \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(w)}{(1 - \bar{w}z)}$$

is the reproducing kernel function of the space $\mathbf{H}_2(\mathbb{D}, \tilde{S})$. Since $\frac{F(z_1, z_2) - F(0, z_2)}{z_1}$ is in $\mathbf{H}_2(\mathbb{D}^2, S)$ for every $F(z_1, z_2) = E(z_2)f(z_1) \in \mathbf{H}_2(\mathbb{D}^2, S)$, $\frac{f(z) - f(0)}{z}$ is in $\mathbf{H}_2(\mathbb{D}, \tilde{S})$ for every $f(z) \in \mathbf{H}_2(\mathbb{D}, \tilde{S})$.

Define $A : \mathbf{H}_2(\mathbb{D}, \tilde{S}) \longrightarrow \mathbf{H}_2(\mathbb{D}, \tilde{S})$ by $Af(z) = \frac{f(z) - f(0)}{z}$. Let $c \in l_2(\mathcal{C})$. Then the identity

$$A(K(w, z)c) = wK(w, z)c - \frac{\tilde{S}(z) - \tilde{S}(0)}{z}\tilde{S}^*(w)c$$

implies that $\frac{\tilde{S}(z) - \tilde{S}(0)}{z}c$ is in $\mathbf{H}_2(\mathbb{D}, \tilde{S})$.

Define
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H}_2(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H}_2(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix}$$
by $Af(z) = \frac{f(z)-f(0)}{z}$, $Bc = \frac{\tilde{S}(z)-\tilde{S}(0)}{z}c$, $Cf(z) = f(0)$ and $Dc = \tilde{S}(0)c$. Let $Af(z) = g(z)$. Then

$$\begin{aligned} \langle g(w), c\rangle_{l_2(\mathcal{C})} =& \langle Af(z), K(w,z)c\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})} \\ =& \langle f(z), w^* K(w,z)c - \frac{\tilde{S}(z) - \tilde{S}(0)}{z} \tilde{S}^*(w)c\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})} \\ =& \langle wf(w) - \tilde{S}(w)B^* f(z), c\rangle_{l_2(\mathcal{C})}, \end{aligned}$$

$A^*(f(z)) = zf(z) - \tilde{S}(z)B^*f(z)$ and $AA^* + BB^* = I_{\mathbf{H}_2(\mathbb{D},\tilde{S})}$. From the identity

$$\langle Cf(z), c\rangle_{l_2(\mathcal{C})} = \langle f(z), K(0,z)c\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})},$$

we have
$$C^*c = K(0,z)c = [I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(0)]c.$$
It implies that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a unitary linear system. □

3. The extension space $\mathcal{D}(\mathbb{D}, \tilde{S})$ of $\mathbf{H}_2(\mathbb{D}, \tilde{S})$

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the unitary linear system on $\begin{pmatrix} \mathbf{H}_2(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix}$ and let $f(z) \in \mathbf{H}_2(\mathbb{D}, \tilde{S})$ as defined in the theorem 2.1. Define $f_n(z) = A^{*n} f(z)$ for every positive integer n. Then

$$\begin{aligned} f_n(z) &= zf_{n-1}(z) - \tilde{S}(z)a_{n-1} \\ &= z^n f(z) - \tilde{S}(z)(a_0 z^{n-1} + \cdots + a_{n-1}) \end{aligned}$$

belongs to $\mathbf{H}_2(\mathbb{D}^2, \tilde{S})$ where $c^* a_{n-1} = \langle B^* f_{n-1}(z), c\rangle_{l_2(\mathcal{C})}$ and

$$\|f_n\|^2_{\mathbf{H}_2(\mathbb{D},\tilde{S})} = \|f\|^2_{\mathbf{H}_2(\mathbb{D},\tilde{S})} - \sum_{k=0}^{n-1} \|a_n\|^2_{l_2(\mathcal{C})}.$$

Note that the sequence is decreasing. If $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $g(z) \in \mathbf{H}_2(\mathbb{D}, l_2(\mathcal{C}))$.

Let the extension space $\mathcal{D}(\mathbb{D},\tilde{S})$ be a Hilbert space whose elements are pairs $(f(z),g(z))$ with the scalar product

$$\|(f(z),g(z))\|^2_{\mathcal{D}(\mathbb{D},\tilde{S})}=\lim_{n\to\infty}[\|z^nf(z)-\tilde{S}(z)(a_0z^{n-1}+\cdots+a_{n-1})\|^2_{\mathbf{H}_2(\mathbb{D},\tilde{S})}+\sum_{k=0}^{n-1}\|a_n\|^2_{l_2(\mathcal{C})}].$$

Then $\|(f(z),g(z))\|_{\mathcal{D}(\mathbb{D},\tilde{S})}=\|f(z)\|_{\mathbf{H}_2(\mathbb{D},\tilde{S})}$.

