Honam Mathematical J. **29** (2007), No. 4, pp. 523–533

SOME PROPERTIES OF INVARIANT SUBSPACES IN BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{B}$ be a reflexive Banach space of functions analytic on the open unit disc and M be an invariant subspace of the multiplication operator by the independent variable, M_z. Suppose that $\varphi \in \mathcal{H}^\infty$ and $M_\varphi : M \to M$, defined by $M_\varphi f = \varphi f$, is the operator of multiplication by φ. We would like to investigate the spectrum and the essential spectrum of M_φ and we are looking for the necessary and sufficient conditions for M_φ to be a Fredholm operator. Also we give a sufficient condition for a sequence $\{\omega_n\}$ to be an interpolating sequence for $\mathcal{B}$. At last the commutant of M_φ under certain conditions on M and φ is determined.

1. Introduction

Let $\mathcal{B}$ be a reflexive Banach space consisting of analytic functions defined on the open unit disc $\mathbf{D}$ and satisfying the following conditions:

(1) $1 \in \mathcal{B}$ and $\varphi\mathcal{B} \subset \mathcal{B}$ for every $\varphi \in \mathcal{H}^\infty$;

(2) for every $\lambda \in \mathbf{D}$ the evaluation functional at λ, $e_\lambda : \mathcal{B} \to \mathbf{C}$ given by $e_\lambda(f) = f(\lambda)$, is bounded;

(3) $\mathrm{rang}(M_z - \lambda) = \ker(e_\lambda)$ for every $\lambda \in \mathbf{D}$.

Such a space is called a reflexive Banach space of analytic functions.

In what follows we present some examples of such spaces .

(a) The classical Hardy spaces H^p for $1 < p < \infty$.

Received June 29, 2007. Accepted August 28, 2007.

2000 Mathematics Subject Classification: Primary 47B35; Secondary 47B38.

Key words and phrases: invariant subspaces, multiplication operators, reflexive Banach space of analytic functions, spectrum, bounded point evaluation.

This research was in part supported by a grant (no.82-SC-1635-C251) from Shiraz university research council.

(b) The Bergman space of analytic functions defined on $\mathbf{D}$, $L_a^P(\mathbf{D})$ for $1 < p < \infty$.

(c) The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $\mathbf{D}$, for which $\|f\|^2 = \sum(n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty$ for every $\alpha \leq 0$.

Recall that a bounded linear operator T on a Banach space is called Fredholm if it is invertible modulo the compact operators. It is known that T is Fredholm if the kernel of T is finite dimensional and the range of T is finite codimensional, i.e.; $\dim\frac{\mathcal{B}}{\mathrm{rang}T} < \infty$. It is worthy of attention to remark that rang$T$ is closed, in fact, if a finite codimensional linear subspace Y of a Banach space $\mathcal{B}$ is the range of a continuous linear mapping N on $\mathcal{B}$, then it must be closed [5, Lemma 3.3].

The essential spectrum of T, denoted by $\sigma_e(T)$, is the set of all complex number λ such that $\lambda - T$ is not Fredholm. Since every invertible operator is Fredholm , we have $\sigma_e(T) \subset \sigma(T)$, the spectrum of T. The essential norm of T, denoted $||T||_e$, is the norm distance from T to the set of compact operators. Also $r(T)$ denotes the spectral radius of T and by $r_e(T)$ we mean the essential spectral radius of T; that is, the supremum of the absolute values of numbers in the essential spectrum of T.

Suppose that M is an invariant subspace of M_z on the Bergman space L_a^2. Recently, the necessary and sufficient conditions on the operator $M_\varphi : M \to M$, $\varphi \in \mathcal{H}^\infty$, to be a Fredholm operator, besides the essential spectrum and the essential norm of this operator are given by K. Zhu [8]. Also, the special case $\varphi(z) = z$ is taken into consideration in [7].

