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STABILITY OF A MIXED TYPE FUNCTIONAL EQUATION IN 3-VARIABLES

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Abstract. In this paper, we prove the stability of a mixed type functional equation

$$f(-x+y+z)+f(x-y+z)+f(x+y-z)$$
$$=f(x-y)+f(y-z)+f(z-x)+f(x)+f(y)+f(z).$$

1. Introduction

In 1940, S. M. Ulam ([15]) posed the following question on the stability of homomorphisms: Given a group G_1, a metric group G_2 with a metric d and a number $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d(f(x), h(x)) < \epsilon$$

for all $x \in G_1$? This question became a source of the stability theory in the Hyers-Ulam sense. The case of approximately additive mappings was solved by D. H. Hyers ([3]) under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias ([12]) generalized the Hyers-Ulam stability by considering variables. In 1994, it also generalized to the function case by Gavruta.

A quadratic function $f(x) = x^2$ is a solution of the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

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Hence this equation is called a quadratic functional equation, and every solution of the quadratic functional equation (1.1) is called a quadratic function. A Hyers-Ulam stability for the quadratic functional equation (1.1) was proved by Skof [13]. In [2], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Later, the stability problems of a various quadratic type functional equations have been extensively investigated by a number of mathematicians ([1], [5-10]).

Consider the following functional equation

$$\begin{aligned} &f(-x+y+z)+f(x-y+z)+f(x+y-z) \\ &= f(x-y)+f(y-z)+f(z-x)+f(x)+f(y)+f(z). \end{aligned} \tag{1.2}$$

A function $f(x) = x^2 + x$ is a solution of the functional equation (1.2). We note that the equation (1.2) is called a mixed type functional equation.

The main purpose of the present paper is to prove the stability of a functional equation (1.2). In section 2, we obtain the general solution of a functional equation (1.2). In section 3, we investigate the Hyers-Ulam-Rassias stability of a functional equation (1.2).

2. Solutions of a functional equation (1.2)

THEOREM 1. *Let X and Y be real linear spaces. A function $f : X \to Y$ satisfies (1.2) for all $x, y, z \in X$ if and only if there exist an additive function $A : X \to Y$ and a quadratic function $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.*

Proof. Necessity. Putting $x = y = z = 0$ in (1.2) we have $f(0) = 0$. Let $A : X \to Y$ and $Q : X \to Y$ be the functions defined by $A(x) := \frac{1}{2}[f(x) - f(-x)]$ and $Q(x) := \frac{1}{2}[f(x) + f(-x)]$, respectively. Then we have $A(0) = Q(0) = 0$, $A(-x) = -A(x)$ and $Q(-x) = Q(x)$ for all $x \in X$. Obviously, we have $f(x) = A(x) + Q(x)$ for all $x \in X$. We claim that A is additive and Q is quadratic. Since f satisfies (1.2) for all $x, y, z \in X$, it is immediately seen that

$$\begin{aligned} &A(-x+y+z)+A(x-y+z)+A(x+y-z) \\ &= A(x-y)+A(y-z)+A(z-x)+A(x)+A(y)+A(z) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &Q(-x+y+z)+Q(x-y+z)+Q(x+y-z) \\ &= Q(x-y)+Q(y-z)+Q(z-x)+Q(x)+Q(y)+Q(z) \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$.

Putting $z = 0$ in (2.1) we have

$$A(-x+y)+A(x-y)+A(x+y) = A(x-y)+A(y)+A(-x)+A(x)+A(y), \quad \text{or}$$

$$A(-x+y)+A(x+y) = 2A(y) \tag{2.3}$$

for all $x, y \in X$. Putting $x = y$ in (2.3) we have $A(2y) = 2A(y)$ and

$$A(-x+y)+A(x+y) = A(2y) \tag{2.4}$$

for all $x, y \in X$. Letting $-x+y = u$ and $x+y = v$ in (2.4) we have $A(u)+A(v) = A(u+v)$ for all $u, v \in X$. Consequently, A is additive as claimed.

Putting $z = 0$ in (2.2) we have

$$Q(-x+y)+Q(x-y)+Q(x+y) = Q(x-y)+Q(y)+Q(-x)+Q(x)+Q(y), \quad \text{or}$$

$$Q(-x+y)+Q(x+y) = 2Q(x)+2Q(y)$$

for all $x, y \in X$. Consequently, Q is quadratic as claimed.

Sufficiency. This is obvious. □

In the next section we prove the stability of a functional equation (1.2).

