

ON THE SINGULAR KOBAYASHI PSEUDOMETRICS

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Abstract. In this paper, we'll introduce infinitesimal Finsler metrics induced by the higher order Kobayashi pseudometric and then prove some properties related to these as compared with the usual Kobayashi metric.

1. Introduction

Kobayashi([7]) introduced a pseudodistance on complex manifold to study holomorphic maps complex manifolds. Royden later published the infinitesimal form, what we call the Kobayashi metric, in [9] as a modification of the Carathéodory metric which has a number of advantages. This infinitesimal form has been developed by many mathematicians. After that, the higher order Kobayashi pseudometric is introduced in [11] by Yu as the generalization of the Kobayashi metric. Nikolov also investigated the higher order Kobayashi pseudometric in [8].

We first introduce some notations. By \mathbb{N} and \mathbb{C} we denote the set of natural numbers and the set of complex numbers, respectively. Also, by F_{Ω}^c and K_{Ω} we denote the Carathéodory metric and the usual Kobayashi metric for some domain Ω , respectively. As usual, by a domain we mean the open and connected set. We will also use the notations \langle, \rangle and $\|\cdot\|$ for the usual inner product and norm on complex Euclidean spaces, respectively. Moreover, by $\mathcal{PSH}(\mathbb{C}^n)$ and $\mathcal{C}(D, \mathbb{R})$ we mean the set of

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all plurisubharmonic functions on \mathbb{C}^n and the set of all continuous real-valued functions on D , respectively.

2. The singular Kobayashi pseudometrics

Let $D \subset \mathbb{C}^n$ be a domain. Then we denote by $\mathcal{O}(\Delta, D)$ the space of all holomorphic mappings from the unit disk $\Delta \subset \mathbb{C}$ into D . For $t \in D$, we denote by $\mathcal{O}_t(\Delta, D)$ the set $\{\varphi \in \mathcal{O}(\Delta, D) \mid \varphi(0) = t\}$.

For each $m \in \mathbb{N}$ and $(z, X) \in D \times \mathbb{C}^n$, the m -th order Kobayashi pseudometric is defined by

$$(2.1) \quad K_D^m(z, X) := \inf\{|\alpha|^{-1} \mid \exists \psi \in \mathcal{O}_z(\Delta, D) \\ \text{s.t. } \nu(\psi) \geq m, \psi^{(m)}(0) = m! \alpha X\}$$

where $\nu(\psi)$ stands for the order of vanishing of $\psi - \psi(0)$ at 0. Clearly $K_D^1(z, X)$ is the usual Kobayashi metric.

Proposition 2.1. *Let D be a domain in \mathbb{C}^n . Then for each $a \in D$, there are an $r > 0$ and a positive constant $C_a > 0$ such that*

$$(2.2) \quad K_D^m(z, X) \leq C_a \|X\|$$

for all $(z, X) \in B(a, r) \times \mathbb{C}^n$ and for all $m \in \mathbb{N}$. Here $B(a, r) := \{z \in \mathbb{C}^n \mid \|z - a\| < r\}$.

Proof. Let $a \in D$ be any point. Then there is a real number $r > 0$ such that $B(a, r) \subset B(a, 2r) \subset D$. Now let $m \in \mathbb{N}$ be any natural number. Applying the decreasing property of K_D^m ([5]), we first have the inequality

$$(2.3) \quad K_D^m(z, X) \leq K_{B(a, 2r)}^m(z, X)$$

for all $(z, X) \in B(a, 2r) \times \mathbb{C}^n$. We also have the following explicit expression for balls ([5]);

$$(2.4) \quad K_{B(a, 2r)}^m(z, X) = \left[\frac{\|X\|^2}{4r^2 - \|z - a\|^2} + \frac{|\langle z - a, X \rangle|^2}{(4r^2 - \|z - a\|^2)^2} \right]^{\frac{1}{2}}$$

for all $(z, X) \in B(a, 2r) \times \mathbb{C}^n$. Note that the expression on the right hand of the equation (2.4) is independent of $m \in \mathbb{N}$. On the other hand, applying Cauchy-Schwarz inequality on the right hand side of (2.4), we get

$$K_{B(a,2r)}^m(z, X) \leq \left[\frac{2r}{4r^2 - \|z - a\|^2} \right] \|X\|$$

for all $(z, X) \in B(a, 2r) \times \mathbb{C}^n$. Now by taking

$$C_a := \sup_{z \in B(a,r)} \frac{2r}{4r^2 - \|z - a\|^2},$$

we have

$$(2.5) \quad K_{B(a,2r)}^m(z, X) \leq C_a \|X\|$$

for all $(z, X) \in B(a, r) \times \mathbb{C}^n$. Combining (2.3) and (2.5), we have the required result. \square

The following readily follows from Proposition 2.1.

