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THE ROTATION THEOREM ON ANALOGUE OF WIENER SPACE

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Abstract. Bearman's rotation theorem is not only very important in pure mathematics but also plays the key role for various research areas, related to Wiener measure. In 2002, the author and professor Im introduced the concept of analogue of Wiener measure, a kind of generalization of Wiener measure and they presented the several papers associated with it. In this article, we prove a formula on analogue of Wiener measure, similar to the formula in Bearman's rotation theorem.

1. Introduction

In 1827, Robert Brown, the English botanist, found the quickly irregularly moving small particle in the water and Einstein suggested the probabilistic approach to the way of research for it in 1905. Since then, making the rigorous mathematical model of this movement was done by Wiener in 1923, so-called the Wiener measure space [9]. The theories of Wiener measure arose the many new questions in mathematics. Many new research areas in mathematics and physics accrue from it, in particular, the theory of stochastic integral, the theory of Yeh-Wiener space, the theory of abstract Wiener space, the theory of Brownian process, Fourier-Feynman transform, on and on. In the most part of these theories, Bearman's rotation theorem is an important theorem which plays a key role in the process of deployment of them.

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On the other hand, Ryu and Im presented the theories for analogue of Wiener measure, which is a kind of generalization of concrete Wiener measure[5]. Sequentially, they introduced several papers related to analogue of Wiener space [5, 6, 7, 8].

In this note, we will find the necessary and sufficient conditions, holding the formula on analogue of Wiener measure, similar to the formula in Bearman's rotation theorem under some conditions.

2. Preliminaries

In this section, we give some definitions, notations and base facts which needed to understood for the next sections.
Wiener suggested a measure space $(C_0[0,t],\omega)$ where $C_0[0,t]$ is the space of all continuous functions on a closed interval $[0,t]$ which vanish at origin, the so called Wiener space in 1923 [9]. The authors introduced a new measure space, similar to Wiener measure in [5] as follows; Let $C[0,t]$ be the space of all continuous functions on a closed interval $[0,t]$. Let t be a positive real number and let n be a non-negative integer. For $\vec{t}=(t_0,t_1,\cdots,t_n)$ with $0=t_0<t_1<\cdots<t_n\leq t$, let $J_{\vec{t}}:C[0,t]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) \;=\; (x(t_0),x(t_1),\cdots,x(t_n))\ .$$

For B_j $(j=0,1,2,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$ where $\mathcal{B}(\mathbb{R})$ is the sets of all Borel subsets of the real number system $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a non-negative finite Borel measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we let

$$\text{(2.1)}\quad m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j)) \;=\; \int_{B_0}\Big[\int_{\prod_{j=1}^{n}B_j}W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\; d\prod_{j=1}^{n}m_L(u_1,\cdots,u_n)\Big]\;d\varphi(u_0)$$

where

$$W(n+1;\vec{t};u_0,u_1,\cdots,u_n)$$
$$= \Big(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\Big)exp\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}\ .$$

By [3], $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique measure ω_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure ω_φ is called analogue of Wiener measure associated with the measure φ. If φ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, then ω_φ is the classical Wiener measure.

By the change of variable formula, we can easily prove the following theorem.

Theorem 2.1 ([5]). *(Integration formula for analogue of Wiener measure) If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then the following equality holds.*

$$\int_{C[0,t]} f(x(t_0),x(t_1),\cdots,x(t_n))\ d\omega_\varphi(x)$$
$$\overset{*}{=} \int_{\mathbb{R}}\Big[\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,u_1,\cdots,u_n)$$
$$d\prod_{j=1}^{n} m_L((u_1,u_2,\cdots,u_n))\Big]d\varphi(u_0)$$

where $\overset{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Remark 2.2. *(Wiener integration formula for the concrete Wiener measure) If $f : \mathbb{R}^n \to \mathbb{C}$ is a Borel measurable function, then the following equality holds.*

$$\int_{C_0[0,t]} f(x(t_1), x(t_2), \cdots, x(t_n))\ d\omega(x)$$
$$\stackrel{*}{=} \frac{1}{\prod_{j=1}^n \sqrt{2\pi(t_j - t_{j-1})}} \int_{\mathbb{R}^n} f(u_1, \cdots, u_n) exp\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\}$$
$$d\prod_{j=1}^n m_L(u_1, u_2, \cdots, u_n)$$

where $u_0 = 0$, and $\stackrel{*}{=}$ means as above.