3. Stability of a functional equation (1.2)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$\begin{aligned} Df(x,y,z) :=& f(-x+y+z) + f(x-y+z) + f(x+y-z) \\ & - f(x-y) - f(y-z) - f(z-x) - f(x) - f(y) - f(z) \end{aligned}$$

for all $x, y, z \in X$.

THEOREM 2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi_1(x,y,z) := \sum_{i=0}^{\infty} 4^{-(i+1)} \phi(2^i x, 2^i y, 2^i z) < \infty \tag{3.1}$$

or

$$\Phi_2(x,y,z) := \sum_{i=0}^{\infty} 4^{i} \phi(2^{-(i+1)} x, 2^{-(i+1)} y, 2^{-(i+1)} z) < \infty \tag{3.1'}$$

for all $x, y, z \in X$. If an even function $f : X \to Y$ satisfies

$$||Df(x,y,z)|| \le \phi(x,y,z) \tag{3.2}$$

for all $x, y, z \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \le \begin{cases} \Phi_1(x,x,0), & \text{if } \phi \text{ satisfies (3.1)} \\ \Phi_2(x,x,0), & \text{if } \phi \text{ satisfies (3.1')} \end{cases} \tag{3.3}$$

for all $x \in X$. The function Q is given by $Q(x) := \lim_{n\to\infty} 4^{-n} f(2^n x)$ if ϕ satisfies (3.1) and $Q(x) := \lim_{n\to\infty} 4^{n} f(2^{-n} x)$ if ϕ satisfies (3.1′).

Proof. (I) The case ϕ satisfies (3.1). Letting $y = x$ and $z = 0$ in (3.2) we have

$$||f(2x) - 3f(x) - f(-x)|| \le \phi(x,x,0), \quad \text{or}$$

$$||4^{-1} f(2x) - f(x)|| \le 4^{-1} \phi(x,x,0) \tag{3.4}$$

for all $x \in X$, since f is even. Replacing x by $2^{n-1}x$ in (3.4) we get

$$||4^{-1}f(2^n x) - f(2^{n-1}x)|| \le 4^{-1}\phi(2^{n-1}x, 2^{n-1}x, 0), \quad \text{or}$$

$$(3.5) \qquad ||4^{-n}f(2^n x) - 4^{-(n-1)}f(2^{n-1}x)|| \le 4^{-n}\phi(2^{n-1}x, 2^{n-1}x, 0)$$

for all $x \in X$ and all nonnegative integers n. By (3.5) and using the triangle inequality, we have

$$(3.6) \qquad ||4^{-n}f(2^n x) - 4^{-m}f(2^m x)|| \le \sum_{i=m}^{n-1} 4^{-(i+1)}\phi(2^i x, 2^i x, 0)$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. By virtue of (3.1), the right hand side of the inequality (3.6) tends to zero as m tends to infinity. Hence, the sequence $\{4^{-n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^{-n}f(2^n x)$$

for all $x \in X$. Putting $m = 0$ in (3.6) and letting $n \to \infty$ we have the formula (3.3). We have $Q(-x) = Q(x)$ and

$$\begin{aligned} ||DQ(x,y,z)|| &= \lim_{n\to\infty} 4^{-n}||Df(2^n x, 2^n y, 2^n z)|| \\ &\le \lim_{n\to\infty} 4^{-n}\phi(2^n x, 2^n y, 2^n z) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X$. As in the proof of Theorem 1, Q is quadratic. Now, let $Q' : X \to Y$ be another quadratic function satisfying (3.3). Then we have

$$\begin{aligned} ||Q(x) - Q'(x)|| &= 4^{-n}||Q(2^n x) - Q'(2^n x)|| \\ &\le 4^{-n}(||Q(2^n x) - f(2^n x)|| + ||Q'(2^n x) - f(2^n x)||) \\ &\le 2 \cdot 4^{-n}\Phi_1(2^n x, 2^n x, 0) \\ &= 2 \cdot 4^{-n} \sum_{i=0}^{\infty} 4^{-(i+1)}\phi(2^i \cdot 2^n x, 2^i \cdot 2^n x, 0) \\ &= 2 \cdot \sum_{i=n}^{\infty} 4^{-(i+1)}\phi(2^i x, 2^i x, 0) \end{aligned}$$

for all $x \in X$ and all positive integers n. But, by virtue of (3.1) we have

$$\lim_{n\to\infty}\sum_{i=n}^{\infty} 4^{-(i+1)}\phi(2^i x, 2^i x, 0) = 0.$$

Therefore we can conclude that $Q(x) = Q'(x)$ for all $x \in X$.