Proposition 2.2. *Let D be a domain in \mathbb{C}^n . Then the sequence $(K_D^m)_{m \in \mathbb{N}}$ is locally bounded from above on $D \times \mathbb{C}^n$.*

For a domain $D \subset \mathbb{C}^n$, we define three maps K_D^∞ , \hat{K}_D and \tilde{K}_D from $D \times \mathbb{C}^n$ into \mathbb{R} by

$$\begin{aligned} K_D^\infty(z, X) &:= \inf_{m \in \mathbb{N}} K_D^m(z, X), \\ \hat{K}_D(z, X) &:= \left(\liminf_{m \rightarrow \infty} K_D^m(z, X) \right)^*, \\ \tilde{K}_D(z, X) &:= \left(\limsup_{m \rightarrow \infty} K_D^m(z, X) \right)^*, \end{aligned}$$

where $*$ means the upper semicontinuous regularization.¹

¹If $v : D \rightarrow [-\infty, +\infty)$ is locally bounded from above, then we define a map v^* for each $z \in D$ by

$$v^*(z) := \limsup_{z' \rightarrow z} v(z') = \inf \{ \phi(z) : \phi \in C(D, \mathbb{R}), v \leq \phi \}$$

The followings are easily proved from properties of K_D^m and the definition of the upper semicontinuous regularization. We refer to [5] and [11] for the properties of K_D^m .

Proposition 2.3. *Let $D \subset \mathbb{C}^n$ be a domain. Then the followings hold;*

- (1) K_D^∞, \hat{K}_D and \tilde{K}_D are upper semicontinuous on $D \times \mathbb{C}^n$.
- (2) K_D^∞, \hat{K}_D and \tilde{K}_D have the length decreasing property. In particular, K_D^∞, \hat{K}_D and \tilde{K}_D are biholomorphically invariant.
- (3) $K_\Delta^\infty = \hat{K}_\Delta = \tilde{K}_\Delta = K_\Delta$. Here K_Δ means the Kobayashi metric for the unit disc Δ .
- (4) $F_D^c \leq K_D^\infty \leq \hat{K}_D \leq \tilde{K}_D \leq K_D$ on $D \times \mathbb{C}^n$.
- (5) K_D^∞, \hat{K}_D and \tilde{K}_D are absolutely homogeneous on $D \times \mathbb{C}^n$. In other words, for $K_D^\infty, K_D^\infty(z, \mu X) = |\mu|K_D^\infty(z, X)$ holds for all $(z, X) \in D \times \mathbb{C}^n$ and for all $\mu \in \mathbb{C}$.

Proof. It follows from their definitions and properties([5][11]) of K_D^m that (1), (3) and (4) hold. Therefore, we simply sketch the proofs of (2) and (5).

(2). Let $f \in \mathcal{O}(D, \Omega)$ and let $l \geq m$. Then we have, for each $(z, X) \in D \times \mathbb{C}^n$,

$$\inf_{k \geq m} K_\Omega^k(f(z), f'(z)X) \leq K_D^l(z, X)$$

and therefore

$$\inf_{k \geq m} K_\Omega^k(f(z), f'(z)X) \leq \inf_{l \geq m} K_D^l(z, X).$$

Thus we reach at the following conclusion;

$$\begin{aligned} \hat{K}_\Omega(f(z), f'(z)X) &= \left(\liminf_{m \rightarrow \infty} K_\Omega^m(f(z), f'(z)X) \right)^* \\ &\leq \left(\liminf_{m \rightarrow \infty} K_D^m(z, X) \right)^* = \hat{K}_D(z, X) \end{aligned}$$

for all $(z, X) \in D \times \mathbb{C}^n$. The proofs for K_D^∞ and \tilde{K}_D are all similar to that of \hat{K}_D .