2. Spectrum and Essential Spectrum

In all that follows, let $\mathcal{B}$ be a reflexive Banach space of analytic functions on $\mathbf{D}$ and M be an invariant subspace of M_z. Also, let $\mathcal{M}(M)$ denote the set of all multipliers of M; that is, the analytic functions φ

on $\mathbf{D}$ such that $\varphi M \subset M$. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ stand for $x^*(x)$, where $x \in \mathcal{B}$ and $x^* \in \mathcal{B}^*$.

Lemma 2.1 Let $\|M_\varphi\|$ denote the norm of operator $M_\varphi : M \to M$. The following statements are equivalent:

(A1) There is a constant $c > 0$ such that $||M_p|| \leq c||p||_\infty$ for every polynomial p.

(A2) $\mathcal{M}(M) = \mathcal{H}^\infty$.

(A3)There is a constant $c > 0$ such that $||M_\varphi|| \leq c||\varphi||_\infty$ for every $\varphi \in \mathcal{H}^\infty$.

Proof. (A1) implies (A2): Suppose that $\varphi \in \mathcal{H}^\infty$. Then there is a sequence $\{p_n\}$ of polynomials such that $\sup_n||p_n||_\infty < \infty$ and $p_n(z) \to \varphi(z)$ for all $z \in \mathbf{D}$. So for every $f \in M$,

$$\sup_n||p_n f|| = \sup_n||M_{p_n} f|| \leq c||f||\sup_n||p_n||_\infty < \infty.$$

It follows that $p_n f$ converges to φf weakly [1, page 272]. So φf is in the weak closure of M which, in turn, implies that $\varphi f \in M$, because of the convexity of M. Hence $\mathcal{H}^\infty \subset \mathcal{M}(M)$. To see the other inclusion, if $\varphi \in \mathcal{M}(M)$, then an application of the closed graph theorem shows that M_φ is bounded. Also for every $f \in M$ and $z \in \mathbf{D}$, $\langle f, M_\varphi^* e_z \rangle = \langle \varphi f, e_z \rangle = \varphi(z) f(z) = \langle f, \varphi(z) e_z \rangle$. Hence $M_\varphi^* e_z = \varphi(z) e_z$ and $||\varphi||_\infty \leq ||M_\varphi^*|| = ||M_\varphi||$.

(A2) implies (A3): Define the norm $|||.|||$ on $\mathcal{H}^\infty$ by $|||\varphi|||_\infty = ||M_\varphi||$. If $\{\varphi_n\}$ is a Cauchy sequence in $(\mathcal{H}^\infty, |||.|||)$ then M_{φ_n} converges to an operator A in $L(\mathcal{B})$, the set of all bounded operators on $\mathcal{B}$. It is easy to see that $\{M_z\}'$, the commutant of M_z, is $\{M_\phi \mid \phi \in \mathcal{H}^\infty\}$ (see [3]); so $A = M_\varphi$ for some $\varphi \in \mathcal{H}^\infty$. It follows that $(\mathcal{H}^\infty, |||.|||)$ is a Banach space. Since $||\varphi||_\infty \leq |||\varphi|||$ for every $\varphi \in \mathcal{H}^\infty$, an application of the inverse mapping theorem to the identity map $i : (\mathcal{H}^\infty, |||.|||) \to (\mathcal{H}^\infty, ||.||_\infty)$ shows that there is a constant c such that $||M_\varphi|| \leq c||\varphi||_\infty$ for every $\varphi \in \mathcal{H}^\infty$.

Clearly, $(A3)$ implies $(A1)$. □

Theorem 2.2 The set of all multipliers of M is $\mathcal{H}^\infty$, and for each $\varphi \in \mathcal{H}^\infty$ the operator $M_\varphi : M \to M$ is a bounded operator. Also, M_φ is invertible if and only if φ is invertible in $\mathcal{H}^\infty$.

