Honam Mathematical J. **29** (2007), No. 4, pp. 535–542

OPTIMAL POLYNOMIAL LOWER BOUNDS FOR THE EXPONENTIAL FUNCTION

JAEGUG BAE

ABSTRACT. In this paper, for each natural number n, we construct a polynomial $p_n(x)$ of degree n so that $p_n(x) \le p_{n+1}(x) \le e^x$ for $x \ge -1$. These polynomials are optimal in the sense that if $p(x)$ is a polynomial of degree n with $p_{n-1}(x) \le p(x) \le e^x$, then $p(x) \le p_n(x)$.

1. Introduction

A typical bound of the exponential function is

$$1 + x \le e^x \quad \text{or} \quad e^x \le \frac{1}{1-x}. \tag{1.1}$$

Here the first inequality holds on the whole real line but the second holds for $x < 1$. Pólya proved the inequality between the arithmetic and geometric means using this simple inequality (see [2, §4.2], [7]). In fact, almost the same proof can be applied to show the inequality between the generalized arithmetic and geometric means:

$$\sum_{j=1}^{n} \lambda_j a_j \ge \prod_{j=1}^{n} {a_j}^{\lambda_j}$$

where λ_j's and a_j's are nonnegative real numbers with $\sum_{j=1}^{n} \lambda_j = 1$.

As a part of an effort to approximate e^x by other (algebraic) functions, various bounds for the exponential function have been developed (see [3], [5, pp. 266-270], [6]). Recently, Kim established the following densely algebraic bounds in [1], [4]

$$e^x \le 1 - \frac{1}{n} + \frac{1}{n}\left(\frac{1 + \left(1 - \frac{1}{n}\right)x}{1 - \frac{x}{n}}\right)^n$$

Received July 26, 2007. Revised October 11, 2007.
2000 Mathematics Subject Classification : Primary 33B10; Secondary 11A99.
Key words and phrases : bounds, polynomials, the exponential function.

for real $n \geq 1$ and $-\frac{n}{n-1} < x < n$. In this paper, we present the following simple polynomial bounds of e^x extending (1.1).

THEOREM 1.1. *For* $n \geq 1$, *let*

$$p_n(x) = 1 + \alpha_1 x + \sum_{j=2}^{n} \alpha_j \, x^2 (x+1)^{j-2}$$

where $\alpha_1 = 1$ *and*

$$\alpha_n = \frac{n-2}{e} \left(\sum_{j=0}^{n-2} \frac{1}{j!} - e \right) + \frac{1}{(n-2)!\, e}, \qquad n \geq 2.$$

Then $p_n(x) \leq p_{n+1}(x) \leq e^x$ *for* $x \geq -1$. *Furthermore, these are optimal in the sense that if* $p(x)$ *is a polynomial of degree* n *with* $p_{n-1}(x) \leq p(x) \leq e^x$ *for* $x \geq -1$, *then* $p(x) \leq p_n(x)$ *for* $x \geq -1$.

This theorem gives lower bounds of e^x but by writing $-x$ for x, we have the following rational upper bound version.

THEOREM 1.2. *For all natural numbers* n, *we have*

$$e^x \leq \frac{1}{p_{n+1}(-x)} \leq \frac{1}{p_n(-x)}, \qquad x < 1.$$

If $p(x)$ *is a polynomial of degree* n *with* $e^x \leq \frac{1}{p(x)} \leq \frac{1}{p_{n-1}(-x)}$, $x < 1$, *then*

$$\frac{1}{p_n(-x)} \leq \frac{1}{p(x)}, \qquad x < 1.$$

With some preliminaries in the next section, we will prove the theorem. We assume that all functions in this paper are real valued on $\mathbf{R}$ and analytic on $\mathbf{C}$. Also, note that when we say the number of zeros of a function, it includes all the multiplicities.

2. Preliminaries

At first, we give a couple of observations on calculus.

PROPOSITION 2.1. *Let* a *be a fixed real number. Suppose* $f^{(k-1)}(a) = 0$ *and* $f^{(k)}(a) > 0$ *for some positive integer* k.

(i) *If* k *is even, then there exists an interval around* a *on which* $f(x)$ *achieves the unique minimum* $f(a) = 0$.