(5). Using the definition of the upper semicontinuous regularization and the properties([5][11]) of K_D^m , we have the following equalities ;

$$\begin{aligned} \hat{K}_D(z, \lambda X) &= \left(\liminf_{m \rightarrow \infty} K_D^m(z, \lambda X) \right)^* \\ &= \left(|\lambda| \liminf_{m \rightarrow \infty} K_D^m(z, X) \right)^* = |\lambda| \hat{K}_D(z, X) \end{aligned}$$

for all $(z, X) \in D \times \mathbb{C}^n$ and for all $\lambda \in \mathbb{C}$. The proofs for K_D^∞ and \tilde{K}_D are all similar to that of \hat{K}_D . \square

According to Proposition 2.3, the maps K_D^∞, \hat{K}_D and \tilde{K}_D are infinitesimal Finsler pseudometrics. In fact, they are infinitesimal Finsler metrics (refer to [2] for the infinitesimal Finsler metric) as we shall recognize later (Theorem 2.7). We call K_D^∞, \hat{K}_D and \tilde{K}_D *the singular Kobayashi pseudometric, the lower singular Kobayashi pseudometric and the upper singular Kobayashi pseudometric* on $D \times \mathbb{C}^n$, respectively.

A set $A \subset \mathbb{C}^k$ is called a *balanced set* if $\lambda z \in A$ for arbitrary $\lambda \in \bar{\Delta}$ and $z \in A$.

Theorem 2.4. *Let $G \subset \mathbb{C}^n$ be a balanced pseudoconvex domain given by*

$$G := \{z \in \mathbb{C}^n \mid h(z) < 1\}$$

with Minkowski function h , i.e., $h : \mathbb{C}^n \rightarrow [0, \infty)$ is a plurisubharmonic function² for which $h(\lambda z) = |\lambda|h(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then we have

$$h(X) = K_G^\infty(0, X) = \hat{K}_G(0, X) = \tilde{K}_G(0, X) = K_G(0, X)$$

for all $X \in \mathbb{C}^n$.

Proof. By Proposition 2.3, it is sufficient to show that $h(X) \leq K_G^\infty(0, X)$ and $K_G(0, X) \leq h(X)$ hold. To show that $K_G(0, X) \leq h(X)$,

²Refer to [3] and [4] for plurisubharmonic functions and more their informations

let us first assume that $h(X) \neq 0$. If we define a map $\phi : \Delta \rightarrow G$ by $\phi(\lambda) = \lambda X/h(X)$, then we have

$$\phi \in \mathcal{O}_0(\Delta, G), \nu(\phi) \geq 1 \text{ and } \phi'(0) = \frac{X}{h(X)}.$$

Hence $K_G(0, X) \leq h(X)$.

Now let us consider the case $h(X) = 0$. For any $t > 1$, if we define a map $\phi_t : \Delta \rightarrow G$ by $\phi_t(\lambda) = t\lambda X$, then we know that

$$\phi_t \in \mathcal{O}_0(\Delta, G), \nu(\phi_t) \geq 1 \text{ and } \phi_t'(0) = tX.$$

It follows from this fact that

$$K_G(0, X) \leq \frac{1}{t} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

That is to say, we have $K_G(0, X) = 0 = h(X)$. Thus in either case we have the inequality

$$(2.6) \quad K_G(0, X) \leq h(X).$$

Conversely, let $\phi \in \mathcal{O}_0(\Delta, G)$ for which

$$\nu(\phi) \geq m \text{ and } \phi^{(m)}(0)\alpha = m!X \ (\alpha > 0).$$

If we define a map $\tilde{\phi} : \Delta \rightarrow \mathbb{C}^n$ by

$$\tilde{\phi}(\lambda) := \begin{cases} \frac{\phi(\lambda)}{\lambda^m}, & \text{if } \lambda \neq 0 \\ \frac{\phi^{(m)}(0)}{m!}, & \text{if } \lambda = 0 \end{cases}$$