Remark 2.3. *We can found the following theorems for the concrete Wiener measure space in* [1].

Theorem A *(Bearman's rotation theorem) $F(x, y)$ is measurable on $(C_0[0,t] \times C_0[0,t], \mathcal{B}(C_0[0,t]) \times \mathcal{B}(C_0[0,t]))$ if and only if $F(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ is measurable on $(C_0[0,t] \times C_0[0,t], \mathcal{B}(C_0[0,t]) \times \mathcal{B}(C_0[0,t]))$ for all real number θ. Moreover,*

$$\int_{C_0[0,t]\times C_0[0,t]} F(x, y)\ d\omega \times \omega(x, y)$$
$$= \int_{C_0[0,t]\times C_0[0,t]} F(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\ d\omega \times \omega(x, y).$$

Corollary B *For given θ, F is measurable on $C_0[0,t]$ if and only if $F(x\sin\theta + y\cos\theta)$ is measurable on $C_0[0,t] \times C_0[0,t]$. And,*

$$\int_{C_0[0,t]} F(y)\ d\omega(y)$$
$$= \int_{C_0[0,t]\times C_0[0,t]} F(x\sin\theta + y\cos\theta)\ d\omega \times \omega(x, y).$$

Corollary C *For real numbers p, q, $F(\sqrt{p^2 + q^2}z)$ is a measurable function of z on $C_0[0,t]$ if and only if $F(p\ x + q\ y)$ is a measurable* *function of* (x,y) *on* $C_0[0,t]\times C_0[0,t]$. *And*

$$\int_{C_0[0,t]} F(\sqrt{p^2+q^2}z)\ d\omega(y) = \int_{C_0[0,t]\times C_0[0,t]} F(p\ x+q\ y)\ d\omega\times\omega(x,y).$$

One of our goals in this paper is to find the necessary and sufficient conditions, satisfied with the formula for analogue of Wiener integral, similar to the formula in above Theorem A, Corollary B and Corollary C.

3. The rotation theorem on analogue of Wiener space

As far as we know Bearman's 1952 paper [1] is the first article, treating the rotation theorem for Wiener measure. His theorem is not only very important in mathematics itself but also plays the key role for various research parts related to Wiener measure. In this section, we investigate the rotation theorem on analogue of Wiener measure. In particular, we will find the necessary and sufficient conditions satisfied with the rotation theorem, essentially similar formula to Bearman's rotation theorem, held on analogue of Wiener measure.

For a Borel measure φ on $\mathbb{R}$, we can find the decomposition of φ as $\varphi=\varphi^d+\varphi^{a,c}+\varphi^{e,s}$ where φ^d is a discrete measure, that is, φ^d has a form $\varphi^d=\sum_{k=1}^{\infty}\alpha_k\delta_{p_k}$, $\varphi^{a,c}$ is an absolutely continuous with respect to the Lebesgue measure m_L on $\mathbb{R}$, that is, there is a function f in $L_1(\mathbb{R})$ such that $\varphi^{a,c}(E)=\int_E f(x)dm_L(x)$ and $\varphi^{e,s}$ is an essentially singular measure for φ [2]. Here, we know that $\varphi^{e,s}$ means the state with no physical interpretation, that is, certain measure has $\varphi^{e,s}=0$ in physics, so we suppose $\varphi^{e,s}=0$ in this article [4]. Throughout this section, without special comments, we assume that φ is a positive Borel measure on $\mathbb{R}$ having a non-zero Radon-Nikodym derivative with respect to $\frac{d\varphi}{dm_L}=f$.