THEOREM 3.1. *Let $S(z_1,z_2)$ be a power series with operator coefficients such that multiplication by $S(z_1,z_2)$ is a contractive transformation in $\mathbf{H}_2(\mathbb{D}^2,\mathcal{C})$. The extension space $\mathcal{D}(\mathbb{D},\tilde{S})$ is a state space of a unitary linear system.*

Proof. Let $S(z_1,z_2)=\sum_{i,j=0}^{\infty}s_{ij}z_1^iz_2^j$. $\tilde{S}(z)$ can be written by $\tilde{S}(z)=\sum_{n=0}^{\infty}S_nz^n$ where

$$S_n=\begin{pmatrix} s_{n0} & 0 & 0 & 0 & \cdots \\ s_{n1} & s_{n0} & 0 & 0 & \cdots \\ s_{n2} & s_{n1} & s_{n0} & 0 & \cdots \\ \vdots & \vdots & & & \end{pmatrix}.$$

Since the identity

$$\langle \tilde{S}(z)u(z),cz^n\rangle_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))}=\langle u(z),(S_0^*z^n+\cdots+S_n^*)c\rangle_{\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))}$$

holds for every $u(z)\in\mathbf{H}_2(\mathbb{D},l_2(\mathcal{C}))$ and $c\in l_2(\mathcal{C})$, $z^nc-\tilde{S}(z)[S_0^*z^n++\cdots+S_n^*]c$ is also in $\mathcal{D}(\mathbb{D},\tilde{S})$. Therefore $([I_{l_2(\mathcal{C})}-\tilde{S}(z)\tilde{S}^*(0)]c,\frac{[\tilde{S}^*(z)-\tilde{S}^*(0)]c}{z})\in\mathcal{D}(\mathbb{D},\tilde{S})$. And the identity

$$\bar{c}f(0)=\langle (f(z),g(z)),([I_{l_2(\mathcal{C})}-\tilde{S}(z)\tilde{S}^*(0)]c,\frac{[\tilde{S}^*(z)-\tilde{S}^*(0)]c}{z})\rangle_{\mathcal{D}(\mathbb{D},\tilde{S})}$$

holds for every $(f(z),g(z))$ in $\mathcal{D}(\mathbb{D},\tilde{S})$. If $G:\mathcal{D}(\mathbb{D},\tilde{S})\longrightarrow l_2(\mathcal{C})$ is defined by $Gf(z)=f(0)$, then

$$G^*C=([I_{l_2(\mathcal{C})}-\tilde{S}(z)\tilde{S}^*(0)]c,\frac{[\tilde{S}^*(z)-\tilde{S}^*(0)]c}{z}).$$

Now we show that $(\frac{f(z)-f(0)}{z},zg(z)-\tilde{S}^*(z)f(0))$ belongs to $\mathcal{D}(\mathbb{D},\tilde{S})$ if $(f(z),g(z))$ is in $\mathcal{D}(\mathbb{D},\tilde{S})$. From the definition of the space $\mathcal{D}(\mathbb{D},\tilde{S})$, we have

$$(f(z),g(z))\in\mathcal{D}(\mathbb{D},\tilde{S})\text{ if and only if }(zf(z)-\tilde{S}(z)g(0),\frac{g(z)-g(0)}{z})\in\mathcal{D}(\mathbb{D},\tilde{S})$$

and

$$\|(zf(z) - \tilde{S}(z)g(0), \frac{g(z) - g(0)}{z})\|_{\mathcal{D}(\mathbb{D},\tilde{S})} = \|(f(z), g(z))\|_{\mathcal{D}(\mathbb{D},\tilde{S})} - \|g(0)\|_{l_2(\mathcal{C})}.$$