Proof. By Property (1), $\mathcal{H}^\infty \subset \mathcal{M}(\mathcal{B})$, also in the proof of (A1)$\Rightarrow$(A2) in previous Lemma we see that $\mathcal{M}(\mathcal{B}) \subset \mathcal{H}^\infty$. So there is a constant $c > 0$ such that $||M_p|| \leq c||p||_\infty$, where $||M_p||$ denotes the norm of M_p on $\mathcal{B}$ which is bigger than the norm of M_p on M. By Lemma 2.1, $\mathcal{M}(M) = \mathcal{H}^\infty$. The boundedness of M_φ follows from the closed graph theorem. □

In this section the techniques of some proofs are slight refinements of the similar ones, used in [8]. For the reader's convenience, we include some of the proofs.

Proposition 2.3 Suppose p is a polynomial and

$$q(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

where $z_1, z_2, \cdots z_n$ are the zeros of p in $\mathbf{D}$. Then the closure of pM in $\mathcal{B}$ is qM.

Proof. Let

$$p(z) = q(z)r(z)s(z),$$

where r is a polynomial with no zero on the closed unit disc, and s is a polynomial with zeros on the unit circle. Since $r(z)$ and $\frac{1}{r(z)}$ are in $\mathcal{H}^\infty$ Theorem 2.2 implies that $rM = M$. Also, since M_q is bounded below, qM is closed in M. Hence it is sufficient to show that sM is dense in M.

Let

$$s(z) = (z - a_1)\cdots(z - a_m)$$

and for any $n \geq 1$ and $k = 1, 2, \cdots m$ let $p_{k,n}$ be the n-th Taylor polynomial of $(z - a_k)^{-1}$ at $z = 0$; moreover, suppose that

$$p_n = p_{1,n}p_{2,n}\cdots p_{m,n}.$$

If $f \in M$ then $s(z)p_n(z)f(z) \to f(z)$ for all $z \in \mathbf{D}$ and

$$||sp_n f|| \le c||sp_n||_\infty ||f|| \le c2^m ||f|| < \infty.$$

Consequently, f is in the weak closure of sM which is a convex set, and so $f \in \overline{sM}$. Hence $\overline{sM} = M$. $\square$

Proposition 2.4 Suppose φ is a non-vanishing function in $\mathcal{H}^\infty$ and M_φ is the multiplication operator on M. Then M_φ is Fredholm if and only if M_φ is invertible.

Proof. The method of proof is similar to the one used in [8, Lemma 6] and so is omitted.

We recall that the index of M is $\dim\frac{M}{zM}$. It is well known that the index of M can assume any value in $\{1, 2, 3 \cdots, \infty\}$ also it follows from the Fredholm theory that $\dim\frac{M}{(z-\lambda)M}$ does not depend on $\lambda \in \mathbf{D}$.

Theorem 2.5 Suppose $\varphi \in \mathcal{H}^\infty$ and M_φ is Fredholm on M. If $\varphi(0) = 0$, then M has finite index.

Proof. Let $\varphi = z\psi$ for some $\psi \in \mathcal{H}^\infty$. If $x^* \in \ker M_z^*$ and $f \in M$ then

$$0 = \langle \psi f, M_z^* x^* \rangle = \langle \varphi f, x^* \rangle = \langle f, M_\varphi^* x^* \rangle$$

so $\ker M_z^* \subset \ker M_\varphi^*$. On the other hand,

$$\dim\frac{M}{zM} = \dim(\frac{M}{zM})^* = \dim(zM)^\perp = \dim \ker M^*{}_z,$$

and

$$\dim \ker M_z^* \le \dim \ker M_\varphi^* = \dim\,(\varphi M)^\perp = \dim\,(\frac{M}{\varphi M})^* = \dim\,\frac{M}{\varphi M} < \infty.$$

Theorem 2.6 Suppose that $\varphi \in \mathcal{H}^\infty$ and M_φ is the multiplication operator on M. Then

(a) $\sigma(M_\varphi) = \overline{\varphi(\mathbf{D})}$.