(ii) *If* k *is odd, then there exists an interval around* a *on which* $f(x)$ *is strictly increasing.*

Proof. Let $h(x)$ be a function defined on an open interval I containing a. Let's say that $h(x)$ has unique minimum (UM) property when $h(a) = 0$ and it is the unique minimum on I, and $h(x)$ has strictly increasing (SI) property when $h(a) = 0$ and $h(x)$ is strictly increasing on I. It's easy to see that if $h'(x)$ has UM-property and $h(a) = 0$ then $h(x)$ has SI-property. On the other hand, if $h'(x)$ has SI-property and $h(a) = 0$ then $h(x)$ has UM-property. Since $f^{(k)}(a) > 0$, by the continuity of $f^{(k)}(x)$, there exists an open interval I containing a on which $f^{(k)}(x) > 0$. Hense, on the interval I, $f^{(k-1)}(x)$ has SI-property, $f^{(k-2)}(x)$ has UM-property, $f^{(k-3)}(x)$ has SI-property, $\cdots$, and so on. Pursuing these alternations k times, we have the conclusions of the proposition. □

PROPOSITION 2.2. *Let* $p(x)$ *be a polynomialof degree* n. *Then* $e^x - p(x)$ *has at most* $n+1$ *zeros.*

Proof. Suppose a function $f(x)$ has k zeros. Applying Roll's Theorem, we know that its derivative $f'(x)$ has at least $k-1$ zeros. In other words, if $f'(x)$ has $k-1$ zeros, then $f(x)$ has at most k zeros. Now the proof is straightforwards by using induction on n, the degree of $p(x)$. □

We introduce one more proposition. We need the following lemma for the proof of it.

LEMMA 2.3. *Suppose there exists an interval* $[a, a+\delta]$ *on which* $f(x) \geq 0$ *and* $f(a) = 0$. *Then there exists* λ $(0 < \lambda < \delta)$ *such that* $f'(x) \geq 0$ *on* $[a, a+\lambda]$.

Proof. We may assume $f(x)$ is not constant. Let $\beta = \inf\{x \in [a, a+\delta] \mid f'(x) < 0\}$. If $\beta > a$, then we may take any real number between a and β as our λ. In fact, $\beta = a$ is impossible because (as an analytic function) $f(x)$ has only finitely many zeros on $[a, a+\delta]$ and for any $\epsilon > 0$, there is no open interval $(a, a+\epsilon)$ on which $f(x) < 0$. □

DEFINITION 2.4. We say that a is an m-multiple zero of $f(x)$ when $f^{(j)}(a) = 0$ for $j = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(a) \neq 0$. In this case, $Z(a, f) = m$ is called the multiplicity of the zero a of f.

PROPOSITION 2.5. *Suppose* $0 \leq f(x) \leq g(x)$ *for* $x \geq a$ *and* $Z(a, g) = m$. *Then* $Z(a, f) \geq m$ *and* $f^{(m)}(a) \leq g^{(m)}(a)$.

Proof. By the definition of a multiple zero, $g^{(j)}(a) = 0$ for $j = 0, 1, 2, \cdots, m-1$. Hence, for each $j \in \{0, 1, 2, \cdots, m\}$, it suffices to show that $0 \leq f^{(j)}(x) \leq g^{(j)}(x)$ on the interval $[a, a+\lambda_j]$ for some $\lambda_j > 0$. We use an induction on j. Assume $0 \leq f^{(j-1)}(x) \leq g^{(j-1)}(x)$ on the interval $[a, a+\lambda_{j-1}]$. Then note $f^{(j-1)}(a) = g^{(j-1)}(a) = 0$. Now applying Lemma 2.3 twice, one for $f^{(j-1)}(x)$ and another for $g^{(j-1)}(x) - f^{(j-1)}(x)$, we can find an interval $[a, a+\lambda_j]$ on which $0 \leq f^{(j)}(x) \leq g^{(j)}(x)$. □

3. Proof of the main theorem

In this section, we prove Theorem 1.1. At first, we establish some properties of the sequence α_n.

LEMMA 3.1. *For* $n \geq 1$, *we have*

(i) $\alpha_n > 0$

(ii) $\alpha_{n+2} = 2\alpha_{n+1} - \alpha_n + \frac{1}{n!\,e}$

(iii) $\sum_{n=1}^{\infty} \alpha_n = \frac{3}{2}$.