Let $f_1(z) = \frac{f(z)-f(0)}{z}$ and $g_1(z) = zg(z) - \tilde{S}^*(z)f(0)$. Since the identities $zf_1(z) - \tilde{S}(z)g_1(0) = f(z) - [I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(0)]f(0)$ and $\frac{g_1(z)-g_1(0)}{z} = g(z) - \frac{[\tilde{S}^*(z)-\tilde{S}^*(0)]}{z}f(0)$ hold, $(zf_1(z) - \tilde{S}(z)g_1(0), \frac{g_1(z)-g_1(0)}{z}) \in \mathcal{D}(\mathbb{D}, \tilde{S})$. It implies that $(f_1(z), g_1(z))$ is in $\mathcal{D}(\mathbb{D}, \tilde{S})$. Define $T: \mathcal{D}(\mathbb{D}, \tilde{S}) \longrightarrow \mathcal{D}(\mathbb{D}, \tilde{S})$ by

$$T(f(z), g(z)) = (\frac{f(z) - f(0)}{z}, zg(z) - \tilde{S}^*(z)f(0)).$$

The identity

$$\begin{aligned}
&\langle T(f(z), g(z)), (u(z), v(z))\rangle_{\mathcal{D}(\mathbb{D},\tilde{S})} \\
=&\langle(\frac{f(z) - f(0)}{z}, zg(z) - \tilde{S}^*(z)f(0)), (u(z), v(z))\rangle_{\mathcal{D}(\mathbb{D},\tilde{S})} \\
=&\langle\frac{f(z) - f(0)}{z}, u(z)\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})} \\
=&\langle f(z), zu(z) - \tilde{S}(z)v(0)\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})} \\
=&\langle(f(z), g(z)), (zu(z) - \tilde{S}(z)v(0), \frac{v(z) - v(0)}{z})\rangle_{\mathcal{D}(\mathbb{D},\tilde{S})}
\end{aligned}$$

holds for any $(u(z), v(z))$ in $\mathcal{D}(\mathbb{D}, \tilde{S})$. Hence we have

$$T^*(f(z), g(z)) = (zf(z) - \tilde{S}(z)g(0), \frac{g(z) - g(0)}{z}).$$

From the definition of the space $\mathcal{D}(\mathbb{D}, \tilde{S})$, $(\frac{\tilde{S}(z)-\tilde{S}(0)}{z}c, [I_{l_2(\mathcal{C})} - \tilde{S}^*(z)\tilde{S}(0)]c)$ is in the space $\mathcal{D}(\mathbb{D}, \tilde{S})$ for any $c \in l_2(\mathcal{C})$.

If we define $F: \mathcal{D}(\mathbb{D}, \tilde{S}) \longrightarrow \mathcal{D}(\mathbb{D}, \tilde{S})$ by

$$Fc = (\frac{\tilde{S}(z) - \tilde{S}(0)}{z}c, [I_{l_2(\mathcal{C})} - \tilde{S}^*(z)\tilde{S}(0)]c),$$

then $F^*(u(z), v(z)) = v(0)$ from the identity

$$\begin{aligned}
\langle Fc, (u(z), v(z))\rangle_{\mathcal{D}(\mathbb{D},\tilde{S})} &= \langle\frac{\tilde{S}(z) - \tilde{S}(0)}{z}c, u(z)\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})} \\
&= \langle c, v(0)\rangle_{\mathbf{H}_2(\mathbb{D},\tilde{S})}.
\end{aligned}$$

And

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix}$$

where $Hc = \tilde{S}(0)c$ is a unitary linear system with

$$\tilde{S}(z) = H + zG(I_{\mathcal{D}(\mathbb{D},\tilde{S})} - zT)^{-1}F.$$

This completes the proof. □

The reproducing kernel function of the space $\mathcal{D}(\mathbb{D}, \tilde{S})$ can be found as follows.

THEOREM 3.2. *Let $S(z_1, z_2)$ be a power series with operator coefficients such that multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbf{H}_2(\mathbb{D}^2, \mathcal{C})$. The reproducing kernel function of $\mathcal{D}(\mathbb{D}, \tilde{S})$ is*

$$\begin{pmatrix} \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(w)}{(1-\bar{w}z)} & \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} \\ \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} & \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(\bar{w})}{1 - w^* z} \end{pmatrix}.$$

Proof. Let

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix}$$

be a unitary linear system defined as in the theorem 2.1. Let $w \in \mathbb{D}$. Since the transformations G and F are continuous, the mappings which take $(f(z), g(z))$ to $f(w)$ and $g(w)$ are continuous from $\mathcal{D}(\mathbb{D}, \tilde{S})$ into $l_2(\mathcal{C})$. There are $(K(w, z)c, L(w, z)c)$ and $(P(w, z), Q(w, z))$ in $\mathcal{D}(\mathbb{D}, \tilde{S})$ for any $c \in l_2(\mathcal{C})$ such that

$$c^* f(w) = \langle (f(z), g(z)), (K(w, z)c, L(w, z)c) \rangle_{\mathcal{D}(\mathbb{D},\tilde{S})}$$

and

$$c^* g(w) = \langle (f(z), g(z)), (P(w, z)c, Q(w, z)c) \rangle_{\mathcal{D}(\mathbb{D},\tilde{S})}$$

for every $(f(z), g(z))$ in $\mathcal{D}(\mathbb{D}, \tilde{S})$. We have

$$K(0, z) = [I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(0)] \text{ and } L(0, z) = \frac{[\tilde{S}^*(z) - \tilde{S}^*(0)]c}{z}.$$