If the index of M is finite, then

(b) M_φ is Fredholm if and only if φ is invertible in $\mathcal{H}^\infty$.

(c) $\sigma_e(M_\varphi) = \overline{\varphi(\mathbf{D})}$ and $r(M_\varphi) = r_e(M_\varphi) \leq ||M_\varphi||_e \leq ||M_\varphi|| \leq c||\varphi||_\infty$, for some $c > 0$.

Proof. (a) By Theorem 2.2, M_φ is invertible on M if and only if φ is invertible in $\mathcal{H}^\infty$. Also $\lambda - M_\varphi = M_{\lambda-\varphi}$ for every complex number λ; so $\sigma(M_\varphi) = \overline{\varphi(\mathbf{D})}$.

(b) If M_φ is Fredholm, Theorem 2.5 implies that $\varphi(0) \neq 0$. Since $\dim\frac{M}{(z-\lambda)M} = \dim\frac{M}{zM}$ for every $\lambda \in \mathbf{D}$, by the same technique as in the proof of the previous theorem one can show that $\varphi(\lambda) \neq 0$. Then Proposition 2.4 and Theorem 2.2 implies that φ is invertible in $\mathcal{H}^\infty$. The converse follows from Theorem 2.2.

(c) By Part (b) $\sigma_e(M_\varphi) = \overline{\varphi(\mathbf{D})}$. Since $\sigma_e(M_\varphi) = \sigma(M_\varphi)$, $r_e(M_\varphi) = r(M_\varphi)$. Using Theorem 2.2 and Lemma 2.1 the proof is complete. □

Lemma 2.7 If $B(z)$ is a finite Blaschke product then M_B is bounded below on $\mathcal{B}$.

Proof. Let

$$B(z) = z^k \prod_{n=1}^{m} \frac{|a_n|}{a_n} \frac{(a_n - z)}{(1 - \bar{a_n} z)}$$

and put $\varphi_n(z) = a_n - z$, $\psi_n(z) = \frac{1}{1-\bar{a_n}z}$ and $\varphi(z) = z^k$. It is known that M_{φ_n} and M_φ are bounded below [4]. Moreover, by Theorem 2.2, M_{ψ_n}, $n = 1, 2, \cdots m$ are invertible; hence M_B is a finite composition of bounded below operators which is certainly bounded below.□

Lemma 2.8 If the index of M is finite and B is a finite Blaschke product, then M_B is Fredholm on M.

Proof. Let $a_1, a_2, \cdots, a_m$ be the zeros of B according to the multiplicity and put $p(z) = (z - a_1)(z - a_2)\cdots(z - a_m)$. Proposition 2.3 implies that the closure of BM is pM, and by the previous lemma, BM is closed; thus $BM = pM$. Since $\dim \frac{M}{(z-\lambda)M}$ is independent of $\lambda \in \mathbf{D}$, if M is of index n then $\dim \frac{M}{BM} = \dim \frac{M}{pM} = nm$. □

Theorem 2.9 Suppose that $\varphi \in \mathcal{H}^\infty$ and M_φ is the multiplication operator on M. If the index of M is finite, then the following conditions are equivalent:

(i) The operator M_φ is Fredholm on M.

(ii)There exist constants $\varepsilon > 0$ and $0 < \delta < 1$ such that $|\varphi(z)| \geq \varepsilon$ for all z with $\delta \leq |z| < 1$.

(iii) The function ψ admits a factorization $\varphi = B\psi$, where B is a finite Blaschke product and ψ is invertible in $\mathcal{H}^\infty$.