Furthermore, we assume that for any Borel subset B of $C[0,t]\times C[0,t]$ and for any real number θ

$$(3.2)\ \int_{C[0,t]\times C[0,t]} \chi_B(x,y)\ d\omega \times \omega(x,y) = \int_{C[0,t]\times C[0,t]} \chi_B(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) d\omega_\varphi \times \omega_\varphi(x,y).$$

Letting a function $J_0 : C[0,t] \to \mathbb{R}$ with $J_0(x) = x_0$, for any Borel subsets E and F of $\mathbb{R}$, $J_0^{-1}(E) \times J_0^{-1}(F)$ is a Borel subset of $C[0,t] \times C[0,t]$. Then by the integration formula for analogue of Wiener measure,

$$\begin{aligned}(3.3)\quad & \omega_\varphi \times \omega_\varphi(J_0^{-1}(E) \times J_0^{-1}(F)) \\ &= \int_{E\times F} f(u)f(v)\ dm_L \times m_L(u,v) \\ &= \int_{E\times F} f(u\cos\theta - v\sin\theta)f(u\sin\theta + v\cos\theta)\ dm_L \times m_L(u,v) \\ &= \omega_\varphi \times \omega_\varphi(R_\theta(J_0^{-1}(E) \times J_0^{-1}(F)))\end{aligned}$$

for all Borel subsets E, F of $\mathbb{R}$ and for all real number θ. Here $R_\theta(u,v) = (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$. Furthermore, by the Tonelli's theorem, applying (3.3), we have for m_L-*a.e.* u and for m_L-*a.e.* v,

(3.4)

$$f(u)f(v) = f(u\cos\theta - v\sin\theta)f(u\sin\theta + v\cos\theta) \quad m_L \times m_L\text{-a.e.} \quad (u,v)$$

holds for all real number θ. Hence for $\theta = \frac{\pi}{4}$, we can choose u, v satisfy $f(u)^2 = f(0)f(\sqrt{2}u)$ and $f(\frac{v}{\sqrt{2}})^2 = f(0)f(v)$. This is the logicality of choosing u, v in the proof of the following lemmas.

Lemma 3.1. *Assume that f holds (3.4). Then*

(1) $f(0) > 0$,

(2) if $f(t) = 0$ for some $t > 0$ then $f(u) = 0$ for $u > t$,

(3) if $f(t) = 0$ for some $t > 0$ then $f(u) = 0$ for $u > 0$ and

(4) if $f(t) = 0$ for some $t < 0$ then $f(u) = 0$ for $u < 0$.

Proof. (1) Putting $u = v$ and $\theta = \frac{\pi}{4}$ in (3.4), $f(u)^2 = f(0)f(\sqrt{2}u)$. So, if $f(0) = 0$, $f(u) = 0$ for real numbers u, contradicted by the assumption.
(2) Letting $s = \sqrt{u^2 - t^2}$, $v = 0$, $\cos\theta = \frac{s}{u}$ and $\sin\theta = \frac{t}{u}$ in (3.4), $f(0)f(u) = f(s)f(t) = 0$, so $f(u) = 0$ by (1).
(3) Letting $u = v = \frac{t}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$ in (3.4), $f(\frac{t}{\sqrt{2}})^2 = f(0)f(t)$, so $f(\frac{t}{\sqrt{2}}) = 0$. For $v > 0$, there is a natural number n_0 such that $\frac{t}{(\sqrt{2})^{n_0}} < v$ because $\lim_{n\to\infty} \frac{t}{(\sqrt{2})^n} = 0$. Then $f(\frac{t}{(\sqrt{2})^{n_0}}) = 0$, so by (2), $f(v) = 0$.
(4) By the essentially same methods in the proofs of (2) and (3), directly, we can show that (4) is true. □

Lemma 3.2. *Under the assumption* (3.4), $f(u) = f(-u)$ *for all real number* u.