Since the identities

$$\langle Fc, (K(w, z)d, L(w, z)d) \rangle_{\mathcal{D}} = d^* \frac{[\tilde{S}(w) - \tilde{S}(0)]c}{w}$$

and

$$\langle G^* c, (K(w,z)d, L(w,z)d)\rangle_{\mathcal{D}} = d^*[I_{l_2(\mathcal{C})} - \tilde{S}(w)\tilde{S}^*(0)]c]$$

hold for every c in $l_2(\mathcal{C})$, we have

$$K(w,0) = I_{l_2(\mathcal{C})} - \tilde{S}^*(\bar{w})\tilde{S}(0) \text{ and } L(w,0) = \frac{[\tilde{S}^*(\bar{w}) - \tilde{S}^*(0)]}{\bar{w}}.$$

Since the identity

$$\begin{aligned}
&\langle A(f(z), g(z)), (K(w,z)c, L(w,z)c)\rangle_{\mathcal{D}} \\
=&\bar{c}[f(w) - f(0)]/z \\
=&\langle (f(z), g(z)), ([K(w,z) - K(0,z)]c/\bar{w}, [L(w,z) - L(0,z)]c/\bar{w})\rangle_{\mathcal{D}} \\
=&\langle (f(z), g(z)), (zK(w,z) - S(z)L(w,0), [L(w,z) - L(w,0]/z)\rangle_{\mathcal{D}}
\end{aligned}$$

holds for every $(f(z), g(z))$ in $\mathcal{D}$, we have

$$\frac{[K(w,z) - K(0,z)]c}{\bar{w}} = zK(w,z) - \tilde{S}(z)L(w,0)$$

and

$$\frac{[L(w,z) - L(0,z)]c}{\bar{w}} = \frac{[L(w,z) - L(w,0)]}{z}.$$

The reproducing kernel function of $\mathcal{D}(\mathbb{D}, \tilde{S})$ is then

$$\begin{pmatrix} \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(w)}{(1-\bar{w}z)} & \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z-\bar{w}} \\ \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z-\bar{w}} & \frac{I_{l_2(\mathcal{C})} - \tilde{S}(z)\tilde{S}^*(\bar{w})}{1-w^* z} \end{pmatrix}.$$

This completes the proof. □

References

[1] D. Alpay, V. Bolotnikov, A. Dijksma and C. Sadosky, *Hilbert spaces contractively included in the Hardy space of the Bidisk*, Positivity **5** (2001), 25-50.

[2] D. Alpay, A. Dijksma and J. Rovnyak, *A theorem of Beurling-Lax type for Hilbert spaces of functions analytic in the unit ball*, Integral Equations and Operator Theory **47** (2003), 251-274.

[3] D. Alpay, A. Dijksma, J. Rovnyak and H. de Snoo, *Schur functions, operator colligations and reproducing kernel Pontryagin spaces*, Operator Theory: Advances and Applications **96** (1997), Birkhäuser Verlag, Basel.

[4] V. Bolotnikov, *Interpolation for multipliers on reproducing kernel Hilbert spaces*, Proc. Amer. Math. Soc. **131** (2002), 1373-1383.

[5] L. de Branges, *Krein spaces of analytical functions*, J. Functional Analysis **81** (1988), 219–259.

[6] ———, *Complementation in Krein spaces*, Trans. Amer. Math. Soc. **305** (1988), 277-291.

[7] ———, *Factorization in Krein spaces*, J. Functional Analysis **124** (1994), 228–262.

[8] L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics, Wiley, New York, 1966.

[9] M. Yang, *A construction of unitary linear systems*, Integral Equations and Operator Theory **19** (1994), 477–499.

[10] ———, *Reproducing Kernel Krein spaces*, J. Appl. Math. & Computing **8** (2001), 659-668.

Meehyea Yang
Department of Mathematics,
University of Incheon,
Incheon 402-749, Korea
E-mail: mhyang@ incheon.ac.kr

Bum Il Hong
Department of Mathematics,
Kyung Hee University,
Yongin 449-701, Korea
E-mail: bihong@ khu.ac.kr