Proof. $(i) \Rightarrow (ii)$. If φ is non- vanishing then by Proposition 2.4, M_φ is invertible. So $\frac{1}{\varphi} \in \mathcal{H}^\infty$ and (ii) holds. On the other hand, if $\varphi(a) = 0$ then there is a function $S(z)$ in $\mathcal{H}^\infty$ such that $\varphi(z) = (z-a)S(z)$. Let $f \in M$ then $\varphi f(z) = (z-a)S(z)f(z) \in (z-a)M$ (Theorem 2.2) so $\varphi M \subset (z-a)M$. Similarly, if $a_1, a_2, \cdots, a_k$ are zeros of φ in $\mathbf{D}$ then $\varphi M \subset pM$ where $p(z) = (z-a_1)(z-a_2)\cdots(z-a_k)$. Consequently, $(pM)^\perp \subset (\varphi M)^\perp$ and so $\dim(pM)^\perp = \dim\frac{M}{pM} = \dim(\frac{M}{pM})^* = \dim(pM)^\perp \leq \dim(\varphi M)^\perp = \dim(\frac{M}{\varphi M})^* = \dim\frac{M}{\varphi M}$. But $\dim\frac{M}{pM} = kn$ where n is the index of M. This implies that φ has only a finite number of zeros in $\mathbf{D}$; hence (ii) holds.

$(ii) \Rightarrow (i)$. If φ is non-vanishing then, clearly, $\frac{1}{\varphi} \in \mathcal{H}^\infty$ and by Theorem 2.2 and Proposition 2.4, M_φ is Fredholm. Otherwise, φ has a finite number of zeros $a_1, a_2, \cdots, a_m$ in $\mathbf{D}$ listed according to their multiplicities . Thus $\varphi(z) = (z-a_1)(z-a_2)\cdots(z-a_m)S(z)$ for some function $S(z)$ in $\mathcal{H}^\infty$, with no zeros in $\mathbf{D}$. It is obvious that $\frac{1}{S} \in \mathcal{H}^\infty$, and using Theorem 2.2, we see that $SM = M$. Hence $\dim\frac{M}{\varphi M} = mn < \infty$.

$(ii) \Rightarrow (iii)$. Clearly, φ has a finite number of zeros $a_1, a_2, \cdots, a_m$ in $\mathbf{D}$. If $\varphi(z) = (z-a_1)(z-a_2)\cdots(z-a_m)S(z)$ for some nonzero function S in $\mathcal{H}^\infty$ then considering

$$B(z) = \prod_{n=1}^{m} \frac{|a_n|}{a_n}\frac{(a_n - z)}{(1-\bar{a_n}z)} \text{ and } \psi(z) = S(z)\prod_{i=1}^{m} \frac{|a_n|}{a_n}\frac{(a_n - z)}{(1-\bar{a_n}z)}$$

(iii) holds.

$(iii) \Rightarrow (ii)$. It is obvious. $\square$

Recall that for $\varphi \in \mathcal{H}^\infty$ $cluster(\varphi)$ is the set of all complex numbers ω such that there exists a sequence $\{z_n\}$ in $\mathbf{D}$ such that $|z_n| \to 1$ and $\varphi(z_n) \to \omega$ as $n \to \infty$.

Corollary 2.10 Under the assumptions of the above theorem, we have $\sigma_e(M_\varphi) = cluster(\varphi)$ and

$$||\varphi||_\infty \leq ||M_\varphi||_e \leq c||\varphi||_\infty$$

for some positive constant c.

Proof. The formula for the essential spectrum follows from the above theorem and the identity $\lambda - M_\varphi = M_{\lambda-\varphi}$, where λ is any complex number. Theorem 2.2 and Lemma 2.1, along with the maximum modulus principle imply that

$$||\varphi||_\infty \leq \mathrm{r}_e(M_\varphi) \leq ||M_\varphi||_e \leq ||M_\varphi|| \leq c||\varphi||_\infty$$

for some $c > 0$. □

3. Interpolating Sequences

In this section, our aim is to reduce the conditions of the main theorem in [6] but to give more results and a very simpler proof of it. As in [6] we have the following definition.