then we have $\tilde{\phi} \in \mathcal{O}(\Delta, \mathbb{C}^n)$ and $\phi(\lambda) = \lambda^m \tilde{\phi}(\lambda)$ for all $\lambda \in \Delta$. On the other hand, since $1 > h(\phi(\lambda)) = |\lambda|^m h(\tilde{\phi}(\lambda))$ for all $\lambda \in \Delta$ and $h \circ \tilde{\phi}$ is a subharmonic function on Δ , it follows from the maximum principle for subharmonic function that $h \circ \tilde{\phi} \leq 1$ on Δ . Hence

$$m! \frac{1}{\alpha} h(X) = h(\phi^{(m)}(0)) = h(m! \tilde{\phi}(0)) = m!(h \circ \tilde{\phi})(0) \leq m!$$

and so $h(X) \leq \alpha$. By the assumption for ϕ and α , we obtain $h(X) \leq K_G^m(0, X)$ for all $m \in \mathbb{N}$. It follows from this fact that

$$(2.7) \quad h(X) \leq \inf_m K_G^m(0, X) = K_G^\infty(0, X).$$

Thus by (2.6), (2.7) and Proposition 2.3, we reach at our conclusion. \square

Let $B \subset \mathbb{C}^n$ be the open unit ball with center 0 in \mathbb{C}^n . Then the Minkowski function for B is the same as the usual Euclidean norm. Recall that B is a balanced pseudoconvex domain and that the group, $\text{Aut}(B)$, of all automorphisms of B acts transitively. Furthermore, note that \hat{K}_B and K_B^∞ are biholomorphically invariant. Thus we have the following ;

Corollary 2.5. *Let B be the open unit ball in \mathbb{C}^n with center 0. Then we have*

$$K_B(z, X) = \left[\frac{\|X\|^2}{1 - \|z\|^2} + \frac{|\langle z, X \rangle|^2}{(1 - \|z\|^2)^2} \right]^{\frac{1}{2}} = K_B^\infty(z, X)$$

for all $(z, X) \in B \times \mathbb{C}^n$.

By applying Theorem 2.4, we also have the corresponding formula for the open unit polydisc.

Corollary 2.6. *Let Δ^n be the open unit polydisc in \mathbb{C}^n . Then we have*

$$K_{\Delta^n}(z, X) = \max \left\{ \frac{|X_1|}{1 - |z_1|^2}, \frac{|X_2|}{1 - |z_2|^2}, \dots, \frac{|X_n|}{1 - |z_n|^2} \right\} = K_{\Delta^n}^\infty(z, X)$$

for all $z = (z_1, \dots, z_n) \in \Delta^n$ and $X = (X_1, \dots, X_n) \in \mathbb{C}^n$.

In fact, we know from the following Theorem that K_D^∞, \hat{K}_D and \tilde{K}_D are infinitesimal Finsler metrics whenever D is a bounded domain.

Theorem 2.7. *Let D be a bounded domain in \mathbb{C}^n . Then there is a positive constant $C > 0$ such that*

$$\hat{K}_D(z, X) \geq K_D^\infty(z, X) \geq C\|X\|$$

for all $(z, X) \in D \times \mathbb{C}^n$.

Proof. The first inequality is obtained from Proposition 2.3. Hence we have to show the second inequality.

Since D is a bounded domain in \mathbb{C}^n , there is a positive real number $R > 0$ such that $\overline{D} \subset B_R =: B(0, R)$. Making use of the fact that K^∞ is invariant under biholomorphism, we have

$$K_{B_R}^\infty(z, X) = \left[\frac{\|X\|^2}{R^2 - \|z\|^2} + \frac{|\langle z, X \rangle|^2}{(R^2 - \|z\|^2)^2} \right]^{\frac{1}{2}}$$

for all $(z, X) \in B_R \times \mathbb{C}^n$. On the other hand, we have, by Proposition 2.3,

$$K_D^\infty(z, X) \geq K_{B_R}^\infty(z, X)$$

for all $(z, X) \in D \times \mathbb{C}^n$. Now taking

$$C = \inf_{z \in D} \frac{1}{\sqrt{R^2 - \|z\|^2}},$$

we have the required inequality $K_D^\infty(z, X) \geq C\|X\|$ for all $(z, X) \in D \times \mathbb{C}^n$. \square

Theorem 2.8. *Let D be a domain in \mathbb{C}^n and let $\Phi : D \times \mathbb{C}^n \rightarrow \mathbb{R}$ be a nonnegative, absolutely homogeneous and upper semicontinuous function. If F is a compact subset of D , then there is a positive constant $C > 0$ such that*

$$\Phi(z, X) \leq C\|X\|$$

for all $(z, X) \in F \times \mathbb{C}^n$.