(II) The case ϕ satisfies (3.1′). We omit the proof, since the proof is similar to the case (I). □

THEOREM 3. *Let $\phi : X^3 \to [0,\infty)$ be a function such that*

$$\Phi_3(x,y,z) := \sum_{i=0}^{\infty} 2^{-(i+1)}\phi(2^i x, 2^i y, 2^i z) < \infty \tag{3.7}$$

or

$$\Phi_4(x,y,z) := \sum_{i=0}^{\infty} 2^{i}\phi(2^{-(i+1)} x, 2^{-(i+1)} y, 2^{-(i+1)} z) < \infty \tag{3.7'}$$

for all $x, y, z \in X$. If an odd function $f : X \to Y$ satisfies (3.2) for all $x, y, z \in X$, then there exists a unique additive function $A : X \to Y$ such that

$$||f(x) - A(x)|| \le \begin{cases} \Phi_3(x,x,0), & \text{if } \phi \text{ satisfies (3.7)} \\ \Phi_4(x,x,0), & \text{if } \phi \text{ satisfies (3.7')} \end{cases} \tag{3.8}$$

for all $x \in X$. The function A is given by $A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$ if ϕ satisfies (3.7) and $A(x) := \lim_{n\to\infty} 2^{n} f(2^{-n} x)$ if ϕ satisfies (3.7′).

Proof. As in the proof of Theorem 2, we can find inequalities

$$||2^{-1}f(2x) - f(x)|| \le 2^{-1}\phi(x,x,0)$$

and

$$||f(x) - 2f(2^{-1}x)|| \le \phi(2^{-1}x, 2^{-1}x, 0)$$

for all $x \in X$.

If ϕ satisfies (3.7), then as in the proof of Theorem 2 it can be proved that the sequence $\{2^{-n}f(2^n x)\}$ is a Cauchy and the function A, defined by $A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$, is the unique additive mapping satisfying (3.8).

If ϕ satisfies (3.7′), then the sequence $\{2^n f(2^{-n}x)\}$ is a Cauchy and the function A, defined by $A(x) := \lim_{n\to\infty} 2^n f(2^{-n}x)$, is the unique additive mapping satisfying (3.8).

We omit the details. □

THEOREM 4. *Let $\phi : X^3 \to [0,\infty)$ be a function satisfying (3.1′) or (3.7) for all $x, y, z \in X$. If a function $f : X \to Y$ satisfies (3.2) for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

(3.9)
$$||f(x) - Q(x) - A(x)|| \le \begin{cases} \widetilde{\Phi}_1(x,x,0) + \widetilde{\Phi}_3(x,x,0), & \text{if } \phi \text{ satisfies (3.7)} \\ \widetilde{\Phi}_2(x,x,0) + \widetilde{\Phi}_4(x,x,0), & \text{if } \phi \text{ satisfies (3.1')}, \end{cases}$$

$$(3.10) \qquad ||\frac{1}{2}(f(x) + f(-x)) - Q(x)|| \le \begin{cases} \widetilde{\Phi}_1(x,x,0), & \text{if } \phi \text{ satisfies (3.7)} \\ \widetilde{\Phi}_2(x,x,0), & \text{if } \phi \text{ satisfies (3.1')} \end{cases}$$

and

$$(3.11) \qquad ||\frac{1}{2}(f(x) - f(-x)) - A(x)|| \le \begin{cases} \widetilde{\Phi}_3(x,x,0), & \text{if } \phi \text{ satisfies (3.7)} \\ \widetilde{\Phi}_4(x,x,0), & \text{if } \phi \text{ satisfies (3.1')} \end{cases}$$

for all $x \in X$, where $\widetilde{\Phi}_i(x,y,z) = \frac{1}{2}(\Phi_i(x,y,z) + \Phi_i(-x,-y,-z))$, $i = 1,2,3,4$.

Proof. Let $g(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $g(-x) = g(x)$ and

$$\begin{aligned} ||Dg(x,y,z)|| &\le \frac{1}{2}(||Df(x,y,z)|| + ||Df(-x,-y,-z)||) \\ &\le \frac{1}{2}(\phi(x,y,z) + \phi(-x,-y,-z)) \\ &= \widetilde{\phi}(x,y,z) \end{aligned}$$

for all $x, y, z \in X$, where $\widetilde{\phi}(x,y,z) = \frac{1}{2}(\phi(x,y,z) + \phi(-x,-y,-z))$.

(I) If ϕ satisfies (3.7). Let $\widetilde{\Phi}_1(x,y,z) := \sum_{i=0}^{\infty} 4^{-(i+1)}\widetilde{\phi}(2^i x, 2^i y, 2^i z)$. Then we get $\widetilde{\Phi}_1(x,y,z) = \frac{1}{2}(\Phi_1(x,y,z) + \Phi_1(-x,-y,-z)) < \infty$, since $\Phi_1(x,y,z) \le \Phi_3(x,y,z) < \infty$ for all $x, y, z \in X$.