Proof. Letting $u = v$ and $\theta = \frac{\pi}{2}$ in (3.4), $f(u)^2 = f(u)f(-u)$, so if $f(u) \neq 0$, then $f(u) = f(-u)$. Suppose $f(u) = 0$ and $f(-u) \neq 0$. From $f(-u)^2 = f(-u)f(u)$, $f(-u) = f(u)$, this is a contradiction. □

From the above lemmas, directly, we have following theorem.

Theorem 3.3. *Under the assumption* (3.4), $f(u) > 0$ *and* $f(u) = f(-u)$ *for all real number* u.

Lemma 3.4. *Under the assumption* (3.4), *letting* $g(u) = \ln\frac{f(\sqrt{u})}{f(0)}$ *for* $u \geq 0$ *and* $g(u) = -\ln\frac{f(\sqrt{-u})}{f(0)}$ *for* $u < 0$, g *is additive.*

Proof. For all real numbers u and v, since $f(u) = f(-u)$, we have $g(u) = -g(-u)$. Here we consider all cases for u and v as four cases.
(i) When $u > 0$ and $v > 0$,

$$g(u) + g(v) = \ln\frac{f(\sqrt{u})f(\sqrt{v})}{f(0)^2} = \ln\frac{f(\sqrt{u+v})}{f(0)} = g(u+v).$$

(ii) When $u < 0$ and $v < 0$,

$$g(u) + g(v) = -\ln\frac{f(\sqrt{-u})f(\sqrt{-v})}{f(0)^2} = -\ln\frac{f(\sqrt{-(u+v)})}{f(0)} = g(u+v).$$

(iii) Suppose $u > 0$, $v < 0$ and $u \geq |v|$.
Let $\sin\theta = -\frac{\sqrt{u+v}}{\sqrt{u}}$, $\cos\theta = -\frac{\sqrt{-v}}{\sqrt{u}}$, $\alpha = 0$ and $\beta = \sqrt{u}$. Then from $f(\alpha)f(\beta) = f(\alpha\cos\theta - \beta\sin\theta)f(\alpha\sin\theta + \beta\cos\theta)$, we have $f(0)f(\sqrt{u}) = f(\sqrt{u+v})f(-\sqrt{-v})$. So,

$$g(u) + g(v) = \ln\frac{f(\sqrt{u})}{f(-\sqrt{-v})} = \ln\frac{f(\sqrt{u+v})}{f(0)} = g(u+v)$$

(iv) Suppose $u > 0$, $v < 0$ and $u < |v|$.
Let $\bar{u} = -v$ and $\bar{v} = -u$, then, by (iii), $g(\bar{u}) + g(\bar{v}) = g(\bar{u} + \bar{v})$. So, $g(u) + g(v) = -(g(\bar{u}) + g(\bar{v})) = g(u+v)$. □

For a long time, many mathematicians studied the properties of the additive function. In 1825, Cauchy proved that if F is continuous from $\mathbb{R}$ to itself with $F(x+y) = F(x) + F(y)$ then $F(t) = tF(1)$ for all t in $\mathbb{R}$. In 1875, Darboux proved that if F is bounded above (or bounded below) on some interval with $F(x+y) = F(x)+F(y)$ then F is continuous. In 1905, Hamel proved that if F is a function from $\mathbb{R}$ to itself with $F(x+y) = F(x) + F(y)$ then either F is continuous or $G_F = \{(x, F(x)) | x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$. That is, he proved that there is an additive function which is not continuous. In 1913, Frechet proved that every additive Lebesgue measurable function is continuous. Therefore, by Cauchy's result and Frechet's result, if g is additive measurable, we can write $g(u) = ug(1)$ for any real number u. Hence by the definition of the function g in above, $f(u) = f(0)e^{au^2}$ for all real number u. Since f is in $L_1(\mathbb{R})$, a is negative. Hence, we have the following two theorems.