Definition 3.1 A sequence $\{\omega_n\}$ in $\mathbf{D}$ is said to be an interpolating sequence for $\mathcal{B}$ if there exists a positive weight sequence $\{k_n\}$ so that the sequence $\{f(\omega_n)k_n\}$ belongs to ℓ^∞ for all f in $\mathcal{B}$ and, conversely, every sequence in ℓ^∞ can be written in this form.

Theorem 3.2 If $\{\omega_n\}$ is a sequence in $\mathbf{D}$ such that

$$\frac{1-|\omega_n|}{1-|\omega_{n-1}|} < \delta < 1 \qquad (1)$$

then $\{\omega_n\}$ is an interpolating sequence for $\mathcal{B}$.

Proof. It is known that if (1) holds then $\{\omega_n\}$ is an interpolating sequence for $\mathcal{H}^\infty$ (see Page 203 of [2]). But $\mathcal{H}^\infty \subset \mathcal{B}$, and

$$|f(\omega_n)| = |\langle f, e_{\omega_n}\rangle| \leq ||f||\,||e_{\omega_n}|| \quad \forall f \in \mathcal{B}.$$

So the result holds, considering the weights $k_n = \frac{1}{||e_{\omega_n}||}$. □

We remark that if $\{\omega_n\}$ is a sequence in $\mathbf{D}$ such that $\omega_n \to \partial\mathbf{D}$ then some subsequence of $\{\omega_n\}$ satisfies (1), so Theorem 5 of [6] holds.

4. The commutant of M_φ

In this section, we assume that M is an invariant subspace of $\mathcal{B}$ with codimension 1 property, that is, $\dim(\frac{M}{zM}) = 1$, and φ is a certain function in $\mathcal{H}^\infty$ or in $\mathcal{A}(\mathbf{D})$ the algebra of all continuous functions on $\overline{\mathbf{D}}$ which are analytic in $\mathbf{D}$. Under these conditions we investigate the commutant of M_φ as an operator from M to M. Let Z_M be the intersection of zero sets of functions in M and e'_λ be the evaluation functional at λ from M to $\mathbf{C}$.

Theorem 4.1 Let $\varphi \in \mathcal{H}^\infty$ be a univalent function. Then $\{M_\varphi\}' = \{M_\psi : \ \psi \in \mathcal{H}^\infty\}$.

Proof. If $f \in M$, $\lambda \in \mathbf{D}$ and $T \in \{M_\varphi\}'$, then we have

$$\begin{aligned}\langle f, {M_\varphi}^* T^*(e'_\lambda)\rangle &= \langle f, T^* {M_\varphi}^*(e'_\lambda)\rangle = \langle M_\varphi T(f), e'_\lambda\rangle \\ &= \varphi(\lambda) Tf(\lambda) = \varphi(\lambda)\langle T(f), e'_\lambda\rangle \\ &= \varphi(\lambda)\langle f, T^*(e'_\lambda)\rangle = \langle f, \varphi(\lambda) T^*(e'_\lambda)\rangle.\end{aligned}$$

Consequently, $M_\varphi^* T^*(e'_\lambda) = \varphi(\lambda) T^*(e'_\lambda)$, which implies that $T^*(e'_\lambda) \in \ker(M_\varphi - \varphi(\lambda))^*$. Let $\lambda \in \mathbf{D} - Z_M$. We show that $\mathrm{ran}(M_\varphi - \varphi(\lambda)) = \ker e'_\lambda$. If $f \in M$, then

$$\langle (\varphi - \varphi(\lambda)) f, e'_\lambda\rangle = ((\varphi - \varphi(\lambda)) f)(\lambda) = 0,$$

and so $\mathrm{ran}(M_\varphi - \varphi(\lambda)) \subset \ker e'_\lambda$.