(II) If ϕ satisfies (3.1′). Let $\widetilde{\Phi}_2(x,y,z) := \sum_{i=0}^{\infty} 4^i \widetilde{\phi}(2^{-(i+1)}x, 2^{-(i+1)}y, 2^{-(i+1)}z)$. Then we get $\widetilde{\Phi}_2(x,y,z) = \frac{1}{2}(\Phi_2(x,y,z) + \Phi_2(-x,-y,-z)) < \infty$ for all $x,y,z \in X$. By Theorem 2, there exists a unique quadratic function $Q : X \to Y$ satisfying (3.10).

Let $h(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $h(-x) = -h(x)$ and

$$\begin{aligned}||Dh(x,y,z)|| &\leq \frac{1}{2}(||Df(x,y,z)|| + ||Df(-x,-y,-z)||)\\ &\leq \frac{1}{2}(\phi(x,y,z) + \phi(-x,-y,-z))\\ &= \widetilde{\phi}(x,y,z)\end{aligned}$$

for all $x,y,z \in X$, where $\widetilde{\phi}(x,y,z) = \frac{1}{2}(\phi(x,y,z) + \phi(-x,-y,-z))$.

(III) If ϕ satisfies (3.7). Let $\widetilde{\Phi}_3(x,y,z) := \sum_{i=0}^{\infty} 2^{-(i+1)} \widetilde{\phi}(2^i x, 2^i y, 2^i z)$. Then we get $\widetilde{\Phi}_3(x,y,z) = \frac{1}{2}(\Phi_3(x,y,z) + \Phi_3(-x,-y,-z)) < \infty$ for all $x,y,z \in X$.

(IV) If ϕ satisfies (3.1′). Let $\widetilde{\Phi}_4(x,y,z) := \sum_{i=0}^{\infty} 2^i \widetilde{\phi}(2^{-(i+1)}x, 2^{-(i+1)}y, 2^{-(i+1)}z)$. Then we get $\widetilde{\Phi}_4(x,y,z) = \frac{1}{2}(\Phi_4(x,y,z) + \Phi_4(-x,-y,-z)) < \infty$ for all $x,y,z \in X$, since $\Phi_4(x,y,z) \leq \Phi_2(x,y,z) < \infty$. By Theorem 3, there exists a unique additive function $A : X \to Y$ satisfying (3.11). Clearly, we have (3.9). □

Applying Theorem 2, we have the following:

COROLLARY 5. *Let $\delta, \theta \geq 0$ and let a function $\psi : [0,\infty) \to [0,\infty)$ satisfy*
(i) $\psi(st) \leq \psi(s)\psi(t)$ for all $s,t \in [0,\infty)$ and $\psi(2) < 4$, or
(ii) $\psi(st) \geq \psi(s)\psi(t)$ for all $s,t \in [0,\infty)$, $\psi(2) > 4$ and $\delta = 0$.
If an even function $f : X \to Y$ satisfies

$$||Df(x,y,z)|| \leq \delta + \theta(\psi(||x||) + \psi(||y||) + \psi(||z||))$$

for all $x,y,z \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \frac{\delta}{3} + \frac{\theta}{|4 - \psi(2)|}(2\psi(||x||) + \psi(0))$$

for all $x \in X$.

Proof. (I) The case (i). Let $\phi(x,y,z) := \delta + \theta(\psi(||x||) + \psi(||y||) + \psi(||z||))$ for all $x, y, z \in X$. Then we get

$$\begin{aligned}
\Phi_1(x,y,z) &= \sum_{i=0}^{\infty} 4^{-(i+1)}\phi(2^i x, 2^i y, 2^i z) \\
&= \sum_{i=0}^{\infty} 4^{-(i+1)}(\delta + \theta(\psi(||2^i x||) + \psi(||2^i y||) + \psi(||2^i z||))) \\
&\leq \frac{\delta}{3} + \sum_{i=0}^{\infty} 4^{-1}\left(\frac{\psi(2)}{4}\right)^i \theta(\psi(||x||) + \psi(||y||) + \psi(||z||)) \\
&= \frac{\delta}{3} + \frac{\theta}{4-\psi(2)}(\psi(||x||) + \psi(||y||) + \psi(||z||) \\
&< \infty.
\end{aligned}$$