Theorem 3.5. *If φ is a positive measure with $\frac{d\varphi}{dm_L} = f$ and (3.2) holds then $f(x)$ is of the form Ae^{-ax^2} where A and a are positive constants.*

Corollary 3.6. *Let f be in $L_1(\mathbb{R})$ and let $\varphi(E) = \int_E f(x)dm_L(x)$ where $f > 0$ and E is a Borel subset of $\mathbb{R}$. If for any integrable function*

F,

$$(3.5)\quad \int_{C[0,t]\times C[0,t]} F(x,y)\ d\omega_\varphi \times \omega_\varphi(x,y)$$
$$= \int_{C[0,t]\times C[0,t]} F(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) d\omega_\varphi \times \omega_\varphi(x,y),$$

for all real number θ. *Then the function* $f(x)$ *has the form* Ae^{-ax^2} *where* A *and* a *are positive constants.*

Theorem 3.7. *For a Borel subset* E *of* $\mathbb{R}$ *and for two positive real numbers* A *and* a, *let* $\varphi(E) = \int_E Ae^{-ax^2} dm_L(x)$, *then for any integrable function* F, (3.5) *is satisfied.*

Proof. For a partition $0 = t_0 < t_1 < \cdots < t_n = t$ and for real numbers $\alpha_j, \beta_j, \gamma_j$ and η_j $(j = 0, 1, \cdots, n)$, let $I = \{x \in C[0,t] |\ \alpha_j < x(t_j) \le \beta_j\}$, $J = \{x \in C[0,t] |\ \gamma_j < x(t_j) \le \eta_j\}$. Then

$$\begin{aligned}
&\omega_\varphi \times \omega_\varphi(I \times J)\\
&= \int_{C[0,t]\times C[0,t]} \chi_{I\times J}(x,y)\ d\omega_\varphi \times \omega_\varphi(x,y)\\
&= \int_{C[0,t]} \int_{C[0,t]} \chi_I(x)\chi_J(y)\ d\omega_\varphi(x) d\omega_\varphi(y)\\
&= \frac{1}{\prod_{j=1}^n \sqrt{2\pi(t_j - t_{j-1})}} \int_{\alpha_0}^{\beta_0} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} e^{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}}\\
&\quad dm_L(u_n)\cdots dm_L(u_1) d\varphi(u_0)\ \frac{1}{\prod_{j=1}^n \sqrt{2\pi(t_j - t_{j-1})}}\\
&\quad \int_{\gamma_0}^{\eta_0} \int_{\gamma_1}^{\eta_1} \cdots \int_{\gamma_n}^{\eta_n} e^{-\frac{1}{2}\sum_{j=1}^n \frac{(v_j - v_{j-1})^2}{t_j - t_{j-1}}}\ dm_L(v_n)\cdots dm_L(v_1) d\varphi(v_0).
\end{aligned}$$

We define $\widetilde{R_\theta} : C[0,t] \times C[0,t] \to C[0,t] \times C[0,t]$ given by $\widetilde{R_\theta}(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. Then

$$\begin{aligned}
&\omega_\varphi \times \omega_\varphi(\widetilde{R_\theta^{-1}}(I \times J)) \\
&= \int_{C[0,t]\times C[0,t]} \chi_{\widetilde{R_\theta^{-1}}(I\times J)}(x,y)\; d\omega_\varphi \times \omega_\varphi(x,y) \\
&= \int_{C[0,t]\times C[0,t]} \prod_{j=0}^{n} [\chi_{(\alpha_j,\beta_j]}(x(t_j)\cos\theta - y(t_j)\sin\theta) \\
&\qquad \chi_{(\gamma_j,\eta_j]}(x(t_j)\sin\theta + y(t_j)\cos\theta)]\; d\omega_\varphi \times \omega_\varphi(x,y) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \prod_{j=0}^{n} [\chi_{(\alpha_j,\beta_j]}(U_j\cos\theta - V_j\sin\theta) \\
&\quad \chi_{(\gamma_j,\eta_j]}(U_j\sin\theta + V_j\cos\theta)] \frac{1}{\prod_{j=1}^{n}\sqrt{2\pi(t_j - t_{j-1})}} e^{-\frac{1}{2}\sum_{j=1}^{n}\frac{(U_j-U_{j-1})^2}{t_j-t_{j-1}}} \\
&\quad dm_L(U_n)\cdots dm_L(U_1)d\varphi(U_0) \frac{1}{\prod_{j=1}^{n}\sqrt{2\pi(t_j - t_{j-1})}} e^{-\frac{1}{2}\sum_{j=1}^{n}\frac{(V_j-V_{j-1})^2}{t_j-t_{j-1}}} \\
&\quad dm_L(V_n)\cdots dm_L(V_1)d\varphi(V_0).
\end{aligned}$$