Proof. If Φ is identically zero on $F \times \mathbb{C}^n$, there is nothing to prove. Hence we may assume that Φ is not identically zero on $F \times \mathbb{C}^n$.

Let B be the open unit ball with center 0 in \mathbb{C}^n . Then the set $F \times \partial B$ is a compact subset in $D \times \mathbb{C}^n$. But since Φ is an upper semicontinuous function on $D \times \mathbb{C}^n$, there is a positive constant $C > 0$ such that $\Phi(z, X) \leq C$ for all $(z, X) \in F \times \partial B$.

Let us now take any element $(z, X) \in F \times (\mathbb{C}^n \setminus \{0\})$. Then $\left(z, \frac{X}{\|X\|}\right) \in F \times \partial B$ and so we have $\Phi(z, X) \leq C\|X\|$. Thus we get the required conclusion. \square

By Proposition 2.3, note that the same conclusion as Theorem 2.8 holds for K_D^∞, \hat{K}_D and \tilde{K}_D .

Remark 2.9. Recall that if $h : \mathbb{C}^n \rightarrow \mathbb{R}$ is the Minkowski function for \mathbb{C}^n , then $h \equiv 0$ on \mathbb{C}^n . Now making use of Proposition 2.3 and Theorem 2.4, we have the following result.

$$K_{\mathbb{C}^n}^\infty(z, X) = \hat{K}_{\mathbb{C}^n}(z, X) = \tilde{K}_{\mathbb{C}^n}(z, X) = K_{\mathbb{C}^n}(z, X) \equiv 0$$

for all $(z, X) \in \mathbb{C}^n \times \mathbb{C}^n$.

Theorem 2.4 shows that $\hat{K}_G(z, \cdot)$ is not generally continuous as a function of the second variable. It follows from Corollary 2.5 and 2.6 that $\hat{K}_G(0, \cdot)$ gives a norm if G is a particular domain. The following example, in general, shows that $\hat{K}_G(z, \cdot)$ does not give a seminorm even though G is bounded.

Example 2.10. Let $G = \{z \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1, |z_1 z_2| < \frac{1}{2}\}$. Then G is a bounded balanced pseudoconvex domain with Minkowski function $h(z) = \max\{|z_1|, |z_2|, \sqrt{2}|z_1 z_2|\}$. By Theorem 2.4, we know that $\hat{K}_G(0, X) = h(X)$ for all $X \in \mathbb{C}^2$. In particular, we have

$$\hat{K}_G\left(0, \left(1, \frac{1}{2}\right)\right) + \hat{K}_G\left(0, \left(\frac{1}{2}, 1\right)\right) = 2 < \frac{3}{\sqrt{2}} = \hat{K}_G\left(0, \left(1, \frac{1}{2}\right) + \left(\frac{1}{2}, 1\right)\right).$$

Let G and D be domains in \mathbb{C}^n . A holomorphic map $\pi : G \rightarrow D$ is called a *holomorphic covering* if for any point $z \in D$ there exists an open neighborhood U of z with the property that each connected components of $\pi^{-1}(U)$ is mapped biholomorphically onto U by π .

Using Theorem 2.4 in [6] and the definition of K^∞, \hat{K} or \tilde{K} , we have the following :

Proposition 2.11. Let \tilde{G} and G be domains in \mathbb{C}^n , and let $\pi : \tilde{G} \rightarrow G$ be a holomorphic covering map³. Then for each $(\tilde{p}, X) \in \tilde{G} \times \mathbb{C}^n$ we

³Refer to [1] and [10] for a covering map and more informations

have the following

$$\begin{aligned} K_{\tilde{G}}^{\infty}(\tilde{p}, X) &= K_G^{\infty}(\pi(\tilde{p}), \pi'(\tilde{p})X), \\ \hat{K}_{\tilde{G}}(\tilde{p}, X) &= \hat{K}_G(\pi(\tilde{p}), \pi'(\tilde{p})X), \\ \tilde{K}_{\tilde{G}}(\tilde{p}, X) &= \tilde{K}_G(\pi(\tilde{p}), \pi'(\tilde{p})X). \end{aligned}$$