(II) The case (ii). Let $\phi(x,y,z) := \theta(\psi(||x||) + \psi(||y||) + \psi(||z||))$ for all $x, y, z \in X$. Then we get

$$\begin{aligned}
\Phi_2(x,y,z) &= \sum_{i=0}^{\infty} 4^i \phi(2^{-(i+1)}x, 2^{-(i+1)}y, 2^{-(i+1)}z) \\
&= \sum_{i=0}^{\infty} 4^i \theta(\psi(||2^{-(i+1)}x||) + \psi(||2^{-(i+1)}y||) + \psi(||2^{-(i+1)}z||)) \\
&\leq \sum_{i=0}^{\infty} \psi(2)^{-1}\left(\frac{4}{\psi(2)}\right)^i \theta(\psi(||x||) + \psi(||y||) + \psi(||z||)) \\
&= \frac{\theta}{\psi(2)-4}(\psi(||x||) + \psi(||y||) + \psi(||z||) \\
&< \infty.
\end{aligned}$$

Applying Theorem 2, the corollary can be proved. □

Applying Theorem 3, we have the following:

COROLLARY 6. *Let $\delta, \theta \geq 0$ and let a function $\psi : [0,\infty) \to [0,\infty)$ satisfy*
(i) $\psi(st) \leq \psi(s)\psi(t)$ for all $s, t \in [0,\infty)$ and $\psi(2) < 2$, or
(ii) $\psi(st) \geq \psi(s)\psi(t)$ for all $s, t \in [0,\infty)$, $\psi(2) > 2$ and $\delta = 0$.
If an odd function $f : X \to Y$ satisfies

$$||Df(x,y,z)|| \leq \delta + \theta(\psi(||x||) + \psi(||y||) + \psi(||z||))$$

for all $x, y, z \in X$, *then there exists a unique additive function* $A : X \to Y$ *such that*

$$||f(x) - A(x)|| \leq \delta + \frac{\theta}{|2 - \psi(2)|}(2\psi(||x||) + \psi(0))$$

for all $x \in X$.

Proof. We omit the details, since the proof is similar to the proof of Corollary 5. □

Applying Corollary 5 and Corollary 6, we have the following:

COROLLARY 7. *Let* $\delta, \theta \geq 0$ *and let a function* $\psi : [0, \infty) \to [0, \infty)$ *satisfy*
(i) $\psi(st) \leq \psi(s)\psi(t)$ *for all* $s, t \in [0, \infty)$ *and* $\psi(2) < 2$, *or*
(ii) $\psi(st) \geq \psi(s)\psi(t)$ *for all* $s, t \in [0, \infty)$, $\psi(2) > 4$ *and* $\delta = 0$, *or*
(iii) $\psi(st) = \psi(s)\psi(t)$ *for all* $s, t \in [0, \infty)$, $2 < \psi(2) < 4$ *and* $\delta = 0$.
If a function $f : X \to Y$ *satisfies*

$$||Df(x, y, z)|| \leq \delta + \theta(\psi(||x||) + \psi(||y||) + \psi(||z||))$$

for all $x, y, z \in X$, *then there exist a unique additive function* $A : X \to Y$ *and a unique quadratic function* $Q : X \to Y$ *such that*

$$||f(x) - Q(x) - A(x)|| \leq \frac{4\delta}{3} + \left(\frac{1}{|2 - \psi(2)|} + \frac{1}{|4 - \psi(2)|}\right)\theta(2\psi(||x||) + \psi(0)),$$

$$||\frac{1}{2}(f(x) + f(-x)) - Q(x)|| \leq \frac{\delta}{3} + \frac{\theta}{|4 - \psi(2)|}(2\psi(||x||) + \psi(0))$$

and

$$||\frac{1}{2}(f(x) - f(-x)) - A(x)|| \leq \delta + \frac{\theta}{|2 - \psi(2)|}(2\psi(||x||) + \psi(0))$$

for all $x \in X$.

Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^p$. Applying Corollary 7, we have the following:

COROLLARY 8. *Let $\delta, \theta \geq 0$, $p \in (0, \infty) \setminus \{1, 2\}$ and $\delta = 0$ if $p > 1$. If a function $f : X \to Y$ satisfies*

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x) - A(x)|| \leq \frac{4\delta}{3} + \left(\frac{1}{|2 - 2^p|} + \frac{1}{|4 - 2^p|}\right) 2\theta ||x||^p,$$

$$||\frac{1}{2}(f(x) + f(-x)) - Q(x)|| \leq \frac{\delta}{3} + \frac{2\theta}{|4 - 2^p|}||x||^p$$

and

$$||\frac{1}{2}(f(x) - f(-x)) - A(x)|| \leq \delta + \frac{2\theta}{|2 - 2^p|}||x||^p$$

for all $x \in X$.

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