Proof. In this proof, we will use the following inequality frequently.

$$e - \sum_{j=0}^{n-1} \frac{1}{j!} = \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right) \tag{3.1}$$
$$< \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) = \frac{n+1}{n!\,n}$$

By this inequality, we have

$$\alpha_n = \frac{n-2}{e}\left(\sum_{j=0}^{n-2} \frac{1}{j!} - e\right) + \frac{1}{(n-2)!\,e}$$
$$> \frac{n-2}{e}\frac{-n}{(n-1)!\,(n-1)} + \frac{1}{(n-2)!\,e} = \frac{1}{(n-2)!\,e}\left(1 - \frac{n(n-2)}{(n-1)^2}\right) > 0.$$

For (ii), the direct calculation shows that

$$2\alpha_{n+1} - \alpha_n + \frac{1}{n!\,e}$$
$$= \frac{2n-2}{e}\left(\sum_{j=0}^{n-1} \frac{1}{j!} - e\right) + \frac{2}{(n-1)!\,e} - \frac{n-2}{e}\left(\sum_{j=0}^{n-2} \frac{1}{j!} - e\right) - \frac{1}{(n-2)!\,e} + \frac{1}{n!\,e}$$
$$= \frac{n}{e}\left(\sum_{j=0}^{n} \frac{1}{j!} - e\right) + \frac{1}{n!\,e} = \alpha_{n+2}.$$

For (iii), it's enough to show that $\sum_{n=1}^{\infty} n\left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) = \frac{1}{2}e$ since $\alpha_1 = 1$ and $\sum_{n=2}^{\infty} \frac{1}{(n-2)!\,e} = 1$. Let $\sigma_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$ with $\sigma_0 = 0$. Then

$$\sum_{k=1}^{n} k\left(e - \sum_{j=0}^{k} \frac{1}{j!}\right)$$
$$= \sigma_n e - \sigma_n \frac{1}{0!} - \sum_{j=1}^{n} (\sigma_n - \sigma_{j-1})\frac{1}{j!} = \sigma_n\left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) + \sum_{j=2}^{n} \frac{\sigma_{j-1}}{j!}.$$

Note that, by (3.1), the first term of the last expression goes to zero as n goes to infinity. Thus we obtain

$$\sum_{n=1}^{\infty} n\left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) = \sum_{j=2}^{\infty} \frac{\sigma_{j-1}}{j!} = \sum_{j=1}^{\infty} \frac{\sigma_j}{(j+1)!}$$
$$= \sum_{j=1}^{\infty} \frac{1}{2} j(j+1)\frac{1}{(j+1)!} = \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{j!} = \frac{1}{2}e.$$

□

LEMMA 3.2. *For* $n \geq 1$, *let* $g_n(x) = e^x - p_n(x)$. *Then*

(i) $g_n(0) = g_n'(0) = 0, \quad g_n''(0) = 1 - 2\sum_{j=2}^{n} \alpha_j \; > \; 0 \quad$ (*with* $g_1''(0) = 1$).

(ii) *For* $n \geq 2$, $g_n(x)$ *has* $n-1$*-multiple zeros at* -1 *and* $g_n^{(n-1)}(-1) > 0$.

Proof. Note that

$$p_n'(x) = 1 + 2\sum_{j=2}^{n} \alpha_j x(x+1)^{j-2} + \sum_{j=3}^{n}(j-2)\alpha_j x^2(x+1)^{j-3},$$

$$\begin{aligned} p_n''(x) &= 2\sum_{j=2}^{n} \alpha_j (x+1)^{j-2} \\ &\quad + 4\sum_{j=3}^{n}(j-2)\alpha_j x(x+1)^{j-3} + \sum_{j=4}^{n}(j-2)(j-3)\alpha_j x^2(x+1)^{j-4}. \end{aligned}$$

Invoking Lemma 3.1, (i) is immediate now. For $k \geq 2$, let us define $f_k(x) = \alpha_k x^2 (x+1)^{k-2}$. Then we observe that

$$\begin{aligned} f_k^{(j)}(-1) &= \; 0 \quad \text{if} \quad 0 \leq j < k-2 \quad \text{or} \quad j > k \\ f_k^{(k)}(x) &= \; k!\alpha_k \\ f_k^{(k-1)}(x) &= \; (k-1)!2\alpha_k x + (k-1)!(k-2)\alpha_k(x+1) \\ f_k^{(k-2)}(x) &= \; (k-2)!\alpha_k x^2 + (k-2)!2(k-2)\alpha_k x(x+1) \\ &\quad + \frac{1}{2}(k-2)!(k-2)(k-3)\alpha_k(x+1)^2. \end{aligned} \tag{3.2}$$