Proof. By Theorem 2.4 in [6], we have, for each $(\tilde{p}, X) \in \tilde{G} \times \mathbb{C}^n$ and for all $m \in \mathbb{N}$,

$$K_{\tilde{G}}^m(\tilde{p}, X) = K_G^m(\pi(\tilde{p}), \pi'(\tilde{p})X).$$

Now applying the definition of K^{∞} , \hat{K} or \tilde{K} , we have the required results.

□

3. The singular Kobayashi distances

We knew from Proposition 2.3 that the singular Kobayashi pseudometric, the lower singular Kobayashi pseudometric and the upper singular Kobayashi pseudometric are upper semicontinuous. So we can use them to define the length of a piecewise C^1 -curve. Furthermore, we are aware that the minimal length of all such curves connecting two fixed points will yield a new pseudodistance. To describe this procedure, we'll only deal with the lower singular Kobayashi pseudometric because the case of the singular Kobayashi pseudometric or the upper singular Kobayashi pseudometric is similar to that of the lower singular Kobayashi pseudometric.

For a domain $D \subset \mathbb{C}^n$, let us define the \hat{K}_D -length of a piecewise C^1 -curve $\gamma : [0, 1] \rightarrow D$ by

$$\hat{L}(\gamma) := \int_0^1 \hat{K}_D(\gamma(t), \gamma'(t)) dt.$$

Then $\hat{L}(\gamma) \in [0, \infty)$ and so we may define a map $\hat{k}_D : D \times D \rightarrow \mathbb{R}$, which is called the *integrated form* of \hat{K}_D , by

$$\hat{k}_D(z, w) := \inf_{\gamma} \hat{L}(\gamma)$$

where the infimum is taken over all piecewise C^1 -curves γ joining z and w . As usual, by k_D we mean the usual Kobayashi pseudodistance for D . Let us denote the integrated form of K_D^∞ and \tilde{K}_D by k_D^∞ and \tilde{k}_D , respectively. Then by Proposition 2.3, we have the following ;

Proposition 3.1. *Let $D \subset \mathbb{C}^n$ be a domain. Then the inequalities $k_D^\infty \leq \hat{k}_D \leq \tilde{k}_D \leq k_D$ hold. Furthermore, they are all pseudodistances on D .*

We call \hat{k}_D the *lower singular Kobayashi pseudodistance* on D . Moreover, if \hat{k}_D is a distance, we then call \hat{k}_D the *lower singular Kobayashi distance* on D .

The following holds by Corollary 2.5 as expected from Lempert’s Theorem([4]).

Proposition 3.2. *Let $B \subset \mathbb{C}^n$ be the open unit ball with center 0. Then we have $k_B^\infty(z, w) = k_B(z, w)$ for all $z, w \in B$.*

Proposition 2.3 and the definition of \hat{k} induce the following results. Note also that we have the similar results for k^∞ and \tilde{k} .

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^l$ and $D \subset \mathbb{C}^n$ be two domains. If $f : \Omega \rightarrow D$ is a holomorphic map, then $\hat{k}_\Omega(z, w) \geq \hat{k}_D(f(z), f(w))$ for any $z, w \in \Omega$. That is, \hat{k} has the distance decreasing property under holomorphic mappings. In particular, \hat{k} -distance is invariant under any biholomorphic map.*

Corollary 3.4. *Under the same assumption as Proposition 3.2, we have*

$$k_B^\infty(z, w) = \hat{k}_B(z, w) = \tilde{k}_B(z, w) = k_B(0, h_z(w)) = p(0, \|h_z(w)\|)$$

for all $z, w \in B$. Here p is the Poincaré distance for Δ and $h_z \in \text{Aut}(B)$ is an automorphism of B that sends z to 0.

By Corollary 2.6, we have the corresponding formula for the open unit polydisc as follows.