To show the converse, since $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$, we have $(\varphi - \varphi(\lambda))(z) = (z - \lambda)g(z)$ for some $g \in \mathcal{B}$. Since φ is univalent, g is bounded below on $\mathbf{D}$. Hence $\frac{1}{g}$ is in $\mathcal{H}^\infty$ and, so $\frac{1}{g} \in \mathcal{M}(M)$ and we have $z - \lambda = \frac{\varphi(z)-\varphi(\lambda)}{g(z)}$. By Lemma 2.1 in [4], $\operatorname{ran}((M_z - \lambda)|_M) = \ker e'_\lambda$. So if $f \in \ker e'_\lambda$, then $f = (z - \lambda)\phi$ for some function $\phi \in M$. Hence

$$f = \frac{\varphi - \varphi(\lambda)}{g}\phi = (\varphi - \varphi(\lambda))\frac{\phi}{g}.$$

Since $\phi \in M$ and $\frac{1}{g} \in \mathcal{H}^\infty$, we conclude that $\ker e'_\lambda \subset \operatorname{ran}(M_\varphi - \varphi(\lambda))$.

Now, we have $(M_\varphi - \varphi(\lambda))^*(e'_\lambda) = (M_\varphi - \varphi(\lambda))^* T^*(e'_\lambda) = 0$. Since $\dim \ker(M_\varphi - \varphi(\lambda))^* = 1$, we conclude that $T^*(e'_\lambda) = \psi(\lambda)e'_\lambda$ for some constant $\psi(\lambda)$. Thus, for every $\lambda \in \mathbf{D} - Z_M$ we have

$$T(f)(\lambda) = \langle T(f), e'_\lambda\rangle = \langle f, T^*(e'_\lambda)\rangle = \psi(\lambda)\langle f, e_\lambda\rangle = \psi(\lambda)f(\lambda);$$

in particular, $\psi(\lambda) = T(1)(\lambda)$. Let ψ denote $T(1)$. Since $\mathbf{D} - Z_M$ is dense in $\mathbf{D}$, we conclude that $T(f)(\lambda) = \psi(\lambda)f(\lambda)$ for every λ in $\mathbf{D}$, and the proof is complete. □

The proof of the above theorem is a modification of the proof of Theorem 2.2 in [3].

Theorem 4.2 Let $\varphi \in \mathcal{A}(\mathbf{D})$, S be a subset of $\mathbf{D} - Z_M$ which has a limit point in $\mathbf{D}$. Suppose further that for every $\lambda \in S$, we have $\varphi^{-1}(\{\varphi(\lambda)\}) = \{\lambda\}$. Then $\{M_\varphi\}' = \{M_\psi : \ \psi \in \mathcal{H}^\infty\}$.

Proof. Without loss of generality, we can assume that $\varphi'(\lambda) \neq 0$ for every $\lambda \in S$. As in the proof of Theorem 4.1, for every $\lambda \in \mathbf{D}$ we have $T^*(e'_\lambda) \in \ker(M_\varphi - \varphi(\lambda))^*$ and for $\lambda \in \mathbf{D} - Z_M$, $\operatorname{ran}(M_\varphi - \varphi(\lambda)) \subset \ker e'_\lambda$.

Now, let $\lambda \in S$. Since $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$, we have $(\varphi - \varphi(\lambda))(z) = (z - \lambda)g(z)$ for some $g \in \mathcal{A}(\mathbf{D})$ which implies that $\ker e'_\lambda \subset \operatorname{ran}(M_\varphi - \varphi(\lambda))$. Now, by assumption g is bounded below on $\mathbf{D}$ and as in the remainder of the proof of Theorem 4.1 we conclude that $\psi(\lambda) = T(1)(\lambda)$. Let ψ denote $T(1)$ since S has a limit point in $\mathbf{D}$, we conclude that $T(f)(\lambda) = \psi(\lambda)f(\lambda)$ for every λ in $\mathbf{D}$ as desired. □

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