Thus, by the change of variable, let $u_j = U_j\cos\theta - V_j\sin\theta$, $v_j = U_j\sin\theta + V_j\cos\theta$, $\frac{d\varphi}{dm_L}(U_0) = Ae^{-aU_0^2}$ and $\frac{d\varphi}{dm_L}(V_0) = Ae^{-aV_0^2}$, then $\omega_\varphi \times \omega_\varphi(I \times J) = \omega_\varphi \times \omega_\varphi(\widetilde{R_\theta^{-1}}(I \times J))$, that is,

$$\begin{aligned}
&\int_{C[0,t]\times C[0,t]} \chi_{I\times J}(x,y)\; d\omega_\varphi \times \omega_\varphi(x,y) \\
&= \int_{C[0,t]\times C[0,t]} \chi_{\widetilde{R_\theta^{-1}}(I\times J)}(x,y)\; d\omega_\varphi \times \omega_\varphi(x,y) \\
&= \int_{C[0,t]\times C[0,t]} \chi_{I\times J}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\; d\omega_\varphi \times \omega_\varphi(x,y).
\end{aligned}$$

By the general method as in the measure theory, we obtain the equality of (3.5) as desired. □

Corollary 3.8. *For given θ, F is integrable on $C[0,t]$ if and only if $F(x\sin\theta + y\cos\theta)$ is measurable on $C[0,t] \times C[0,t]$. Moreover, if φ is* *given Theorem 3.7, then*

$$\int_{C[0,t]} F(y)\, d\omega_\varphi(y) = \int_{C[0,t]\times C[0,t]} F(x\sin\theta + y\cos\theta)\, d\omega_\varphi \times \omega_\varphi(x,y).$$

Corollary 3.9. *For a Borel subset E of $\mathbb{R}$ and positive real numbers A and a, let $\varphi(E) = \int_E Ae^{-ax^2} dm_L(x)$. If $f(px+qy)$ is integrable on $(C[0,t]\times C[0,t], \omega_\varphi \times \omega_\varphi)$ for real numbers p, q with $p^2+q^2 \neq 0$, then*

$$\int_{C[0,t]}\int_{C[0,t]} f(px+qy)\, d\omega_\varphi(x) d\omega_\varphi(y) = \int_{C[0,t]} f(\sqrt{p^2+q^2}z) d\omega_\varphi(z). \tag{3.6}$$

Proof. In Theorem 3.7, let $F(x,y) = f(y)$ and $g(z) = \sqrt{p^2+q^2}z$, and we choose θ such that $sin\theta = \frac{p}{\sqrt{p^2+q^2}}$ and $cos\theta = \frac{q}{\sqrt{p^2+q^2}}$. Then for $f\circ g$, (3.6) holds. □

Remark 3.10. *Theorem 3.7, 3.8 and 3.9 in above are similar forms to Theorem A, B, C in Remark 2.3, respectively.*

Remark 3.11. *Theorem 3.7, 3.8 and 3.9 in above hold for either $\varphi = \delta_0$ or $\varphi(E) = \int_E Ae^{-\alpha x^2} dm_L(x)$.*

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