Corollary 3.5. *For the open unit polydisc Δ^n , we have*

$$k_{\Delta^n}^\infty(z, w) = \hat{k}_{\Delta^n}(z, w) = \tilde{k}_{\Delta^n}(z, w) = \max_{1 \leq j \leq n} p \left(0, \left| \frac{w_j - z_j}{1 - w_j \bar{z}_j} \right| \right)$$

for all $z, w \in \Delta^n$. Here p is the Poincaré distance for Δ .

Note that the conclusions of Proposition 3.3 hold with k^∞ and \tilde{k} in place of \hat{k} . Recalling also that any bounded domain is Kobayashi hyperbolic([4]), it follows from Proposition 3.2 and 3.3 that the following Corollary holds.

Corollary 3.6. *Let D be any open ball with a finite radius in \mathbb{C}^n . Then \hat{k}_D is the lower singular Kobayashi distance.*

We have the following results from Proposition 3.1.

Proposition 3.7. *Let $D \subset \mathbb{C}^n$ be a domain. If k_D^∞ is a distance, then \hat{k}_D is the lower singular Kobayashi distance.*

Proposition 3.8. *Let $D \subset \mathbb{C}^n$ be a domain. If one of k_D^∞, \hat{k}_D and \tilde{k}_D is a distance, then D is a Kobayashi hyperbolic domain.*

Theorem 3.9. *Let $\pi : \tilde{G} \rightarrow G$ be a holomorphic covering map and let $p, q \in G$. If $\tilde{p} \in \tilde{G}$ is an element such that $\pi(\tilde{p}) = p$, then the following holds;*

$$\hat{k}_G(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} \hat{k}_{\tilde{G}}(\tilde{p}, \tilde{q}).$$

Proof. By the holomorphic contraction property (Proposition 3.3), we have

$$\hat{k}_G(p, q) \leq \inf_{\tilde{q} \in \pi^{-1}(q)} \hat{k}_{\tilde{G}}(\tilde{p}, \tilde{q}).$$

Hence to show the reverse inequality, suppose that there exists an $\epsilon > 0$ such that the inequality

$$\hat{k}_G(p, q) + 2\epsilon \leq \inf_{\tilde{q} \in \pi^{-1}(q)} \hat{k}_{\tilde{G}}(\tilde{p}, \tilde{q})$$

holds. Then by the definition of $\hat{k}_G(p, q)$, there is a piecewise C^1 -curve $\gamma : [0, 1] \rightarrow G$ connecting p and q such that

$$\int_0^1 \hat{K}_G(\gamma(t), \gamma'(t)) dt < \hat{k}_G(p, q) + \epsilon.$$

Since $\pi : \tilde{G} \rightarrow G$ is a holomorphic covering, there are a $\tilde{q} \in \pi^{-1}(q)$ and a piecewise C^1 -curve $\tilde{\gamma} : [0, 1] \rightarrow \tilde{G}$ connecting \tilde{p} and \tilde{q} such that $\pi \circ \tilde{\gamma} = \gamma$.

On the other hand, by Proposition 2.11 for the lower singular Kobayashi pseudometric, we have

$$\begin{aligned} \int_0^1 \hat{K}_G(\gamma(t), \gamma'(t)) dt &= \int_0^1 \hat{K}_G((\pi \circ \tilde{\gamma})(t), (\pi \circ \tilde{\gamma})'(t)) dt \\ &= \int_0^1 \hat{K}_{\tilde{G}}(\tilde{\gamma}(t), \tilde{\gamma}'(t)) dt. \end{aligned}$$

Hence we have

$$\begin{aligned} \hat{k}_{\tilde{G}}(\tilde{p}, \tilde{q}) &\leq \int_0^1 \hat{K}_{\tilde{G}}(\tilde{\gamma}(t), \tilde{\gamma}'(t)) dt \\ &= \int_0^1 \hat{K}_G(\gamma(t), \gamma'(t)) dt \\ &< \hat{k}_G(p, q) + \epsilon, \end{aligned}$$

which is a contradiction to our assumption. \square

Taking account of k^∞ or \tilde{k} instead of \hat{k} and proceeding with the same way as Theorem 3.9, we can get the following.

Corollary 3.10. *Under the same assumption as Theorem 3.9, we have*

$$k_G^\infty(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} k_G^\infty(\tilde{p}, \tilde{q}),$$

$$\tilde{k}_G(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} \tilde{k}_G(\tilde{p}, \tilde{q}).$$

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