To prove (ii), let $n \geq 2$. Since $g_n(x) = e^x - 1 - \alpha_1 x - \sum_{k=2}^{n} f_k(x)$, we have

$$\begin{aligned} g_n^{(n-1)}(-1) &= \frac{1}{e} - f_{n-1}^{(n-1)}(-1) - f_n^{(n-1)}(-1) \\ &= \frac{1}{e} - (n-1)!\alpha_{n-1} + (n-1)!2\alpha_n \\ &= (n-1)!\left(2\alpha_n - \alpha_{n-1} + \frac{1}{(n-1)!e}\right) \; = \; (n-1)!\alpha_{n+1} \; > 0 \end{aligned}$$

by (3.2) and Lemma 3.1 (ii). Finally, it remains to show $g_n^{(j)}(-1) = 0$, for $0 \leq j \leq n-2$. Clearly, $g_n^{(0)}(-1) = g_n(-1) = \frac{1}{e} - 1 + \alpha_1 - \alpha_2 = 0$ and $g_n^{(1)}(-1) = \frac{1}{e} - \alpha_1 + 2\alpha_2 - \alpha_3 = 0$. Using (3.2) and Lemma 3.1 (ii) again for

$2 \le j \le n-2$, we obtain

$$\begin{aligned} g_n^{(j)}(-1) &= \frac{1}{e} - \sum_{k=2}^{n} f_k^{(j)}(-1) = \frac{1}{e} - f_j^{(j)}(-1) - f_{j+1}^{(j)}(-1) - f_{j+2}^{(j)}(-1) \\ &= \frac{1}{e} - j!\,\alpha_j + j!\,2\,\alpha_{j+1} - j!\,\alpha_{j+2} \\ &= j!\left(-\alpha_{j+2} + 2\alpha_{j+1} - \alpha_j + \frac{1}{j!e}\right) = 0. \end{aligned}$$

□

Now we are ready to prove the main theorem.

Proof of Theorem 1.1.
Since α_n is a positive sequence by Lemma 3.1 and $x^2(x+1) \ge 0$ for $x \ge -1$, it's clear that $p_n(x) \le p_{n+1}(x)$ for $x \ge -1$. We are to show $g_n(x) \ge 0$ for $x \ge -1$. Lemma 3.2 shows that $g_n(x)$ has double zeros at $x = 0$ and $n-1$-multiple zeros at $x = -1$. Also, Proposition 2.2 says that these are all the zeros of $g_n(x)$. Lemma 3.2 (i) implies that $g_n(0) = 0$ is a local minimum. Moreover, combining Lemma 3.2 (ii) with Proposition 2.1, we realize that $g_n(-1) = 0$ is a local minimum for odd $n(>1)$ and $g_n(x)$ is increasing in some neighborhood of $x = -1$ for even n. Based on these analyses, we may conclude $g_n(x) \ge 0$ for $x \ge -1$. Finally, suppose $p(x)$ is a polynomial of degree n with $p_{n-1}(x) \le p(x) \le e^x$ for $x \ge -1$. This means $0 \le p(x) - p_{n-1}(x) \le g_{n-1}(x)$ for $x \ge -1$. Let $f(x) = p(x) - p_{n-1}(x)$. Then $f(x)$ is a polynomial of degree n and by Lemma 3.2 and Proposition 2.5, it has double zeros at 0 and $n-2$-multiple zeros at -1. Hence

$$f(x) = \alpha x^2 (x+1)^{n-2}$$

for some $\alpha \in \mathbf{R}$. Proposition 2.5 also says that

$$(n-2)!\,\alpha = f^{(n-2)}(-1) \le g_{n-1}^{(n-2)}(-1) = (n-2)!\,\alpha_n$$

which implies $\alpha \le \alpha_n$. Therefore we have

$$p_n(x) - p(x) = p_n(x) - p_{n-1}(x) - f(x) = (\alpha_n - \alpha) x^2 (x+1)^{n-2} \ge 0$$

for $x \ge -1$. □

References

[1] J. Bae, S. Kim, *On a generalization of an upper bound for the exponential function (to appear).*

[2] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge, 1975.

[3] J. Karamata, *Sur l'approximation de e^x par des fonctions rationnelles (in Serbian)*, Bull. Soc. Math. Phys. Serbie **1** (1949), 7–19.

[4] S. Kim, *Densely algebraic bounds for the exponential function*, Proc. Amer. Math. Soc. **135** (2007), 237–241.

[5] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, New York, 1970.

[6] W. E. Sewell, *Some inequalities connected with exponential function (in Spanish)*, Rev. Ci (Lima) **40** (1938), 453–456.

[7] J. E. Wetzel, *On the functional inequality* $f(x+y) \geq f(x)f(y)$, Amer. Math. Monthly **74** (1967), 1065–1068.

Jaegug Bae
Department of Applied Mathematics,
Korea Maritime University,
1 Dongsam-Dong, Yeongdo-Gu, Pusan 606-791, S. Korea
E-mail: jgbae@hhu.ac.kr